Multivariable Advanced Calculus

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Introduction

This book is directed to people who have a good understanding of the concepts of one variable calculus including the notions of limit of a sequence and completeness of $\mathbb{R}$. It develops multivariable advanced calculus.

In order to do multivariable calculus correctly, you must first understand some linear algebra. Therefore, a condensed course in linear algebra is presented first, emphasizing those topics in linear algebra which are useful in analysis, not those topics which are primarily dependent on row operations.

Many topics could be presented in greater generality than I have chosen to do. I have also attempted to feature calculus, not topology. This means I introduce the topology as it is needed rather than using the possibly more efficient practice of placing it right at the beginning in more generality than will be needed. I think it might make the topological concepts more memorable by linking them in this way to other concepts.

Some Fundamental Concepts

2.1 Set Theory

2.1.1 Basic Definitions

A set is a collection of things called elements of the set. For example, the set of integers, the collection of signed whole numbers such as 1,2,-4, etc. This set whose existence will be assumed is denoted by $\mathbb{Z}$. Other sets could be the set of people in a family or the set of donuts in a display case at the store. Sometimes parentheses, { } specify a set by listing the things which are in the set between the parentheses. For example the set of integers between -1 and 2, including these numbers could be denoted as $\{-1,0,1,2\}$. The notation signifying x is an element of a set S, is written as $x \in S$. Thus, $1 \in \{-1,0,1,2,3\}$. Here are some axioms about sets. Axioms are statements which are accepted, not proved.

1. Two sets are equal if and only if they have the same elements.
2. To every set, A, and to every condition $S(x)$ there corresponds a set, B, whose elements are exactly those elements x of A for which $S(x)$ holds.
3. For every collection of sets there exists a set that contains all the elements that belong to at least one set of the given collection.
4. The Cartesian product of a nonempty family of nonempty sets is nonempty.
5. If A is a set there exists a set, $\mathcal{P}(A)$ such that $\mathcal{P}(A)$ is the set of all subsets of A. This is called the power set.

These axioms are referred to as the axiom of extension, axiom of specification, axiom of unions, axiom of choice, and axiom of powers respectively.

It seems fairly clear you should want to believe in the axiom of extension. It is merely saying, for example, that $\{1,2,3\} = \{2,3,1\}$ since these two sets have the same elements in them. Similarly, it would seem you should be able to specify a new set from a given set using some "condition" which can be used as a test to determine whether the element in question is in the set. For example, the set of all integers which are multiples of 2. This set could be specified as follows.

$$\{x \in \mathbb{Z} : x = 2y \text{ for some } y \in \mathbb{Z}\}.$$

In this notation, the colon is read as "such that" and in this case the condition is being a multiple of 2.

Another example of political interest, could be the set of all judges who are not judicial activists. I think you can see this last is not a very precise condition since there is no way

to determine to everyone's satisfaction whether a given judge is an activist. Also, **just because something is grammatically correct does not mean it makes any sense.** For example consider the following nonsense.

$$S = \{x \in \text{set of dogs} \ : \ \text{it is colder in the mountains than in the winter}\}.$$

So what is a condition?

We will leave these sorts of considerations and assume our conditions make sense. The axiom of unions states that for any collection of sets, there is a set consisting of all the elements in each of the sets in the collection. Of course this is also open to further consideration. What is a collection? Maybe it would be better to say "set of sets" or, given a set whose elements are sets there exists a set whose elements consist of exactly those things which are elements of at least one of these sets. If $\mathcal{S}$ is such a set whose elements are sets,

$$\cup\{A : A \in \mathcal{S}\} \text{ or } \cup\mathcal{S}$$

signify this union.

Something is in the Cartesian product of a set or "family" of sets if it consists of a single thing taken from each set in the family. Thus $(1,2,3) \in \{1,4,.2\} \times \{1,2,7\} \times \{4,3,7,9\}$ because it consists of exactly one element from each of the sets which are separated by $\times$. Also, this is the notation for the Cartesian product of finitely many sets. If $\mathcal{S}$ is a set whose elements are sets,

$$\prod_{A\in\mathcal{S}} A$$

signifies the Cartesian product.

The Cartesian product is the set of choice functions, a choice function being a function which selects exactly one element of each set of $\mathcal{S}$. You may think the axiom of choice, stating that the Cartesian product of a nonempty family of nonempty sets is nonempty, is innocuous but there was a time when many mathematicians were ready to throw it out because it implies things which are very hard to believe, things which never happen without the axiom of choice.

A is a subset of B, written $A \subseteq B$, if every element of A is also an element of B. This can also be written as $B \supseteq A$. A is a proper subset of B, written $A \subset B$ or $B \supset A$ if A is a subset of B but A is not equal to $B, A \neq B$. $A \cap B$ denotes the intersection of the two sets, A and B and it means the set of elements of A which are also elements of B. The axiom of specification shows this is a set. The empty set is the set which has no elements in it, denoted as $\emptyset$. $A \cup B$ denotes the union of the two sets, A and B and it means the set of all elements which are in either of the sets. It is a set because of the axiom of unions.

The complement of a set, (the set of things which are not in the given set) must be taken with respect to a given set called the universal set which is a set which contains the one whose complement is being taken. Thus, the complement of A, denoted as A^C (or more precisely as $X \setminus A$) is a set obtained from using the axiom of specification to write

$$A^C \equiv \{x \in X : x \notin A\}$$

The symbol $\notin$ means: "is not an element of". Note the axiom of specification takes place relative to a given set. Without this universal set it makes no sense to use the axiom of specification to obtain the complement.

Words such as "all" or "there exists" are called quantifiers and they must be understood relative to some given set. For example, the set of all integers larger than 3. Or there exists an integer larger than 7. Such statements have to do with a given set, in this case the integers. Failure to have a reference set when quantifiers are used turns out to be illogical

even though such usage may be grammatically correct. Quantifiers are used often enough that there are symbols for them. The symbol $\forall$ is read as "for all" or "for every" and the symbol $\exists$ is read as "there exists". Thus $\forall\forall\exists\exists$ could mean for every upside down A there exists a backwards E.

DeMorgan's laws are very useful in mathematics. Let $\mathcal{S}$ be a set of sets each of which is contained in some universal set, U. Then

$$\cup\left\{A^C : A \in \mathcal{S}\right\} = \left(\cap\left\{A : A \in \mathcal{S}\right\}\right)^C$$

and

$$\cap\left\{A^C : A \in \mathcal{S}\right\} = \left(\cup\left\{A : A \in \mathcal{S}\right\}\right)^C.$$

These laws follow directly from the definitions. Also following directly from the definitions are:

Let $\mathcal{S}$ be a set of sets then

$$B \cup \cup\left\{A : A \in \mathcal{S}\right\} = \cup\left\{B \cup A : A \in \mathcal{S}\right\}.$$

and: Let $\mathcal{S}$ be a set of sets show

$$B \cap \cup\left\{A : A \in \mathcal{S}\right\} = \cup\left\{B \cap A : A \in \mathcal{S}\right\}.$$

Unfortunately, there is no single universal set which can be used for all sets. Here is why: Suppose there were. Call it S. Then you could consider A the set of all elements of S which are not elements of themselves, this from the axiom of specification. If A is an element of itself, then it fails to qualify for inclusion in A. Therefore, it must not be an element of itself. However, if this is so, it qualifies for inclusion in A so it is an element of itself and so this can't be true either. Thus the most basic of conditions you could imagine, that of being an element of, is meaningless and so allowing such a set causes the whole theory to be meaningless. The solution is to not allow a universal set. As mentioned by Halmos in Naive set theory, "Nothing contains everything". Always beware of statements involving quantifiers wherever they occur, even this one. This little observation described above is due to Bertrand Russell and is called Russell's paradox.

2.1.2 The Schroder Bernstein Theorem

It is very important to be able to compare the size of sets in a rational way. The most useful theorem in this context is the Schroder Bernstein theorem which is the main result to be presented in this section. The Cartesian product is discussed above. The next definition reviews this and defines the concept of a function.

Definition 2.1.1 *Let X and Y be sets.*

$$X \times Y \equiv \{(x, y) : x \in X \text{ and } y \in Y\}$$

A relation is defined to be a subset of $X \times Y$. A function, f, also called a mapping, is a relation which has the property that if (x, y) and (x, y_1) are both elements of the f, then $y = y_1$. The domain of f is defined as

$$D(f) \equiv \{x : (x, y) \in f\},$$

written as $f : D(f) \to Y$.

It is probably safe to say that most people do not think of functions as a type of relation which is a subset of the Cartesian product of two sets. A function is like a machine which takes inputs, x and makes them into a unique output, $f(x)$. Of course, that is what the above definition says with more precision. An ordered pair, (x,y) which is an element of the function or mapping has an input, x and a unique output, y,denoted as $f(x)$ while the name of the function is f. "mapping" is often a noun meaning function. However, it also is a verb as in "f is mapping A to B ". That which a function is thought of as doing is also referred to using the word "maps" as in: f maps X to Y. However, a set of functions may be called a set of maps so this word might also be used as the plural of a noun. There is no help for it. You just have to suffer with this nonsense.

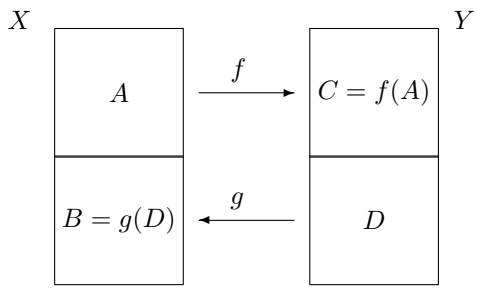
The following theorem which is interesting for its own sake will be used to prove the Schroder Bernstein theorem.

Theorem 2.1.2 *Let $f : X \to Y$ and $g : Y \to X$ be two functions. Then there exist sets A, B, C, D, such that*

$$A \cup B = X,\ C \cup D = Y,\ A \cap B = \emptyset,\ C \cap D = \emptyset,$$

$$f(A) = C,\ g(D) = B.$$

The following picture illustrates the conclusion of this theorem.



Proof: Consider the empty set, $\emptyset \subseteq X$. If $y \in Y \setminus f(\emptyset)$, then $g(y) \notin \emptyset$ because $\emptyset$ has no elements. Also, if A, B, C, and D are as described above, A also would have this same property that the empty set has. However, A is probably larger. Therefore, say $A_0 \subseteq X$ satisfies $\mathcal{P}$ if whenever $y \in Y \setminus f(A_0)$, $g(y) \notin A_0$.

$$\mathcal{A} \equiv \{A_0 \subseteq X : A_0 \text{ satisfies } \mathcal{P}\}.$$

Let $A = \cup\mathcal{A}$. If $y \in Y \setminus f(A)$, then for each $A_0 \in \mathcal{A}$, $y \in Y \setminus f(A_0)$ and so $g(y) \notin A_0$. Since $g(y) \notin A_0$ for all $A_0 \in \mathcal{A}$, it follows $g(y) \notin A$. Hence A satisfies $\mathcal{P}$ and is the largest subset of X which does so. Now define

$$C \equiv f(A),\ D \equiv Y \setminus C,\ B \equiv X \setminus A.$$

It only remains to verify that $g(D) = B$.

Suppose $x \in B = X \setminus A$. Then $A \cup \{x\}$ does not satisfy $\mathcal{P}$ and so there exists $y \in Y \setminus f(A \cup \{x\}) \subseteq D$ such that $g(y) \in A \cup \{x\}$. But $y \notin f(A)$ and so since A satisfies $\mathcal{P}$, it follows $g(y) \notin A$. Hence $g(y) = x$ and so $x \in g(D)$ and this proves the theorem.

Theorem 2.1.3 *(Schroder Bernstein) If $f : X \to Y$ and $g : Y \to X$ are one to one, then there exists $h : X \to Y$ which is one to one and onto.*

Proof: Let A, B, C, D be the sets of Theorem2.1.2 and define

$$h(x) \equiv \begin{cases} f(x) & \text{if } x \in A \\ g^{-1}(x) & \text{if } x \in B \end{cases}$$

Then h is the desired one to one and onto mapping.

Recall that the Cartesian product may be considered as the collection of choice functions.

Definition 2.1.4 *Let I be a set and let X_i be a set for each $i \in I$. f is a choice function written as*

$$f \in \prod_{i \in I} X_i$$

if $f(i) \in X_i$ for each $i \in I$.

The axiom of choice says that if $X_i \neq \emptyset$ for each $i \in I$, for I a set, then

$$\prod_{i \in I} X_i \neq \emptyset.$$

Sometimes the two functions, f and g are onto but not one to one. It turns out that with the axiom of choice, a similar conclusion to the above may be obtained.

Corollary 2.1.5 *If $f : X \to Y$ is onto and $g : Y \to X$ is onto, then there exists $h : X \to Y$ which is one to one and onto.*

Proof: For each $y \in Y$, $f^{-1}(y) \equiv \{x \in X : f(x) = y\} \neq \emptyset$. Therefore, by the axiom of choice, there exists $f_0^{-1} \in \prod_{y \in Y} f^{-1}(y)$ which is the same as saying that for each $y \in Y$, $f_0^{-1}(y) \in f^{-1}(y)$. Similarly, there exists $g_0^{-1}(x) \in g^{-1}(x)$ for all $x \in X$. Then f_0^{-1} is one to one because if $f_0^{-1}(y_1) = f_0^{-1}(y_2)$, then

$$y_1 = f\left(f_0^{-1}(y_1)\right) = f\left(f_0^{-1}(y_2)\right) = y_2.$$

Similarly g_0^{-1} is one to one. Therefore, by the Schroder Bernstein theorem, there exists $h : X \to Y$ which is one to one and onto.

Definition 2.1.6 *A set S, is finite if there exists a natural number n and a map θ which maps $\{1, \cdots, n\}$ one to one and onto S. S is infinite if it is not finite. A set S, is called countable if there exists a map θ mapping $\mathbb{N}$ one to one and onto S.(When θ maps a set A to a set B, this will be written as $\theta : A \to B$ in the future.) Here $\mathbb{N} \equiv \{1, 2, \cdots\}$, the natural numbers. S is at most countable if there exists a map $\theta : \mathbb{N} \to S$ which is onto.*

The property of being at most countable is often referred to as being countable because the question of interest is normally whether one can list all elements of the set, designating a first, second, third etc. in such a way as to give each element of the set a natural number. The possibility that a single element of the set may be counted more than once is often not important.

Theorem 2.1.7 *If X and Y are both at most countable, then $X \times Y$ is also at most countable. If either X or Y is countable, then $X \times Y$ is also countable.*

Proof: It is given that there exists a mapping $\eta : \mathbb{N} \to X$ which is onto. Define $\eta(i) \equiv x_i$ and consider X as the set $\{x_1, x_2, x_3, \cdots\}$. Similarly, consider Y as the set $\{y_1, y_2, y_3, \cdots\}$. It follows the elements of $X \times Y$ are included in the following rectangular array.

$$\begin{array}{llll} (x_1,y_1) & (x_1,y_2) & (x_1,y_3) & \cdots \quad \leftarrow \text{Those which have } x_1 \text{ in first slot.} \\ (x_2,y_1) & (x_2,y_2) & (x_2,y_3) & \cdots \quad \leftarrow \text{Those which have } x_2 \text{ in first slot.} \\ (x_3,y_1) & (x_3,y_2) & (x_3,y_3) & \cdots \quad \leftarrow \text{Those which have } x_3 \text{ in first slot.} \\ \vdots & \vdots & \vdots & \qquad\qquad\qquad\qquad \vdots \end{array}$$

Follow a path through this array as follows.

$$\begin{array}{ccccc} (x_1,y_1) & \rightarrow & (x_1,y_2) & & (x_1,y_3) \quad \rightarrow \\ & \swarrow & & \nearrow & \\ (x_2,y_1) & & (x_2,y_2) & & \\ \downarrow & \nearrow & & & \\ (x_3,y_1) & & & & \end{array}$$

Thus the first element of $X \times Y$ is (x_1, y_1), the second element of $X \times Y$ is (x_1, y_2), the third element of $X \times Y$ is (x_2, y_1) etc. This assigns a number from $\mathbb{N}$ to each element of $X \times Y$. Thus $X \times Y$ is at most countable.

It remains to show the last claim. Suppose without loss of generality that X is countable. Then there exists $\alpha : \mathbb{N} \to X$ which is one to one and onto. Let $\beta : X \times Y \to \mathbb{N}$ be defined by $\beta((x,y)) \equiv \alpha^{-1}(x)$. Thus β is onto $\mathbb{N}$. By the first part there exists a function from $\mathbb{N}$ onto $X \times Y$. Therefore, by Corollary 2.1.5, there exists a one to one and onto mapping from $X \times Y$ to $\mathbb{N}$. This proves the theorem.

Theorem 2.1.8 *If X and Y are at most countable, then $X \cup Y$ is at most countable. If either X or Y are countable, then $X \cup Y$ is countable.*

Proof: As in the preceding theorem,

$$X = \{x_1, x_2, x_3, \cdots\}$$

and

$$Y = \{y_1, y_2, y_3, \cdots\}.$$

Consider the following array consisting of $X \cup Y$ and path through it.

$$\begin{array}{ccccc} x_1 & \rightarrow & x_2 & & x_3 \quad \rightarrow \\ & \swarrow & & \nearrow & \\ y_1 & \rightarrow & y_2 & & \end{array}$$

Thus the first element of $X \cup Y$ is x_1, the second is x_2 the third is y_1 the fourth is y_2 etc.

Consider the second claim. By the first part, there is a map from $\mathbb{N}$ onto $X \times Y$. Suppose without loss of generality that X is countable and $\alpha : \mathbb{N} \to X$ is one to one and onto. Then define $\beta(y) \equiv 1$, for all $y \in Y$,and $\beta(x) \equiv \alpha^{-1}(x)$. Thus, β maps $X \times Y$ onto $\mathbb{N}$ and this shows there exist two onto maps, one mapping $X \cup Y$ onto $\mathbb{N}$ and the other mapping $\mathbb{N}$ onto $X \cup Y$. Then Corollary 2.1.5 yields the conclusion. This proves the theorem.

2.1.3 Equivalence Relations

There are many ways to compare elements of a set other than to say two elements are equal or the same. For example, in the set of people let two people be equivalent if they have the

same weight. This would not be saying they were the same person, just that they weighed the same. Often such relations involve considering one characteristic of the elements of a set and then saying the two elements are equivalent if they are the same as far as the given characteristic is concerned.

Definition 2.1.9 *Let S be a set. $\sim$ is an equivalence relation on S if it satisfies the following axioms.*

1. *$x \sim x$ for all $x \in S$. (Reflexive)*
2. *If $x \sim y$ then $y \sim x$. (Symmetric)*
3. *If $x \sim y$ and $y \sim z$, then $x \sim z$. (Transitive)*

Definition 2.1.10 *$[x]$ denotes the set of all elements of S which are equivalent to x and $[x]$ is called the equivalence class determined by x or just the equivalence class of x.*

With the above definition one can prove the following simple theorem.

Theorem 2.1.11 *Let $\sim$ be an equivalence class defined on a set, S and let $\mathcal{H}$ denote the set of equivalence classes. Then if $[x]$ and $[y]$ are two of these equivalence classes, either $x \sim y$ and $[x] = [y]$ or it is not true that $x \sim y$ and $[x] \cap [y] = \emptyset$.*

2.2 lim sup and lim inf

It is assumed in all that is done that $\mathbb{R}$ is complete. There are two ways to describe completeness of $\mathbb{R}$. One is to say that every bounded set has a least upper bound and a greatest lower bound. The other is to say that every Cauchy sequence converges. These two equivalent notions of completeness will be taken as given.

The symbol, $\mathbb{F}$ will mean either $\mathbb{R}$ or $\mathbb{C}$.

Definition 2.2.1 *For A a nonempty set of real numbers,* $\sup A$ *is defined as the least upper bound in case A is bounded above and equals ∞ if A is not bounded above. Similarly* $\inf A$ *is defined to equal the greatest lower bound in case A is bounded below and equals $-\infty$ in case A is not bounded below.*

Sometimes the limit of a sequence does not exist. For example, if $a_n = (-1)^n$, then $\lim_{n\to\infty} a_n$ does not exist. This is because the terms of the sequence are a distance of 1 apart. Therefore there can't exist a single number such that all the terms of the sequence are ultimately within $1/4$ of that number. The nice thing about lim sup and lim inf is that they always exist. First here is a simple lemma and definition.

Definition 2.2.2 *Denote by $[-\infty, \infty]$ the real line along with symbols ∞ and $-\infty$. It is understood that ∞ is larger than every real number and $-\infty$ is smaller than every real number. Then if $\{A_n\}$ is an increasing sequence of points of $[-\infty, \infty]$, $\lim_{n\to\infty} A_n$ equals ∞ if the only upper bound of the set $\{A_n\}$ is ∞. If $\{A_n\}$ is bounded above by a real number, then $\lim_{n\to\infty} A_n$ is defined in the usual way and equals the least upper bound of $\{A_n\}$. If $\{A_n\}$ is a decreasing sequence of points of $[-\infty, \infty]$, $\lim_{n\to\infty} A_n$ equals $-\infty$ if the only lower bound of the sequence $\{A_n\}$ is $-\infty$. If $\{A_n\}$ is bounded below by a real number, then $\lim_{n\to\infty} A_n$ is defined in the usual way and equals the greatest lower bound of $\{A_n\}$. More simply, if $\{A_n\}$ is increasing,*

$$\lim_{n\to\infty} A_n \equiv \sup\{A_n\}$$

and if $\{A_n\}$ is decreasing then

$$\lim_{n\to\infty} A_n \equiv \inf\{A_n\}.$$

Lemma 2.2.3 *Let $\{a_n\}$ be a sequence of real numbers and let $U_n \equiv \sup\{a_k : k \geq n\}$. Then $\{U_n\}$ is a decreasing sequence. Also if $L_n \equiv \inf\{a_k : k \geq n\}$, then $\{L_n\}$ is an increasing sequence. Therefore, $\lim_{n\to\infty} L_n$ and $\lim_{n\to\infty} U_n$ both exist.*

Proof: Let W_n be an upper bound for $\{a_k : k \geq n\}$. Then since these sets are getting smaller, it follows that for $m < n$, W_m is an upper bound for $\{a_k : k \geq n\}$. In particular if $W_m = U_m$, then U_m is an upper bound for $\{a_k : k \geq n\}$ and so U_m is at least as large as U_n, the least upper bound for $\{a_k : k \geq n\}$. The claim that $\{L_n\}$ is decreasing is similar. This proves the lemma.

From the lemma, the following definition makes sense.

Definition 2.2.4

Let $\{a_n\}$ be any sequence of points of $[-\infty, \infty]$

$$\limsup_{n\to\infty} a_n \equiv \lim_{n\to\infty} \sup\{a_k : k \geq n\}$$

$$\liminf_{n\to\infty} a_n \equiv \lim_{n\to\infty} \inf\{a_k : k \geq n\}.$$

Theorem 2.2.5

Suppose $\{a_n\}$ is a sequence of real numbers and that $\limsup_{n\to\infty} a_n$ and $\liminf_{n\to\infty} a_n$ are both real numbers. Then $\lim_{n\to\infty} a_n$ exists if and only if $\liminf_{n\to\infty} a_n = \limsup_{n\to\infty} a_n$ and in this case,

$$\lim_{n\to\infty} a_n = \liminf_{n\to\infty} a_n = \limsup_{n\to\infty} a_n.$$

Proof: First note that

$$\sup\{a_k : k \geq n\} \geq \inf\{a_k : k \geq n\}$$

and so from Theorem 4.1.7,

$$\begin{aligned} \limsup_{n\to\infty} a_n &\equiv \lim_{n\to\infty} \sup\{a_k : k \geq n\} \\ &\geq \lim_{n\to\infty} \inf\{a_k : k \geq n\} \\ &\equiv \liminf_{n\to\infty} a_n. \end{aligned}$$

Suppose first that $\lim_{n\to\infty} a_n$ exists and is a real number. Then by Theorem 4.4.3 $\{a_n\}$ is a Cauchy sequence. Therefore, if $\varepsilon > 0$ is given, there exists N such that if $m, n \geq N$, then

$$|a_n - a_m| < \varepsilon/3.$$

From the definition of $\sup\{a_k : k \geq N\}$, there exists $n_1 \geq N$ such that

$$\sup\{a_k : k \geq N\} \leq a_{n_1} + \varepsilon/3.$$

Similarly, there exists $n_2 \geq N$ such that

$$\inf\{a_k : k \geq N\} \geq a_{n_2} - \varepsilon/3.$$

It follows that

$$\sup\{a_k : k \geq N\} - \inf\{a_k : k \geq N\} \leq |a_{n_1} - a_{n_2}| + \frac{2\varepsilon}{3} < \varepsilon.$$

Since the sequence, $\{\sup\{a_k : k \geq N\}\}_{N=1}^{\infty}$ is decreasing and $\{\inf\{a_k : k \geq N\}\}_{N=1}^{\infty}$ is increasing, it follows from Theorem 4.1.7

$$0 \leq \lim_{N\to\infty} \sup\{a_k : k \geq N\} - \lim_{N\to\infty} \inf\{a_k : k \geq N\} \leq \varepsilon$$

Since ε is arbitrary, this shows

$$\lim_{N\to\infty} \sup\{a_k : k \geq N\} = \lim_{N\to\infty} \inf\{a_k : k \geq N\} \tag{2.1}$$

Next suppose 2.1. Then

$$\lim_{N\to\infty} (\sup\{a_k : k \geq N\} - \inf\{a_k : k \geq N\}) = 0$$

Since $\sup\{a_k : k \geq N\} \geq \inf\{a_k : k \geq N\}$ it follows that for every $\varepsilon > 0$, there exists N such that

$$\sup\{a_k : k \geq N\} - \inf\{a_k : k \geq N\} < \varepsilon$$

Thus if $m, n > N$, then

$$|a_m - a_n| < \varepsilon$$

which means $\{a_n\}$ is a Cauchy sequence. Since $\mathbb{R}$ is complete, it follows that $\lim_{n\to\infty} a_n \equiv a$ exists. By the squeezing theorem, it follows

$$a = \lim \inf_{n\to\infty} a_n = \lim \sup_{n\to\infty} a_n$$

and this proves the theorem.

With the above theorem, here is how to define the limit of a sequence of points in $[-\infty, \infty]$.

Definition 2.2.6 *Let $\{a_n\}$ be a sequence of points of $[-\infty, \infty]$. Then $\lim_{n\to\infty} a_n$ exists exactly when*

$$\lim \inf_{n\to\infty} a_n = \lim \sup_{n\to\infty} a_n$$

and in this case

$$\lim_{n\to\infty} a_n \equiv \lim \inf_{n\to\infty} a_n = \lim \sup_{n\to\infty} a_n.$$

The significance of lim sup and lim inf, in addition to what was just discussed, is contained in the following theorem which follows quickly from the definition.

Theorem 2.2.7 *Suppose $\{a_n\}$ is a sequence of points of $[-\infty, \infty]$. Let*

$$\lambda = \lim \sup_{n\to\infty} a_n.$$

Then if $b > \lambda$, it follows there exists N such that whenever $n \geq N$,

$$a_n \leq b.$$

If $c < \lambda$, then $a_n > c$ for infinitely many values of n. Let

$$\gamma = \lim \inf_{n\to\infty} a_n.$$

Then if $d < \gamma$, it follows there exists N such that whenever $n \geq N$,

$$a_n \geq d.$$

If $e > \gamma$, it follows $a_n < e$ for infinitely many values of n.

The proof of this theorem is left as an exercise for you. It follows directly from the definition and it is the sort of thing you must do yourself. Here is one other simple proposition.

Proposition 2.2.8 *Let* $\lim_{n\to\infty} a_n = a > 0$. *Then*

$$\limsup_{n\to\infty} a_n b_n = a \limsup_{n\to\infty} b_n.$$

Proof: This follows from the definition. Let $\lambda_n = \sup\{a_k b_k : k \geq n\}$. For all n large enough, $a_n > a - \varepsilon$ where ε is small enough that $a - \varepsilon > 0$. Therefore,

$$\lambda_n \geq \sup\{b_k : k \geq n\}(a - \varepsilon)$$

for all n large enough. Then

$$\begin{aligned} \limsup_{n\to\infty} a_n b_n &= \lim_{n\to\infty} \lambda_n \equiv \limsup_{n\to\infty} a_n b_n \\ &\geq \lim_{n\to\infty} (\sup\{b_k : k \geq n\}(a - \varepsilon)) \\ &= (a - \varepsilon) \limsup_{n\to\infty} b_n \end{aligned}$$

Similar reasoning shows

$$\limsup_{n\to\infty} a_n b_n \leq (a + \varepsilon) \limsup_{n\to\infty} b_n$$

Now since $\varepsilon > 0$ is arbitrary, the conclusion follows.

2.3 Double Series

Sometimes it is required to consider double series which are of the form

$$\sum_{k=m}^{\infty} \sum_{j=m}^{\infty} a_{jk} \equiv \sum_{k=m}^{\infty} \left(\sum_{j=m}^{\infty} a_{jk} \right).$$

In other words, first sum on j yielding something which depends on k and then sum these. The major consideration for these double series is the question of when

$$\sum_{k=m}^{\infty} \sum_{j=m}^{\infty} a_{jk} = \sum_{j=m}^{\infty} \sum_{k=m}^{\infty} a_{jk}.$$

In other words, when does it make no difference which subscript is summed over first? In the case of finite sums there is no issue here. You can always write

$$\sum_{k=m}^{M} \sum_{j=m}^{N} a_{jk} = \sum_{j=m}^{N} \sum_{k=m}^{M} a_{jk}$$

because addition is commutative. However, there are limits involved with infinite sums and the interchange in order of summation involves taking limits in a different order. Therefore, it is not always true that it is permissible to interchange the two sums. A general rule of thumb is this: If something involves changing the order in which two limits are taken, you may not do it without agonizing over the question. In general, limits foul up algebra and also introduce things which are counter intuitive. Here is an example. This example is a little technical. It is placed here just to prove conclusively there is a question which needs to be considered.

Example 2.3.1 *Consider the following picture which depicts some of the ordered pairs (m,n) where m,n are positive integers.*

The numbers next to the point are the values of a_{mn}. You see $a_{nn}=0$ for all n, $a_{21}=a, a_{12}=b, a_{mn}=c$ for (m,n) on the line $y=1+x$ whenever $m>1$, and $a_{mn}=-c$ for all (m,n) on the line $y=x-1$ whenever $m>2$.

Then $\sum_{m=1}^{\infty} a_{mn}=a$ if $n=1$, $\sum_{m=1}^{\infty} a_{mn}=b-c$ if $n=2$ and if $n>2, \sum_{m=1}^{\infty} a_{mn}=0$. Therefore,

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} a_{mn}=a+b-c.$$

Next observe that $\sum_{n=1}^{\infty} a_{mn}=b$ if $m=1, \sum_{n=1}^{\infty} a_{mn}=a+c$ if $m=2$, and $\sum_{n=1}^{\infty} a_{mn}=0$ if $m>2$. Therefore,

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} a_{mn}=b+a+c$$

and so the two sums are different. Moreover, you can see that by assigning different values of a,b, and c, you can get an example for any two different numbers desired.

It turns out that if $a_{ij}\geq 0$ for all i,j, then you can always interchange the order of summation. This is shown next and is based on the following lemma. First, some notation should be discussed.

Definition 2.3.2 *Let $f(a,b)\in[-\infty,\infty]$ for $a\in A$ and $b\in B$ where A,B are sets which means that $f(a,b)$ is either a number, ∞, or $-\infty$. The symbol, $+\infty$ is interpreted as a point out at the end of the number line which is larger than every real number. Of*

course there is no such number. That is why it is called ∞. *The symbol,* $-\infty$ *is interpreted similarly. Then* $\sup_{a\in A} f(a,b)$ *means* $\sup(S_b)$ *where* $S_b \equiv \{f(a,b) : a \in A\}$.

Unlike limits, you can take the sup in different orders.

Lemma 2.3.3 *Let* $f(a,b) \in [-\infty,\infty]$ *for* $a \in A$ *and* $b \in B$ *where* A, B *are sets. Then*

$$\sup_{a\in A}\sup_{b\in B} f(a,b) = \sup_{b\in B}\sup_{a\in A} f(a,b).$$

Proof: Note that for all a, b, $f(a,b) \le \sup_{b\in B}\sup_{a\in A} f(a,b)$ and therefore, for all a, $\sup_{b\in B} f(a,b) \le \sup_{b\in B}\sup_{a\in A} f(a,b)$. Therefore,

$$\sup_{a\in A}\sup_{b\in B} f(a,b) \le \sup_{b\in B}\sup_{a\in A} f(a,b).$$

Repeat the same argument interchanging a and b, to get the conclusion of the lemma.

Lemma 2.3.4 *If* $\{A_n\}$ *is an increasing sequence in* $[-\infty,\infty]$, *then*

$$\sup\{A_n\} = \lim_{n\to\infty} A_n.$$

Proof: Let $\sup(\{A_n : n \in \mathbb{N}\}) = r$. In the first case, suppose $r < \infty$. Then letting $\varepsilon > 0$ be given, there exists n such that $A_n \in (r-\varepsilon, r]$. Since $\{A_n\}$ is increasing, it follows if $m > n$, then $r - \varepsilon < A_n \le A_m \le r$ and so $\lim_{n\to\infty} A_n = r$ as claimed. In the case where $r = \infty$, then if a is a real number, there exists n such that $A_n > a$. Since $\{A_k\}$ is increasing, it follows that if $m > n$, $A_m > a$. But this is what is meant by $\lim_{n\to\infty} A_n = \infty$. The other case is that $r = -\infty$. But in this case, $A_n = -\infty$ for all n and so $\lim_{n\to\infty} A_n = -\infty$.

Theorem 2.3.5 *Let* $a_{ij} \ge 0$. *Then*

$$\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} a_{ij} = \sum_{j=1}^{\infty}\sum_{i=1}^{\infty} a_{ij}.$$

Proof: First note there is no trouble in defining these sums because the a_{ij} are all nonnegative. If a sum diverges, it only diverges to ∞ and so ∞ is the value of the sum. Next note that

$$\sum_{j=r}^{\infty}\sum_{i=r}^{\infty} a_{ij} \ge \sup_n \sum_{j=r}^{\infty}\sum_{i=r}^{n} a_{ij}$$

because for all j,

$$\sum_{i=r}^{\infty} a_{ij} \ge \sum_{i=r}^{n} a_{ij}.$$

Therefore,

$$\sum_{j=r}^{\infty}\sum_{i=r}^{\infty} a_{ij} \ge \sup_n \sum_{j=r}^{\infty}\sum_{i=r}^{n} a_{ij} = \sup_n \lim_{m\to\infty} \sum_{j=r}^{m}\sum_{i=r}^{n} a_{ij}$$

$$= \sup_n \lim_{m\to\infty} \sum_{i=r}^{n}\sum_{j=r}^{m} a_{ij} = \sup_n \sum_{i=r}^{n} \lim_{m\to\infty} \sum_{j=r}^{m} a_{ij}$$

$$= \sup_n \sum_{i=r}^{n}\sum_{j=r}^{\infty} a_{ij} = \lim_{n\to\infty} \sum_{i=r}^{n}\sum_{j=r}^{\infty} a_{ij} = \sum_{i=r}^{\infty}\sum_{j=r}^{\infty} a_{ij}$$

Interchanging the i and j in the above argument proves the theorem.

Basic Linear Algebra

All the topics for calculus of one variable generalize to calculus of any number of variables in which the functions can have values in m dimensional space and there is more than one variable.

The notation, $\mathbb{C}^n$ refers to the collection of ordered lists of n complex numbers. Since every real number is also a complex number, this simply generalizes the usual notion of $\mathbb{R}^n$, the collection of all ordered lists of n real numbers. In order to avoid worrying about whether it is real or complex numbers which are being referred to, the symbol $\mathbb{F}$ will be used. If it is not clear, always pick $\mathbb{C}$.

Definition 3.0.6 *Define*

$$\mathbb{F}^n \equiv \{(x_1, \cdots, x_n) : x_j \in \mathbb{F} \text{ for } j = 1, \cdots, n\}.$$

$(x_1, \cdots, x_n) = (y_1, \cdots, y_n)$ *if and only if for all* $j = 1, \cdots, n$, $x_j = y_j$. *When*

$$(x_1, \cdots, x_n) \in \mathbb{F}^n,$$

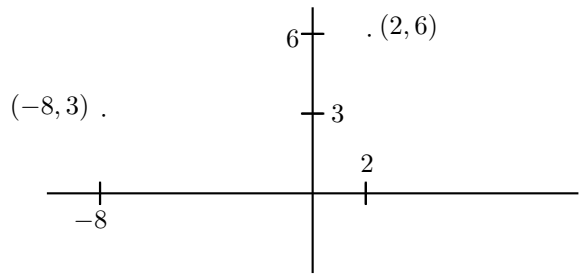
it is conventional to denote $(x_1, \cdots, x_n)$ *by the single bold face letter,* $\mathbf{x}$. *The numbers,* x_j *are called the coordinates. The set*

$$\{(0, \cdots, 0, t, 0, \cdots, 0) : t \in \mathbb{F}\}$$

for t *in the* i^{th} *slot is called the* i^{th} *coordinate axis. The point* $\mathbf{0} \equiv (0, \cdots, 0)$ *is called the origin.*

Thus $(1, 2, 4i) \in \mathbb{F}^3$ and $(2, 1, 4i) \in \mathbb{F}^3$ but $(1, 2, 4i) \neq (2, 1, 4i)$ because, even though the same numbers are involved, they don't match up. In particular, the first entries are not equal.

The geometric significance of $\mathbb{R}^n$ for $n \leq 3$ has been encountered already in calculus or in precalculus. Here is a short review. First consider the case when $n = 1$. Then from the definition, $\mathbb{R}^1 = \mathbb{R}$. Recall that $\mathbb{R}$ is identified with the points of a line. Look at the number line again. Observe that this amounts to identifying a point on this line with a real number. In other words a real number determines where you are on this line. Now suppose $n = 2$ and consider two lines which intersect each other at right angles as shown in the following picture.



Notice how you can identify a point shown in the plane with the ordered pair, $(2,6)$. You go to the right a distance of 2 and then up a distance of 6. Similarly, you can identify another point in the plane with the ordered pair $(-8,3)$. Go to the left a distance of 8 and then up a distance of 3. The reason you go to the left is that there is a $-$ sign on the eight. From this reasoning, every ordered pair determines a unique point in the plane. Conversely, taking a point in the plane, you could draw two lines through the point, one vertical and the other horizontal and determine unique points, x_1 on the horizontal line in the above picture and x_2 on the vertical line in the above picture, such that the point of interest is identified with the ordered pair, (x_1, x_2). In short, points in the plane can be identified with ordered pairs similar to the way that points on the real line are identified with real numbers. Now suppose $n = 3$. As just explained, the first two coordinates determine a point in a plane. Letting the third component determine how far up or down you go, depending on whether this number is positive or negative, this determines a point in space. Thus, $(1,4,-5)$ would mean to determine the point in the plane that goes with $(1,4)$ and then to go below this plane a distance of 5 to obtain a unique point in space. You see that the ordered triples correspond to points in space just as the ordered pairs correspond to points in a plane and single real numbers correspond to points on a line.

You can't stop here and say that you are only interested in $n \leq 3$. What if you were interested in the motion of two objects? You would need three coordinates to describe where the first object is and you would need another three coordinates to describe where the other object is located. Therefore, you would need to be considering $\mathbb{R}^6$. If the two objects moved around, you would need a time coordinate as well. As another example, consider a hot object which is cooling and suppose you want the temperature of this object. How many coordinates would be needed? You would need one for the temperature, three for the position of the point in the object and one more for the time. Thus you would need to be considering $\mathbb{R}^5$. Many other examples can be given. Sometimes n is very large. This is often the case in applications to business when they are trying to maximize profit subject to constraints. It also occurs in numerical analysis when people try to solve hard problems on a computer.

There are other ways to identify points in space with three numbers but the one presented is the most basic. In this case, the coordinates are known as Cartesian coordinates after Descartes[1] who invented this idea in the first half of the seventeenth century. I will often not bother to draw a distinction between the point in n dimensional space and its Cartesian coordinates.

The geometric significance of $\mathbb{C}^n$ for $n > 1$ is not available because each copy of $\mathbb{C}$ corresponds to the plane or $\mathbb{R}^2$.

[1]René Descartes 1596-1650 is often credited with inventing analytic geometry although it seems the ideas were actually known much earlier. He was interested in many different subjects, physiology, chemistry, and physics being some of them. He also wrote a large book in which he tried to explain the book of Genesis scientifically. Descartes ended up dying in Sweden.

3.1 Algebra in $\mathbb{F}^n$, Vector Spaces

There are two algebraic operations done with elements of $\mathbb{F}^n$. One is addition and the other is multiplication by numbers, called scalars. In the case of $\mathbb{C}^n$ the scalars are complex numbers while in the case of $\mathbb{R}^n$ the only allowed scalars are real numbers. Thus, the scalars always come from $\mathbb{F}$ in either case.

Definition 3.1.1 *If $\mathbf{x} \in \mathbb{F}^n$ and $a \in \mathbb{F}$, also called a scalar, then $a\mathbf{x} \in \mathbb{F}^n$ is defined by*

$$a\mathbf{x} = a(x_1, \cdots, x_n) \equiv (ax_1, \cdots, ax_n). \tag{3.1}$$

This is known as scalar multiplication. If $\mathbf{x}, \mathbf{y} \in \mathbb{F}^n$ then $\mathbf{x} + \mathbf{y} \in \mathbb{F}^n$ and is defined by

$$\begin{aligned} \mathbf{x} + \mathbf{y} &= (x_1, \cdots, x_n) + (y_1, \cdots, y_n) \\ &\equiv (x_1 + y_1, \cdots, x_n + y_n) \end{aligned} \tag{3.2}$$

the points in $\mathbb{F}^n$ are also referred to as vectors.

With this definition, the algebraic properties satisfy the conclusions of the following theorem. These conclusions are called the vector space axioms. Any time you have a set and a field of scalars satisfying the axioms of the following theorem, it is called a vector space.

Theorem 3.1.2 *For $\mathbf{v}, \mathbf{w} \in \mathbb{F}^n$ and α, β scalars, (real numbers), the following hold.*

$$\mathbf{v} + \mathbf{w} = \mathbf{w} + \mathbf{v}, \tag{3.3}$$

the commutative law of addition,

$$(\mathbf{v} + \mathbf{w}) + \mathbf{z} = \mathbf{v} + (\mathbf{w} + \mathbf{z}), \tag{3.4}$$

the associative law for addition,

$$\mathbf{v} + \mathbf{0} = \mathbf{v}, \tag{3.5}$$

the existence of an additive identity,

$$\mathbf{v} + (-\mathbf{v}) = \mathbf{0}, \tag{3.6}$$

the existence of an additive inverse, Also

$$\alpha(\mathbf{v} + \mathbf{w}) = \alpha\mathbf{v} + \alpha\mathbf{w}, \tag{3.7}$$

$$(\alpha + \beta)\mathbf{v} = \alpha\mathbf{v} + \beta\mathbf{v}, \tag{3.8}$$

$$\alpha(\beta\mathbf{v}) = \alpha\beta(\mathbf{v}), \tag{3.9}$$

$$1\mathbf{v} = \mathbf{v}. \tag{3.10}$$

In the above $\mathbf{0} = (0, \cdots, 0)$.

You should verify these properties all hold. For example, consider 3.7

$$\begin{aligned} \alpha(\mathbf{v} + \mathbf{w}) &= \alpha(v_1 + w_1, \cdots, v_n + w_n) \\ &= (\alpha(v_1 + w_1), \cdots, \alpha(v_n + w_n)) \\ &= (\alpha v_1 + \alpha w_1, \cdots, \alpha v_n + \alpha w_n) \\ &= (\alpha v_1, \cdots, \alpha v_n) + (\alpha w_1, \cdots, \alpha w_n) \\ &= \alpha\mathbf{v} + \alpha\mathbf{w}. \end{aligned}$$

As usual subtraction is defined as $\mathbf{x} - \mathbf{y} \equiv \mathbf{x} + (-\mathbf{y})$.

3.2 Subspaces Spans And Bases

The concept of linear combination is fundamental in all of linear algebra.

Definition 3.2.1 *Let $\{\mathbf{x}_1, \cdots, \mathbf{x}_p\}$ be vectors in a vector space, Y having the field of scalars $\mathbb{F}$. A linear combination is any expression of the form*

$$\sum_{i=1}^{p} c_i \mathbf{x}_i$$

where the c_i are scalars. The set of all linear combinations of these vectors is called $\text{span}(\mathbf{x}_1, \cdots, \mathbf{x}_n)$. *If $V \subseteq Y$, then V is called a subspace if whenever α, β are scalars and $\mathbf{u}$ and $\mathbf{v}$ are vectors of V, it follows $\alpha\mathbf{u} + \beta\mathbf{v} \in V$. That is, it is "closed under the algebraic operations of vector addition and scalar multiplication" and is therefore, a vector space. A linear combination of vectors is said to be trivial if all the scalars in the linear combination equal zero. A set of vectors is said to be linearly independent if the only linear combination of these vectors which equals the zero vector is the trivial linear combination. Thus $\{\mathbf{x}_1, \cdots, \mathbf{x}_n\}$ is called linearly independent if whenever*

$$\sum_{k=1}^{p} c_k \mathbf{x}_k = \mathbf{0}$$

it follows that all the scalars, c_k equal zero. A set of vectors, $\{\mathbf{x}_1, \cdots, \mathbf{x}_p\}$, is called linearly dependent if it is not linearly independent. Thus the set of vectors is linearly dependent if there exist scalars, $c_i, i = 1, \cdots, n$, not all zero such that $\sum_{k=1}^{p} c_k \mathbf{x}_k = \mathbf{0}$.

Lemma 3.2.2 *A set of vectors $\{\mathbf{x}_1, \cdots, \mathbf{x}_p\}$ is linearly independent if and only if none of the vectors can be obtained as a linear combination of the others.*

Proof: Suppose first that $\{\mathbf{x}_1, \cdots, \mathbf{x}_p\}$ is linearly independent. If

$$\mathbf{x}_k = \sum_{j \neq k} c_j \mathbf{x}_j,$$

then

$$\mathbf{0} = 1\mathbf{x}_k + \sum_{j \neq k} (-c_j)\, \mathbf{x}_j,$$

a nontrivial linear combination, contrary to assumption. This shows that if the set is linearly independent, then none of the vectors is a linear combination of the others.

Now suppose no vector is a linear combination of the others. Is $\{\mathbf{x}_1, \cdots, \mathbf{x}_p\}$ linearly independent? If it is not, there exist scalars, c_i, not all zero such that

$$\sum_{i=1}^{p} c_i \mathbf{x}_i = \mathbf{0}.$$

Say $c_k \neq 0$. Then you can solve for $\mathbf{x}_k$ as

$$\mathbf{x}_k = \sum_{j \neq k} (-c_j)\, / c_k \mathbf{x}_j$$

contrary to assumption. This proves the lemma.

The following is called the exchange theorem.

Theorem 3.2.3 *Let $\{\mathbf{x}_1, \cdots, \mathbf{x}_r\}$ be a linearly independent set of vectors such that each $\mathbf{x}_i$ is in the* $\operatorname{span}\{\mathbf{y}_1, \cdots, \mathbf{y}_s\}$. *Then $r \leq s$.*

Proof: Define $\operatorname{span}\{\mathbf{y}_1, \cdots, \mathbf{y}_s\} \equiv V$, it follows there exist scalars, $c_1, \cdots, c_s$ such that

$$\mathbf{x}_1 = \sum_{i=1}^{s} c_i \mathbf{y}_i. \tag{3.11}$$

Not all of these scalars can equal zero because if this were the case, it would follow that $\mathbf{x}_1 = \mathbf{0}$ and so $\{\mathbf{x}_1, \cdots, \mathbf{x}_r\}$ would not be linearly independent. Indeed, if $\mathbf{x}_1 = \mathbf{0}$, $1\mathbf{x}_1 + \sum_{i=2}^{r} 0\mathbf{x}_i = \mathbf{x}_1 = \mathbf{0}$ and so there would exist a nontrivial linear combination of the vectors $\{\mathbf{x}_1, \cdots, \mathbf{x}_r\}$ which equals zero.

Say $c_k \neq 0$. Then solve (3.11) for $\mathbf{y}_k$ and obtain

$$\mathbf{y}_k \in \operatorname{span}\left(\mathbf{x}_1, \overbrace{\mathbf{y}_1, \cdots, \mathbf{y}_{k-1}, \mathbf{y}_{k+1}, \cdots, \mathbf{y}_s}^{\text{s-1 vectors here}}\right).$$

Define $\{\mathbf{z}_1, \cdots, \mathbf{z}_{s-1}\}$ by

$$\{\mathbf{z}_1, \cdots, \mathbf{z}_{s-1}\} \equiv \{\mathbf{y}_1, \cdots, \mathbf{y}_{k-1}, \mathbf{y}_{k+1}, \cdots, \mathbf{y}_s\}$$

Therefore, $\operatorname{span}\{\mathbf{x}_1, \mathbf{z}_1, \cdots, \mathbf{z}_{s-1}\} = V$ because if $\mathbf{v} \in V$, there exist constants $c_1, \cdots, c_s$ such that

$$\mathbf{v} = \sum_{i=1}^{s-1} c_i \mathbf{z}_i + c_s \mathbf{y}_k.$$

Now replace the $\mathbf{y}_k$ in the above with a linear combination of the vectors, $\{\mathbf{x}_1, \mathbf{z}_1, \cdots, \mathbf{z}_{s-1}\}$ to obtain $\mathbf{v} \in \operatorname{span}\{\mathbf{x}_1, \mathbf{z}_1, \cdots, \mathbf{z}_{s-1}\}$. The vector $\mathbf{y}_k$, in the list $\{\mathbf{y}_1, \cdots, \mathbf{y}_s\}$, has now been replaced with the vector $\mathbf{x}_1$ and the resulting modified list of vectors has the same span as the original list of vectors, $\{\mathbf{y}_1, \cdots, \mathbf{y}_s\}$.

Now suppose that $r > s$ and that $\operatorname{span}\{\mathbf{x}_1, \cdots, \mathbf{x}_l, \mathbf{z}_1, \cdots, \mathbf{z}_p\} = V$ where the vectors, $\mathbf{z}_1, \cdots, \mathbf{z}_p$ are each taken from the set, $\{\mathbf{y}_1, \cdots, \mathbf{y}_s\}$ and $l + p = s$. This has now been done for $l = 1$ above. Then since $r > s$, it follows that $l \leq s < r$ and so $l + 1 \leq r$. Therefore, $\mathbf{x}_{l+1}$ is a vector not in the list, $\{\mathbf{x}_1, \cdots, \mathbf{x}_l\}$ and since $\operatorname{span}\{\mathbf{x}_1, \cdots, \mathbf{x}_l, \mathbf{z}_1, \cdots, \mathbf{z}_p\} = V$ there exist scalars, c_i and d_j such that

$$\mathbf{x}_{l+1} = \sum_{i=1}^{l} c_i \mathbf{x}_i + \sum_{j=1}^{p} d_j \mathbf{z}_j. \tag{3.12}$$

Now not all the d_j can equal zero because if this were so, it would follow that $\{\mathbf{x}_1, \cdots, \mathbf{x}_r\}$ would be a linearly dependent set because one of the vectors would equal a linear combination of the others. Therefore, (3.12) can be solved for one of the $\mathbf{z}_i$, say $\mathbf{z}_k$, in terms of $\mathbf{x}_{l+1}$ and the other $\mathbf{z}_i$ and just as in the above argument, replace that $\mathbf{z}_i$ with $\mathbf{x}_{l+1}$ to obtain

$$\operatorname{span}\left(\mathbf{x}_1, \cdots \mathbf{x}_l, \mathbf{x}_{l+1}, \overbrace{\mathbf{z}_1, \cdots \mathbf{z}_{k-1}, \mathbf{z}_{k+1}, \cdots, \mathbf{z}_p}^{\text{p-1 vectors here}}\right) = V.$$

Continue this way, eventually obtaining

$$\operatorname{span}(\mathbf{x}_1, \cdots, \mathbf{x}_s) = V.$$

But then $\mathbf{x}_r \in \operatorname{span}\{\mathbf{x}_1, \cdots, \mathbf{x}_s\}$ contrary to the assumption that $\{\mathbf{x}_1, \cdots, \mathbf{x}_r\}$ is linearly independent. Therefore, $r \leq s$ as claimed.

Here is another proof in case you didn't like the above proof.

Theorem 3.2.4 *If*

$$\text{span}(\mathbf{u}_1, \cdots, \mathbf{u}_r) \subseteq \text{span}(\mathbf{v}_1, \cdots, \mathbf{v}_s) \equiv V$$

and $\{\mathbf{u}_1, \cdots, \mathbf{u}_r\}$ *are linearly independent, then* $r \leq s$.

Proof: Suppose $r > s$. Let E_p denote a finite list of vectors of $\{\mathbf{v}_1, \cdots, \mathbf{v}_s\}$ and let $|E_p|$ denote the number of vectors in the list. Let F_p denote the first p vectors in $\{\mathbf{u}_1, \cdots, \mathbf{u}_r\}$. In case $p = 0, F_p$ will denote the empty set. For $0 \leq p \leq s$, let E_p have the property

$$\text{span}(F_p, E_p) = V$$

and $|E_p|$ is as small as possible for this to happen. I claim $|E_p| \leq s - p$ if E_p is nonempty.

Here is why. For $p = 0$, it is obvious. Suppose true for some $p < s$. Then

$$\mathbf{u}_{p+1} \in \text{span}(F_p, E_p)$$

and so there are constants, $c_1, \cdots, c_p$ and $d_1, \cdots, d_m$ where $m \leq s - p$ such that

$$\mathbf{u}_{p+1} = \sum_{i=1}^{p} c_i \mathbf{u}_i + \sum_{j=1}^{m} d_i \mathbf{z}_j$$

for

$$\{\mathbf{z}_1, \cdots, \mathbf{z}_m\} \subseteq \{\mathbf{v}_1, \cdots, \mathbf{v}_s\}.$$

Then not all the d_i can equal zero because this would violate the linear independence of the $\{\mathbf{u}_1, \cdots, \mathbf{u}_r\}$. Therefore, you can solve for one of the $\mathbf{z}_k$ as a linear combination of $\{\mathbf{u}_1, \cdots, \mathbf{u}_{p+1}\}$ and the other $\mathbf{z}_j$. Thus you can change F_p to F_{p+1} and include one fewer vector in E_p. Thus $|E_{p+1}| \leq m - 1 \leq s - p - 1$. This proves the claim.

Therefore, E_s is empty and $\text{span}(\mathbf{u}_1, \cdots, \mathbf{u}_s) = V$. However, this gives a contradiction because it would require

$$\mathbf{u}_{s+1} \in \text{span}(\mathbf{u}_1, \cdots, \mathbf{u}_s)$$

which violates the linear independence of these vectors. This proves the theorem.

Definition 3.2.5 *A finite set of vectors,* $\{\mathbf{x}_1, \cdots, \mathbf{x}_r\}$ *is a basis for a vector space* V *if*

$$\text{span}(\mathbf{x}_1, \cdots, \mathbf{x}_r) = V$$

and $\{\mathbf{x}_1, \cdots, \mathbf{x}_r\}$ *is linearly independent. Thus if* $\mathbf{v} \in V$ *there exist unique scalars,* $v_1, \cdots, v_r$ *such that* $\mathbf{v} = \sum_{i=1}^{r} v_i \mathbf{x}_i$. *These scalars are called the components of* $\mathbf{v}$ *with respect to the basis* $\{\mathbf{x}_1, \cdots, \mathbf{x}_r\}$.

Corollary 3.2.6 *Let* $\{\mathbf{x}_1, \cdots, \mathbf{x}_r\}$ *and* $\{\mathbf{y}_1, \cdots, \mathbf{y}_s\}$ *be two bases*[2] *of* $\mathbb{F}^n$. *Then* $r = s = n$.

Proof: From the exchange theorem, $r \leq s$ and $s \leq r$. Now note the vectors,

$$\mathbf{e}_i = \overbrace{(0, \cdots, 0, 1, 0 \cdots, 0)}^{1 \text{ is in the } i^{th} \text{ slot}}$$

for $i = 1, 2, \cdots, n$ are a basis for $\mathbb{F}^n$. This proves the corollary.

[2]This is the plural form of basis. We could say basiss but it would involve an inordinate amount of hissing as in "The sixth shiek's sixth sheep is sick". This is the reason that bases is used instead of basiss.

Lemma 3.2.7 *Let $\{\mathbf{v}_1, \cdots, \mathbf{v}_r\}$ be a set of vectors. Then $V \equiv \operatorname{span}(\mathbf{v}_1, \cdots, \mathbf{v}_r)$ is a subspace.*

Proof: Suppose α, β are two scalars and let $\sum_{k=1}^r c_k \mathbf{v}_k$ and $\sum_{k=1}^r d_k \mathbf{v}_k$ are two elements of V. What about

$$\alpha \sum_{k=1}^r c_k \mathbf{v}_k + \beta \sum_{k=1}^r d_k \mathbf{v}_k?$$

Is it also in V?

$$\alpha \sum_{k=1}^r c_k \mathbf{v}_k + \beta \sum_{k=1}^r d_k \mathbf{v}_k = \sum_{k=1}^r (\alpha c_k + \beta d_k) \mathbf{v}_k \in V$$

so the answer is yes. This proves the lemma.

Definition 3.2.8 *Let V be a vector space. Then $\dim(V)$ read as the dimension of V is the number of vectors in a basis.*

Of course you should wonder right now whether an arbitrary subspace of a finite dimensional vector space even has a basis. In fact it does and this is in the next theorem. First, here is an interesting lemma.

Lemma 3.2.9 *Suppose $\mathbf{v} \notin \operatorname{span}(\mathbf{u}_1, \cdots, \mathbf{u}_k)$ and $\{\mathbf{u}_1, \cdots, \mathbf{u}_k\}$ is linearly independent. Then $\{\mathbf{u}_1, \cdots, \mathbf{u}_k, \mathbf{v}\}$ is also linearly independent.*

Proof: Suppose $\sum_{i=1}^k c_i \mathbf{u}_i + d\mathbf{v} = \mathbf{0}$. It is required to verify that each $c_i = 0$ and that $d = 0$. But if $d \neq 0$, then you can solve for $\mathbf{v}$ as a linear combination of the vectors, $\{\mathbf{u}_1, \cdots, \mathbf{u}_k\}$,

$$\mathbf{v} = -\sum_{i=1}^k \left(\frac{c_i}{d}\right) \mathbf{u}_i$$

contrary to assumption. Therefore, $d = 0$. But then $\sum_{i=1}^k c_i \mathbf{u}_i = 0$ and the linear independence of $\{\mathbf{u}_1, \cdots, \mathbf{u}_k\}$ implies each $c_i = 0$ also. This proves the lemma.

Theorem 3.2.10 *Let V be a nonzero subspace of Y a finite dimensional vector space having dimension n. Then V has a basis.*

Proof: Let $\mathbf{v}_1 \in V$ where $\mathbf{v}_1 \neq 0$. If $\operatorname{span}\{\mathbf{v}_1\} = V$, stop. $\{\mathbf{v}_1\}$ is a basis for V. Otherwise, there exists $\mathbf{v}_2 \in V$ which is not in $\operatorname{span}\{\mathbf{v}_1\}$. By Lemma 3.2.9 $\{\mathbf{v}_1, \mathbf{v}_2\}$ is a linearly independent set of vectors. If $\operatorname{span}\{\mathbf{v}_1, \mathbf{v}_2\} = V$ stop, $\{\mathbf{v}_1, \mathbf{v}_2\}$ is a basis for V. If $\operatorname{span}\{\mathbf{v}_1, \mathbf{v}_2\} \neq V$, then there exists $\mathbf{v}_3 \notin \operatorname{span}\{\mathbf{v}_1, \mathbf{v}_2\}$ and $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is a larger linearly independent set of vectors. Continuing this way, the process must stop before $n+1$ steps because if not, it would be possible to obtain $n+1$ linearly independent vectors contrary to the exchange theorem and the assumed dimension of Y. This proves the theorem.

In words the following corollary states that any linearly independent set of vectors can be enlarged to form a basis.

Corollary 3.2.11 *Let V be a subspace of Y, a finite dimensional vector space of dimension n and let $\{\mathbf{v}_1, \cdots, \mathbf{v}_r\}$ be a linearly independent set of vectors in V. Then either it is a basis for V or there exist vectors, $\mathbf{v}_{r+1}, \cdots, \mathbf{v}_s$ such that $\{\mathbf{v}_1, \cdots, \mathbf{v}_r, \mathbf{v}_{r+1}, \cdots, \mathbf{v}_s\}$ is a basis for V.*

Proof: This follows immediately from the proof of Theorem 3.2.10. You do exactly the same argument except you start with $\{\mathbf{v}_1, \cdots, \mathbf{v}_r\}$ rather than $\{\mathbf{v}_1\}$.

It is also true that any spanning set of vectors can be restricted to obtain a basis.

Theorem 3.2.12 *Let V be a subspace of Y, a finite dimensional vector space of dimension n and suppose* $\mathrm{span}(\mathbf{u}_1 \cdots, \mathbf{u}_p) = V$ *where the* $\mathbf{u}_i$ *are nonzero vectors. Then there exist vectors,* $\{\mathbf{v}_1 \cdots, \mathbf{v}_r\}$ *such that* $\{\mathbf{v}_1 \cdots, \mathbf{v}_r\} \subseteq \{\mathbf{u}_1 \cdots, \mathbf{u}_p\}$ *and* $\{\mathbf{v}_1 \cdots, \mathbf{v}_r\}$ *is a basis for* V.

Proof: Let r be the smallest positive integer with the property that for some set, $\{\mathbf{v}_1 \cdots, \mathbf{v}_r\} \subseteq \{\mathbf{u}_1 \cdots, \mathbf{u}_p\}$,

$$\mathrm{span}(\mathbf{v}_1 \cdots, \mathbf{v}_r) = V.$$

Then $r \leq p$ and it must be the case that $\{\mathbf{v}_1 \cdots, \mathbf{v}_r\}$ is linearly independent because if it were not so, one of the vectors, say $\mathbf{v}_k$ would be a linear combination of the others. But then you could delete this vector from $\{\mathbf{v}_1 \cdots, \mathbf{v}_r\}$ and the resulting list of $r-1$ vectors would still span V contrary to the definition of r. This proves the theorem.

3.3 Linear Transformations

In calculus of many variables one studies functions of many variables and what is meant by their derivatives or integrals, etc. The simplest kind of function of many variables is a linear transformation. You have to begin with the simple things if you expect to make sense of the harder things. The following is the definition of a linear transformation.

Definition 3.3.1 *Let V and W be two finite dimensional vector spaces. A function, L which maps V to W is called a linear transformation and written as $L \in \mathcal{L}(V,W)$ if for all scalars α and β, and vectors $\mathbf{v}, \mathbf{w}$,*

$$L(\alpha\mathbf{v}+\beta\mathbf{w}) = \alpha L(\mathbf{v}) + \beta L(\mathbf{w}).$$

An example of a linear transformation is familiar matrix multiplication, familiar if you have had a linear algebra course. Let $A = (a_{ij})$ be an $m \times n$ matrix. Then an example of a linear transformation $L : \mathbb{F}^n \to \mathbb{F}^m$ is given by

$$(L\mathbf{v})_i \equiv \sum_{j=1}^{n} a_{ij} v_j.$$

Here

$$\mathbf{v} \equiv \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} \in \mathbb{F}^n.$$

In the general case, the space of linear transformations is itself a vector space. This will be discussed next.

Definition 3.3.2 *Given $L, M \in \mathcal{L}(V,W)$ define a new element of $\mathcal{L}(V,W)$, denoted by $L+M$ according to the rule*

$$(L+M)\mathbf{v} \equiv L\mathbf{v} + M\mathbf{v}.$$

For α a scalar and $L \in \mathcal{L}(V,W)$, define $\alpha L \in \mathcal{L}(V,W)$ by

$$\alpha L(\mathbf{v}) \equiv \alpha(L\mathbf{v}).$$

You should verify that all the axioms of a vector space hold for $\mathcal{L}(V,W)$ with the above definitions of vector addition and scalar multiplication. What about the dimension of $\mathcal{L}(V,W)$?

Before answering this question, here is a lemma.

Lemma 3.3.3 *Let V and W be vector spaces and suppose $\{\mathbf{v}_1, \cdots, \mathbf{v}_n\}$ is a basis for V. Then if $L : V \to W$ is given by $L\mathbf{v}_k = \mathbf{w}_k \in W$ and*

$$L\left(\sum_{k=1}^{n} a_k \mathbf{v}_k\right) \equiv \sum_{k=1}^{n} a_k L\mathbf{v}_k = \sum_{k=1}^{n} a_k \mathbf{w}_k$$

then L is well defined and is in $\mathcal{L}(V,W)$. Also, if L, M are two linear transformations such that $L\mathbf{v}_k = M\mathbf{v}_k$ for all k, then $M = L$.

Proof: L is well defined on V because, since $\{\mathbf{v}_1, \cdots, \mathbf{v}_n\}$ is a basis, there is exactly one way to write a given vector of V as a linear combination. Next, observe that L is obviously linear from the definition. If L, M are equal on the basis, then if $\sum_{k=1}^{n} a_k \mathbf{v}_k$ is an arbitrary vector of V,

$$\begin{aligned} L\left(\sum_{k=1}^{n} a_k \mathbf{v}_k\right) &= \sum_{k=1}^{n} a_k L\mathbf{v}_k \\ &= \sum_{k=1}^{n} a_k M\mathbf{v}_k = M\left(\sum_{k=1}^{n} a_k \mathbf{v}_k\right) \end{aligned}$$

and so $L = M$ because they give the same result for every vector in V.

The message is that when you define a linear transformation, it suffices to tell what it does to a basis.

Definition 3.3.4

The symbol, δ_{ij} is defined as 1 if $i = j$ and 0 if $i \neq j$.

Theorem 3.3.5

Let V and W be finite dimensional vector spaces of dimension n and m respectively Then $\dim(\mathcal{L}(V,W)) = mn$.

Proof: Let two sets of bases be

$$\{\mathbf{v}_1, \cdots, \mathbf{v}_n\} \text{ and } \{\mathbf{w}_1, \cdots, \mathbf{w}_m\}$$

for V and W respectively. Using Lemma 3.3.3, let $\mathbf{w}_i\mathbf{v}_j \in \mathcal{L}(V,W)$ be the linear transformation defined on the basis, $\{\mathbf{v}_1, \cdots, \mathbf{v}_n\}$, by

$$\mathbf{w}_i\mathbf{v}_k(\mathbf{v}_j) \equiv \mathbf{w}_i\delta_{jk}.$$

Note that to define these special linear transformations, sometimes called dyadics, it is necessary that $\{\mathbf{v}_1, \cdots, \mathbf{v}_n\}$ be a basis since their definition requires giving the values of the linear transformation on a basis.

Let $L \in \mathcal{L}(V,W)$. Since $\{\mathbf{w}_1, \cdots, \mathbf{w}_m\}$ is a basis, there exist constants d_{jr} such that

$$L\mathbf{v}_r = \sum_{j=1}^{m} d_{jr}\mathbf{w}_j$$

Then from the above,

$$L\mathbf{v}_r = \sum_{j=1}^{m} d_{jr}\mathbf{w}_j = \sum_{j=1}^{m}\sum_{k=1}^{n} d_{jr}\delta_{kr}\mathbf{w}_j = \sum_{j=1}^{m}\sum_{k=1}^{n} d_{jr}\mathbf{w}_j\mathbf{v}_k(\mathbf{v}_r)$$

which shows

$$L = \sum_{j=1}^{m}\sum_{k=1}^{n} d_{jk}\mathbf{w}_j\mathbf{v}_k$$

because the two linear transformations agree on a basis. Since L is arbitrary this shows

$$\{\mathbf{w}_i\mathbf{v}_k : i = 1, \cdots, m,\ k = 1, \cdots, n\}$$

spans $\mathcal{L}(V, W)$.

If

$$\sum_{i,k} d_{ik}\mathbf{w}_i\mathbf{v}_k = \mathbf{0},$$

then

$$\mathbf{0} = \sum_{i,k} d_{ik}\mathbf{w}_i\mathbf{v}_k(\mathbf{v}_l) = \sum_{i=1}^{m} d_{il}\mathbf{w}_i$$

and so, since $\{\mathbf{w}_1, \cdots, \mathbf{w}_m\}$ is a basis, $d_{il} = 0$ for each $i = 1, \cdots, m$. Since l is arbitrary, this shows $d_{il} = 0$ for all i and l. Thus these linear transformations form a basis and this shows the dimension of $\mathcal{L}(V, W)$ is mn as claimed.

Definition 3.3.6 *Let V, W be finite dimensional vector spaces such that a basis for V is*

$$\{\mathbf{v}_1, \cdots, \mathbf{v}_n\}$$

and a basis for W is

$$\{\mathbf{w}_1, \cdots, \mathbf{w}_m\}.$$

Then as explained in Theorem 3.3.5, for $L \in \mathcal{L}(V, W)$, there exist scalars l_{ij} such that

$$L = \sum_{ij} l_{ij}\mathbf{w}_i\mathbf{v}_j$$

Consider a rectangular array of scalars such that the entry in the i^{th} row and the j^{th} column is l_{ij},

$$\begin{pmatrix} l_{11} & l_{12} & \cdots & l_{1n} \\ l_{21} & l_{22} & \cdots & l_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ l_{m1} & l_{m2} & \cdots & l_{mn} \end{pmatrix}$$

This is called the matrix of the linear transformation with respect to the two bases. This will typically be denoted by (l_{ij}). It is called a matrix and in this case the matrix is $m \times n$ because it has m rows and n columns.

Theorem 3.3.7 *Let $L \in \mathcal{L}(V, W)$ and let (l_{ij}) be the matrix of L with respect to the two bases,*

$$\{\mathbf{v}_1, \cdots, \mathbf{v}_n\} \text{ and } \{\mathbf{w}_1, \cdots, \mathbf{w}_m\}.$$

of V and W respectively. Then for $\mathbf{v} \in V$ having components $(x_1, \cdots, x_n)$ with respect to the basis $\{\mathbf{v}_1, \cdots, \mathbf{v}_n\}$, the components of $L\mathbf{v}$ with respect to the basis $\{\mathbf{w}_1, \cdots, \mathbf{w}_m\}$ are

$$\left(\sum_j l_{1j}x_j, \cdots, \sum_j l_{mj}x_j\right)$$

Proof: From the definition of l_{ij},

$$\begin{aligned} L\mathbf{v} &= \sum_{ij} l_{ij}\mathbf{w}_i\mathbf{v}_j(\mathbf{v}) = \sum_{ij} l_{ij}\mathbf{w}_i\mathbf{v}_j\left(\sum_k x_k\mathbf{v}_k\right) \\ &= \sum_{ijk} l_{ij}\mathbf{w}_i\mathbf{v}_j(\mathbf{v}_k)\,x_k = \sum_{ijk} l_{ij}\mathbf{w}_i\delta_{jk}x_k = \sum_i\left(\sum_j l_{ij}x_j\right)\mathbf{w}_i \end{aligned}$$

and this proves the theorem.

Theorem 3.3.8 *Let $(V,\{\mathbf{v}_1,\cdots,\mathbf{v}_n\}),(U,\{\mathbf{u}_1,\cdots,\mathbf{u}_m\}),(W,\{\mathbf{w}_1,\cdots,\mathbf{w}_p\})$ be three vector spaces along with bases for each one. Let $L\in\mathcal{L}(V,U)$ and $M\in\mathcal{L}(U,W)$. Then $ML\in\mathcal{L}(V,W)$ and if (c_{ij}) is the matrix of ML with respect to $\{\mathbf{v}_1,\cdots,\mathbf{v}_n\}$ and $\{\mathbf{w}_1,\cdots,\mathbf{w}_p\}$ and (l_{ij}) and (m_{ij}) are the matrices of L and M respectively with respect to the given bases, then*

$$c_{rj} = \sum_{s=1}^{m} m_{rs}l_{sj}.$$

Proof: First note that from the definition,

$$(\mathbf{w}_i\mathbf{u}_j)(\mathbf{u}_k\mathbf{v}_l)(\mathbf{v}_r) = (\mathbf{w}_i\mathbf{u}_j)\mathbf{u}_k\delta_{lr} = \mathbf{w}_i\delta_{jk}\delta_{lr}$$

and

$$\mathbf{w}_i\mathbf{v}_l\delta_{jk}(\mathbf{v}_r) = \mathbf{w}_i\delta_{jk}\delta_{lr}$$

which shows

$$(\mathbf{w}_i\mathbf{u}_j)(\mathbf{u}_k\mathbf{v}_l) = \mathbf{w}_i\mathbf{v}_l\delta_{jk} \tag{3.13}$$

Therefore,

$$\begin{aligned} ML &= \left(\sum_{rs} m_{rs}\mathbf{w}_r\mathbf{u}_s\right)\left(\sum_{ij} l_{ij}\mathbf{u}_i\mathbf{v}_j\right) \\ &= \sum_{rsij} m_{rs}l_{ij}(\mathbf{w}_r\mathbf{u}_s)(\mathbf{u}_i\mathbf{v}_j) = \sum_{rsij} m_{rs}l_{ij}\mathbf{w}_r\mathbf{v}_j\delta_{is} \\ &= \sum_{rsj} m_{rs}l_{sj}\mathbf{w}_r\mathbf{v}_j = \sum_{rj}\left(\sum_s m_{rs}l_{sj}\right)\mathbf{w}_r\mathbf{v}_j \end{aligned}$$

and this proves the theorem.

The relation 3.13 is a very important cancellation property which is used later as well as in this theorem.

Theorem 3.3.9 *Suppose $(V,\{\mathbf{v}_1,\cdots,\mathbf{v}_n\})$ is a vector space and a basis and $(V,\{\mathbf{v}'_1,\cdots,\mathbf{v}'_n\})$ is the same vector space with a different basis. Suppose $L\in\mathcal{L}(V,V)$. Let (l_{ij}) be the matrix of L taken with respect to $\{\mathbf{v}_1,\cdots,\mathbf{v}_n\}$ and let $\left(l'_{ij}\right)$ be the $n\times n$ matrix of L taken with respect to $\{\mathbf{v}'_1,\cdots,\mathbf{v}'_n\}$ That is,*

$$L = \sum_{ij} l_{ij}\mathbf{v}_i\mathbf{v}_j,\ L = \sum_{rs} l'_{rs}\mathbf{v}'_r\mathbf{v}'_s.$$

Then there exist $n\times n$ matrices (d_{ij}) and $\left(d'_{ij}\right)$ satisfying

$$\sum_j d_{ij}d'_{jk} = \delta_{ik}$$

such that

$$l_{ij} = \sum_{rs} d_{ir} l'_{rs} d'_{sj}$$

Proof: First consider the identity map, id defined by $\mathrm{id}(\mathbf{v}) = \mathbf{v}$ with respect to the two bases, $\{\mathbf{v}_1, \cdots, \mathbf{v}_n\}$ and $\{\mathbf{v}'_1, \cdots, \mathbf{v}'_n\}$.

$$\mathrm{id} = \sum_{tu} d'_{tu} \mathbf{v}'_t \mathbf{v}_u, \ \mathrm{id} = \sum_{ij} d_{ij} \mathbf{v}_i \mathbf{v}'_j \tag{3.14}$$

Now it follows from 3.13

$$\begin{aligned} \mathrm{id} &= \mathrm{id} \circ \mathrm{id} = \sum_{tuij} d'_{tu} d_{ij} \left(\mathbf{v}'_t \mathbf{v}_u\right) \left(\mathbf{v}_i \mathbf{v}'_j\right) = \sum_{tuij} d'_{tu} d_{ij} \delta_{iu} \mathbf{v}'_t \mathbf{v}'_j \\ &= \sum_{tij} d'_{ti} d_{ij} \mathbf{v}'_t \mathbf{v}'_j = \sum_{tj} \left(\sum_i d'_{ti} d_{ij} \right) \mathbf{v}'_t \mathbf{v}'_j \end{aligned}$$

On the other hand,

$$\mathrm{id} = \sum_{tj} \delta_{tj} \mathbf{v}'_t \mathbf{v}'_j$$

because $\mathrm{id}(\mathbf{v}'_k) = \mathbf{v}'_k$ and

$$\sum_{tj} \delta_{tj} \mathbf{v}'_t \mathbf{v}'_j \left(\mathbf{v}'_k\right) = \sum_{tj} \delta_{tj} \mathbf{v}'_t \delta_{jk} = \sum_t \delta_{tk} \mathbf{v}'_t = \mathbf{v}'_k.$$

Therefore,

$$\left(\sum_i d'_{ti} d_{ij} \right) = \delta_{tj}.$$

Switching the order of the above products shows

$$\left(\sum_i d_{ti} d'_{ij} \right) = \delta_{tj}$$

In terms of matrices, this says $\left(d'_{ij}\right)$ is the inverse matrix of $\left(d_{ij}\right)$.

Now using 3.14 and the cancellation property 3.13,

$$\begin{aligned} L &= \sum_{iu} l_{iu} \mathbf{v}_i \mathbf{v}_u = \sum_{rs} l'_{rs} \mathbf{v}'_r \mathbf{v}'_s = \mathrm{id} \sum_{rs} l'_{rs} \mathbf{v}'_r \mathbf{v}'_s \, \mathrm{id} \\ &= \sum_{ij} d_{ij} \mathbf{v}_i \mathbf{v}'_j \sum_{rs} l'_{rs} \mathbf{v}'_r \mathbf{v}'_s \sum_{tu} d'_{tu} \mathbf{v}'_t \mathbf{v}_u \\ &= \sum_{ijturs} d_{ij} l'_{rs} d'_{tu} \left(\mathbf{v}_i \mathbf{v}'_j\right) \left(\mathbf{v}'_r \mathbf{v}'_s\right) \left(\mathbf{v}'_t \mathbf{v}_u\right) \\ &= \sum_{ijturs} d_{ij} l'_{rs} d'_{tu} \mathbf{v}_i \mathbf{v}_u \delta_{jr} \delta_{st} = \sum_{iu} \left(\sum_{js} d_{ij} l'_{js} d'_{su} \right) \mathbf{v}_i \mathbf{v}_u \end{aligned}$$

and since the linear transformations, $\{\mathbf{v}_i \mathbf{v}_u\}$ are linearly independent, this shows

$$l_{iu} = \sum_{js} d_{ij} l'_{js} d'_{su}$$

as claimed. This proves the theorem.

Recall the following definition which is a review of important terminology about matrices.

Definition 3.3.10 *If A is an $m\times n$ matrix and B is an $n\times p$ matrix, $A=(A_{ij}), B=(B_{ij})$, then if $(AB)_{ij}$ is the ij^{th} entry of the product, then*

$$(AB)_{ij}=\sum_k A_{ik}B_{kj}$$

An $n\times n$ matrix, A is said to be invertible if there exists another $n\times n$ matrix, denoted by A^{-1} such that $AA^{-1}=A^{-1}A=I$ where the ij^{th} entry of I is δ_{ij}. Recall also that $\left(A^T\right)_{ij}\equiv A_{ji}$. This is called the transpose of A.

Theorem 3.3.11 *The following are important properties of matrices.*

1. $IA=AI=A$
2. $(AB)C=A(BC)$
3. A^{-1} *is unique if it exists.*
4. *When the inverses exist,* $(AB)^{-1}=B^{-1}A^{-1}$
5. $(AB)^T=B^TA^T$

Proof: I will prove these things directly from the above definition but there are more elegant ways to see these things in terms of composition of linear transformations which is really what matrix multiplication corresponds to.

First, $(IA)_{ij}\equiv\sum_k\delta_{ik}A_{kj}=A_{ij}$. The other order is similar.

Next consider the associative law of multiplication.

$$\begin{aligned}((AB)C)_{ij} &\equiv \sum_k (AB)_{ik}C_{kj}=\sum_k\sum_r A_{ir}B_{rk}C_{kj}\\ &= \sum_r A_{ir}\sum_k B_{rk}C_{kj}=\sum_r A_{ir}(BC)_{rj}=(A(BC))_{ij}\end{aligned}$$

Since the ij^{th} entries are equal, the two matrices are equal.

Next consider the uniqueness of the inverse. If $AB=BA=I$, then using the associative law,

$$B=IB=\left(A^{-1}A\right)B=A^{-1}(AB)=A^{-1}I=A^{-1}$$

Thus if it acts like the inverse, it is the inverse.

Consider now the inverse of a product.

$$AB\left(B^{-1}A^{-1}\right)=A\left(BB^{-1}\right)A^{-1}=AIA^{-1}=I$$

Similarly, $\left(B^{-1}A^{-1}\right)AB=I$. Hence from what was just shown, $(AB)^{-1}$ exists and equals $B^{-1}A^{-1}$.

Finally consider the statement about transposes.

$$\left((AB)^T\right)_{ij}\equiv(AB)_{ji}\equiv\sum_k A_{jk}B_{ki}\equiv\sum_k\left(B^T\right)_{ik}\left(A^T\right)_{kj}\equiv\left(B^TA^T\right)_{ij}$$

Since the ij^{th} entries are the same, the two matrices are equal. This proves the theorem.

In terms of matrix multiplication, Theorem 3.3.9 says that if M_1 and M_2 are matrices for the same linear transformation relative to two different bases, it follows there exists an invertible matrix, S such that

$$M_1=S^{-1}M_2S$$

This is called a similarity transformation and is important in linear algebra but this is as far as the theory will be developed here.

3.4 Block Multiplication Of Matrices

Consider the following problem

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} E & F \\ G & H \end{pmatrix}$$

You know how to do this from the above definition of matrix mutiplication. You get

$$\begin{pmatrix} AE+BG & AF+BH \\ CE+DG & CF+DH \end{pmatrix}.$$

Now what if instead of numbers, the entries, A, B, C, D, E, F, G are matrices of a size such that the multiplications and additions needed in the above formula all make sense. Would the formula be true in this case? I will show below that this is true.

Suppose A is a matrix of the form

$$\begin{pmatrix} A_{11} & \cdots & A_{1m} \\ \vdots & \ddots & \vdots \\ A_{r1} & \cdots & A_{rm} \end{pmatrix} \tag{3.15}$$

where A_{ij} is a $s_i \times p_j$ matrix where s_i does not depend on j and p_j does not depend on i. Such a matrix is called a **block matrix,** also a **partitioned matrix**. Let $n = \sum_j p_j$ and $k = \sum_i s_i$ so A is an $k \times n$ matrix. What is $A\mathbf{x}$ where $\mathbf{x} \in \mathbb{F}^n$? From the process of multiplying a matrix times a vector, the following lemma follows.

Lemma 3.4.1 *Let A be an $m \times n$ block matrix as in 3.15 and let $\mathbf{x} \in \mathbb{F}^n$. Then $A\mathbf{x}$ is of the form*

$$A\mathbf{x} = \begin{pmatrix} \sum_j A_{1j}\mathbf{x}_j \\ \vdots \\ \sum_j A_{rj}\mathbf{x}_j \end{pmatrix}$$

where $\mathbf{x} = (\mathbf{x}_1, \cdots, \mathbf{x}_m)^T$ and $\mathbf{x}_i \in \mathbb{F}^{p_i}$.

Suppose also that B is a block matrix of the form

$$\begin{pmatrix} B_{11} & \cdots & B_{1p} \\ \vdots & \ddots & \vdots \\ B_{r1} & \cdots & B_{rp} \end{pmatrix} \tag{3.16}$$

and A is a block matrix of the form

$$\begin{pmatrix} A_{11} & \cdots & A_{1m} \\ \vdots & \ddots & \vdots \\ A_{p1} & \cdots & A_{pm} \end{pmatrix} \tag{3.17}$$

and that for all i, j, it makes sense to multiply $B_{is}A_{sj}$ for all $s \in \{1, \cdots, m\}$ and that for each $s, B_{is}A_{sj}$ is the same size so that it makes sense to write $\sum_s B_{is}A_{sj}$.

Theorem 3.4.2 *Let B be a block matrix as in 3.16 and let A be a block matrix as in 3.17 such that B_{is} is conformable with A_{sj} and each product, $B_{is}A_{sj}$ is of the same size so they can be added. Then BA is a block matrix such that the ij^{th} block is of the form*

$$\sum_s B_{is}A_{sj}. \tag{3.18}$$

Proof: Let B_{is} be a $q_i \times p_s$ matrix and A_{sj} be a $p_s \times r_j$ matrix. Also let $\mathbf{x} \in \mathbb{F}^n$ and let $\mathbf{x} = (\mathbf{x}_1, \cdots, \mathbf{x}_m)^T$ and $\mathbf{x}_i \in \mathbb{F}^{r_i}$ so it makes sense to multiply $A_{sj}\mathbf{x}_j$. Then from the associative law of matrix multiplication and Lemma 3.4.1 applied twice,

$$\left(\begin{pmatrix} B_{11} & \cdots & B_{1p} \\ \vdots & \ddots & \vdots \\ B_{r1} & \cdots & B_{rp} \end{pmatrix} \begin{pmatrix} A_{11} & \cdots & A_{1m} \\ \vdots & \ddots & \vdots \\ A_{p1} & \cdots & A_{pm} \end{pmatrix}\right) \begin{pmatrix} \mathbf{x}_1 \\ \vdots \\ \mathbf{x}_m \end{pmatrix}$$

$$= \begin{pmatrix} B_{11} & \cdots & B_{1p} \\ \vdots & \ddots & \vdots \\ B_{r1} & \cdots & B_{rp} \end{pmatrix} \begin{pmatrix} \sum_j A_{1j}\mathbf{x}_j \\ \vdots \\ \sum_j A_{rj}\mathbf{x}_j \end{pmatrix}$$

$$= \begin{pmatrix} \sum_s \sum_j B_{1s}A_{sj}\mathbf{x}_j \\ \vdots \\ \sum_s \sum_j B_{rs}A_{sj}\mathbf{x}_j \end{pmatrix} = \begin{pmatrix} \sum_j \left(\sum_s B_{1s}A_{sj}\right)\mathbf{x}_j \\ \vdots \\ \sum_j \left(\sum_s B_{rs}A_{sj}\right)\mathbf{x}_j \end{pmatrix}$$

$$= \begin{pmatrix} \sum_s B_{1s}A_{s1} & \cdots & \sum_s B_{1s}A_{sm} \\ \vdots & \ddots & \vdots \\ \sum_s B_{rs}A_{s1} & \cdots & \sum_s B_{rs}A_{sm} \end{pmatrix} \begin{pmatrix} \mathbf{x}_1 \\ \vdots \\ \mathbf{x}_m \end{pmatrix}$$

By Lemma 3.4.1, this shows that $(BA)\,\mathbf{x}$ equals the block matrix whose ij^{th} entry is given by 3.18 times $\mathbf{x}$. Since $\mathbf{x}$ is an arbitrary vector in $\mathbb{F}^n$, this proves the theorem.

The message of this theorem is that you can formally multiply block matrices as though the blocks were numbers. You just have to pay attention to the preservation of order.

3.5 Determinants

3.5.1 The Determinant Of A Matrix

The following Lemma will be essential in the definition of the determinant.

Lemma 3.5.1 *There exists a unique function,* sgn_n *which maps each list of numbers from* $\{1, \cdots, n\}$ *to one of the three numbers,* $0, 1,$ *or* -1 *which also has the following properties.*

$$\operatorname{sgn}_n(1, \cdots, n) = 1 \tag{3.19}$$

$$\operatorname{sgn}_n(i_1, \cdots, p, \cdots, q, \cdots, i_n) = -\operatorname{sgn}_n(i_1, \cdots, q, \cdots, p, \cdots, i_n) \tag{3.20}$$

In words, the second property states that if two of the numbers are switched, the value of the function is multiplied by -1*. Also, in the case where* $n > 1$ *and* $\{i_1, \cdots, i_n\} = \{1, \cdots, n\}$ *so that every number from* $\{1, \cdots, n\}$ *appears in the ordered list,* $(i_1, \cdots, i_n)$*,*

$$\operatorname{sgn}_n(i_1, \cdots, i_{\theta-1}, n, i_{\theta+1}, \cdots, i_n) \equiv$$

$$(-1)^{n-\theta} \operatorname{sgn}_{n-1}(i_1, \cdots, i_{\theta-1}, i_{\theta+1}, \cdots, i_n) \tag{3.21}$$

where $n = i_\theta$ *in the ordered list,* $(i_1, \cdots, i_n)$*.*

Proof: To begin with, it is necessary to show the existence of such a function. This is clearly true if $n = 1$. Define $\operatorname{sgn}_1(1) \equiv 1$ and observe that it works. No switching is possible. In the case where $n = 2$, it is also clearly true. Let $\operatorname{sgn}_2(1, 2) = 1$ and $\operatorname{sgn}_2(2, 1) = -1$ while $\operatorname{sgn}_2(2, 2) = \operatorname{sgn}_2(1, 1) = 0$ and verify it works. Assuming such a function exists for n,

sgn_{n+1} will be defined in terms of sgn_n. If there are any repeated numbers in $(i_1, \cdots, i_{n+1})$, $\operatorname{sgn}_{n+1}(i_1, \cdots, i_{n+1}) \equiv 0$. If there are no repeats, then $n+1$ appears somewhere in the ordered list. Let θ be the position of the number $n+1$ in the list. Thus, the list is of the form $(i_1, \cdots, i_{\theta-1}, n+1, i_{\theta+1}, \cdots, i_{n+1})$. From 3.21 it must be that

$$\operatorname{sgn}_{n+1}(i_1, \cdots, i_{\theta-1}, n+1, i_{\theta+1}, \cdots, i_{n+1}) \equiv$$

$$(-1)^{n+1-\theta} \operatorname{sgn}_n(i_1, \cdots, i_{\theta-1}, i_{\theta+1}, \cdots, i_{n+1}).$$

It is necessary to verify this satisfies 3.19 and 3.20 with n replaced with $n+1$. The first of these is obviously true because

$$\operatorname{sgn}_{n+1}(1, \cdots, n, n+1) \equiv (-1)^{n+1-(n+1)} \operatorname{sgn}_n(1, \cdots, n) = 1.$$

If there are repeated numbers in $(i_1, \cdots, i_{n+1})$, then it is obvious 3.20 holds because both sides would equal zero from the above definition. It remains to verify 3.20 in the case where there are no numbers repeated in $(i_1, \cdots, i_{n+1})$. Consider

$$\operatorname{sgn}_{n+1}\left(i_1, \cdots, \overset{r}{p}, \cdots, \overset{s}{q}, \cdots, i_{n+1}\right),$$

where the r above the p indicates the number, p is in the r^{th} position and the s above the q indicates that the number, q is in the s^{th} position. Suppose first that $r < \theta < s$. Then

$$\operatorname{sgn}_{n+1}\left(i_1, \cdots, \overset{r}{p}, \cdots, \overset{\theta}{n+1}, \cdots, \overset{s}{q}, \cdots, i_{n+1}\right) \equiv$$

$$(-1)^{n+1-\theta} \operatorname{sgn}_n\left(i_1, \cdots, \overset{r}{p}, \cdots, \overset{s-1}{q}, \cdots, i_{n+1}\right)$$

while

$$\operatorname{sgn}_{n+1}\left(i_1, \cdots, \overset{r}{q}, \cdots, \overset{\theta}{n+1}, \cdots, \overset{s}{p}, \cdots, i_{n+1}\right) =$$

$$(-1)^{n+1-\theta} \operatorname{sgn}_n\left(i_1, \cdots, \overset{r}{q}, \cdots, \overset{s-1}{p}, \cdots, i_{n+1}\right)$$

and so, by induction, a switch of p and q introduces a minus sign in the result. Similarly, if $\theta > s$ or if $\theta < r$ it also follows that 3.20 holds. The interesting case is when $\theta = r$ or $\theta = s$. Consider the case where $\theta = r$ and note the other case is entirely similar.

$$\operatorname{sgn}_{n+1}\left(i_1, \cdots, \overset{r}{n+1}, \cdots, \overset{s}{q}, \cdots, i_{n+1}\right) =$$

$$(-1)^{n+1-r} \operatorname{sgn}_n\left(i_1, \cdots, \overset{s-1}{q}, \cdots, i_{n+1}\right) \tag{3.22}$$

while

$$\operatorname{sgn}_{n+1}\left(i_1, \cdots, \overset{r}{q}, \cdots, \overset{s}{n+1}, \cdots, i_{n+1}\right) =$$

$$(-1)^{n+1-s} \operatorname{sgn}_n\left(i_1, \cdots, \overset{r}{q}, \cdots, i_{n+1}\right). \tag{3.23}$$

By making $s-1-r$ switches, move the q which is in the $s-1^{th}$ position in 3.22 to the r^{th} position in 3.23. By induction, each of these switches introduces a factor of -1 and so

$$\operatorname{sgn}_n\left(i_1, \cdots, \overset{s-1}{q}, \cdots, i_{n+1}\right) = (-1)^{s-1-r} \operatorname{sgn}_n\left(i_1, \cdots, \overset{r}{q}, \cdots, i_{n+1}\right).$$

Therefore,

$$\begin{aligned}\operatorname{sgn}_{n+1}\left(i_1,\cdots,\overset{r}{n+1},\cdots,\overset{s}{q},\cdots,i_{n+1}\right) &= (-1)^{n+1-r}\operatorname{sgn}_n\left(i_1,\cdots,\overset{s-1}{q},\cdots,i_{n+1}\right)\\ &= (-1)^{n+1-r}(-1)^{s-1-r}\operatorname{sgn}_n\left(i_1,\cdots,\overset{r}{q},\cdots,i_{n+1}\right)\\ = (-1)^{n+s}\operatorname{sgn}_n\left(i_1,\cdots,\overset{r}{q},\cdots,i_{n+1}\right) &= (-1)^{2s-1}(-1)^{n+1-s}\operatorname{sgn}_n\left(i_1,\cdots,\overset{r}{q},\cdots,i_{n+1}\right)\\ &= -\operatorname{sgn}_{n+1}\left(i_1,\cdots,\overset{r}{q},\cdots,\overset{s}{n+1},\cdots,i_{n+1}\right).\end{aligned}$$

This proves the existence of the desired function.

To see this function is unique, note that you can obtain any ordered list of distinct numbers from a sequence of switches. If there exist two functions, f and g both satisfying 3.19 and 3.20, you could start with $f(1,\cdots,n)=g(1,\cdots,n)$ and applying the same sequence of switches, eventually arrive at $f(i_1,\cdots,i_n)=g(i_1,\cdots,i_n)$. If any numbers are repeated, then 3.20 gives both functions are equal to zero for that ordered list. This proves the lemma.

In what follows sgn will often be used rather than sgn_n because the context supplies the appropriate n.

Definition 3.5.2 *Let f be a real valued function which has the set of ordered lists of numbers from $\{1,\cdots,n\}$ as its domain. Define*

$$\sum_{(k_1,\cdots,k_n)} f(k_1\cdots k_n)$$

to be the sum of all the $f(k_1\cdots k_n)$ for all possible choices of ordered lists $(k_1,\cdots,k_n)$ of numbers of $\{1,\cdots,n\}$. For example,

$$\sum_{(k_1,k_2)} f(k_1,k_2)=f(1,2)+f(2,1)+f(1,1)+f(2,2).$$

Definition 3.5.3 *Let $(a_{ij})=A$ denote an $n\times n$ matrix. The determinant of A, denoted by $\det(A)$ is defined by*

$$\det(A)\equiv\sum_{(k_1,\cdots,k_n)}\operatorname{sgn}(k_1,\cdots,k_n)\,a_{1k_1}\cdots a_{nk_n}$$

where the sum is taken over all ordered lists of numbers from $\{1,\cdots,n\}$. Note it suffices to take the sum over only those ordered lists in which there are no repeats because if there are, $\operatorname{sgn}(k_1,\cdots,k_n)=0$ and so that term contributes 0 to the sum.

Let A be an $n\times n$ matrix, $A=(a_{ij})$ and let $(r_1,\cdots,r_n)$ denote an ordered list of n numbers from $\{1,\cdots,n\}$. Let $A(r_1,\cdots,r_n)$ denote the matrix whose k^{th} row is the r_k row of the matrix, A. Thus

$$\det(A(r_1,\cdots,r_n))=\sum_{(k_1,\cdots,k_n)}\operatorname{sgn}(k_1,\cdots,k_n)\,a_{r_1k_1}\cdots a_{r_nk_n}\tag{3.24}$$

and

$$A(1,\cdots,n)=A.$$

Proposition 3.5.4 *Let*

$$(r_1, \cdots, r_n)$$

be an ordered list of numbers from $\{1, \cdots, n\}$. *Then*

$$\operatorname{sgn}(r_1, \cdots, r_n) \det(A)$$

$$= \sum_{(k_1, \cdots, k_n)} \operatorname{sgn}(k_1, \cdots, k_n) a_{r_1 k_1} \cdots a_{r_n k_n} \tag{3.25}$$

$$= \det(A(r_1, \cdots, r_n)). \tag{3.26}$$

Proof: Let $(1, \cdots, n) = (1, \cdots, r, \cdots s, \cdots, n)$ so $r < s$.

$$\det(A(1, \cdots, r, \cdots, s, \cdots, n)) = \tag{3.27}$$

$$\sum_{(k_1, \cdots, k_n)} \operatorname{sgn}(k_1, \cdots, k_r, \cdots, k_s, \cdots, k_n) a_{1k_1} \cdots a_{rk_r} \cdots a_{sk_s} \cdots a_{nk_n},$$

and renaming the variables, calling k_s, k_r and k_r, k_s, this equals

$$= \sum_{(k_1, \cdots, k_n)} \operatorname{sgn}(k_1, \cdots, k_s, \cdots, k_r, \cdots, k_n) a_{1k_1} \cdots a_{rk_s} \cdots a_{sk_r} \cdots a_{nk_n}$$

$$= \sum_{(k_1, \cdots, k_n)} -\operatorname{sgn}\left(k_1, \cdots, \overbrace{k_r, \cdots, k_s}^{\text{These got switched}}, \cdots, k_n\right) a_{1k_1} \cdots a_{sk_r} \cdots a_{rk_s} \cdots a_{nk_n}$$

$$= -\det(A(1, \cdots, s, \cdots, r, \cdots, n)). \tag{3.28}$$

Consequently,

$$\det(A(1, \cdots, s, \cdots, r, \cdots, n)) =$$

$$-\det(A(1, \cdots, r, \cdots, s, \cdots, n)) = -\det(A)$$

Now letting $A(1, \cdots, s, \cdots, r, \cdots, n)$ play the role of A, and continuing in this way, switching pairs of numbers,

$$\det(A(r_1, \cdots, r_n)) = (-1)^p \det(A)$$

where it took p switches to obtain$(r_1, \cdots, r_n)$ from $(1, \cdots, n)$. By Lemma 3.5.1, this implies

$$\det(A(r_1, \cdots, r_n)) = (-1)^p \det(A) = \operatorname{sgn}(r_1, \cdots, r_n) \det(A)$$

and proves the proposition in the case when there are no repeated numbers in the ordered list, $(r_1, \cdots, r_n)$. However, if there is a repeat, say the r^{th} row equals the s^{th} row, then the reasoning of 3.27 -3.28 shows that $A(r_1, \cdots, r_n) = 0$ and also $\operatorname{sgn}(r_1, \cdots, r_n) = 0$ so the formula holds in this case also.

Observation 3.5.5 *There are* $n!$ *ordered lists of distinct numbers from* $\{1, \cdots, n\}$.

With the above, it is possible to give a more symmetric description of the determinant from which it will follow that $\det(A) = \det\left(A^T\right)$.

Corollary 3.5.6 *The following formula for* $\det(A)$ *is valid.*

$$\det(A) = \frac{1}{n!}\cdot$$

$$\sum_{(r_1,\cdots,r_n)} \sum_{(k_1,\cdots,k_n)} \operatorname{sgn}(r_1,\cdots,r_n) \operatorname{sgn}(k_1,\cdots,k_n) a_{r_1k_1}\cdots a_{r_nk_n}. \tag{3.29}$$

And also $\det\left(A^T\right) = \det(A)$ *where* A^T *is the transpose of* A*. (Recall that for* $A^T = \left(a_{ij}^T\right)$*,* $a_{ij}^T = a_{ji}$*.)*

Proof: From Proposition 3.5.4, if the r_i are distinct,

$$\det(A) = \sum_{(k_1,\cdots,k_n)} \operatorname{sgn}(r_1,\cdots,r_n) \operatorname{sgn}(k_1,\cdots,k_n) a_{r_1k_1}\cdots a_{r_nk_n}.$$

Summing over all ordered lists, $(r_1,\cdots,r_n)$ where the r_i are distinct, (If the r_i are not distinct, $\operatorname{sgn}(r_1,\cdots,r_n) = 0$ and so there is no contribution to the sum.)

$$n!\det(A) =$$

$$\sum_{(r_1,\cdots,r_n)} \sum_{(k_1,\cdots,k_n)} \operatorname{sgn}(r_1,\cdots,r_n) \operatorname{sgn}(k_1,\cdots,k_n) a_{r_1k_1}\cdots a_{r_nk_n}.$$

This proves the corollary since the formula gives the same number for A as it does for A^T.

Corollary 3.5.7 *If two rows or two columns in an* $n\times n$ *matrix,* A*, are switched, the determinant of the resulting matrix equals* (-1) *times the determinant of the original matrix. If* A *is an* $n\times n$ *matrix in which two rows are equal or two columns are equal then* $\det(A) = 0$*. Suppose the* i^{th} *row of* A *equals* $(xa_1 + yb_1,\cdots,xa_n + yb_n)$*. Then*

$$\det(A) = x\det(A_1) + y\det(A_2)$$

where the i^{th} *row of* A_1 *is* $(a_1,\cdots,a_n)$ *and the* i^{th} *row of* A_2 *is* $(b_1,\cdots,b_n)$*, all other rows of* A_1 *and* A_2 *coinciding with those of* A*. In other words,* det *is a linear function of each row* A*. The same is true with the word "row" replaced with the word "column".*

Proof: By Proposition 3.5.4 when two rows are switched, the determinant of the resulting matrix is (-1) times the determinant of the original matrix. By Corollary 3.5.6 the same holds for columns because the columns of the matrix equal the rows of the transposed matrix. Thus if A_1 is the matrix obtained from A by switching two columns,

$$\det(A) = \det\left(A^T\right) = -\det\left(A_1^T\right) = -\det(A_1).$$

If A has two equal columns or two equal rows, then switching them results in the same matrix. Therefore, $\det(A) = -\det(A)$ and so $\det(A) = 0$.

It remains to verify the last assertion.

$$\begin{aligned}\det(A) &\equiv \sum_{(k_1,\cdots,k_n)} \operatorname{sgn}(k_1,\cdots,k_n) a_{1k_1}\cdots(xa_{k_i} + yb_{k_i})\cdots a_{nk_n}\\ &= x\sum_{(k_1,\cdots,k_n)} \operatorname{sgn}(k_1,\cdots,k_n) a_{1k_1}\cdots a_{k_i}\cdots a_{nk_n}\\ &+y\sum_{(k_1,\cdots,k_n)} \operatorname{sgn}(k_1,\cdots,k_n) a_{1k_1}\cdots b_{k_i}\cdots a_{nk_n}\end{aligned}$$

$$\equiv x \det(A_1) + y \det(A_2).$$

The same is true of columns because $\det\left(A^T\right) = \det(A)$ and the rows of A^T are the columns of A.

The following corollary is also of great use.

Corollary 3.5.8 *Suppose A is an $n \times n$ matrix and some column (row) is a linear combination of r other columns (rows). Then* $\det(A) = 0$.

Proof: Let $A = \begin{pmatrix} \mathbf{a}_1 & \cdots & \mathbf{a}_n \end{pmatrix}$ be the columns of A and suppose the condition that one column is a linear combination of r of the others is satisfied. Then by using Corollary 3.5.7 you may rearrange the columns to have the n^{th} column a linear combination of the first r columns. Thus $\mathbf{a}_n = \sum_{k=1}^{r} c_k \mathbf{a}_k$ and so

$$\det(A) = \det\begin{pmatrix} \mathbf{a}_1 & \cdots & \mathbf{a}_r & \cdots & \mathbf{a}_{n-1} & \sum_{k=1}^{r} c_k \mathbf{a}_k \end{pmatrix}.$$

By Corollary 3.5.7

$$\det(A) = \sum_{k=1}^{r} c_k \det\begin{pmatrix} \mathbf{a}_1 & \cdots & \mathbf{a}_r & \cdots & \mathbf{a}_{n-1} & \mathbf{a}_k \end{pmatrix} = 0.$$

The case for rows follows from the fact that $\det(A) = \det\left(A^T\right)$. This proves the corollary.

Recall the following definition of matrix multiplication.

Definition 3.5.9 *If A and B are $n \times n$ matrices, $A = (a_{ij})$ and $B = (b_{ij})$, $AB = (c_{ij})$ where*

$$c_{ij} \equiv \sum_{k=1}^{n} a_{ik} b_{kj}.$$

One of the most important rules about determinants is that the determinant of a product equals the product of the determinants.

Theorem 3.5.10 *Let A and B be $n \times n$ matrices. Then*

$$\det(AB) = \det(A) \det(B).$$

Proof: Let c_{ij} be the ij^{th} entry of AB. Then by Proposition 3.5.4,

$$\det(AB) =$$

$$\begin{aligned}
&\sum_{(k_1,\cdots,k_n)} \operatorname{sgn}(k_1,\cdots,k_n)\, c_{1k_1} \cdots c_{nk_n} \\
&= \sum_{(k_1,\cdots,k_n)} \operatorname{sgn}(k_1,\cdots,k_n) \left(\sum_{r_1} a_{1r_1} b_{r_1 k_1}\right) \cdots \left(\sum_{r_n} a_{nr_n} b_{r_n k_n}\right) \\
&= \sum_{(r_1\cdots,r_n)} \sum_{(k_1,\cdots,k_n)} \operatorname{sgn}(k_1,\cdots,k_n)\, b_{r_1 k_1} \cdots b_{r_n k_n} \left(a_{1r_1} \cdots a_{nr_n}\right) \\
&= \sum_{(r_1\cdots,r_n)} \operatorname{sgn}(r_1 \cdots r_n)\, a_{1r_1} \cdots a_{nr_n} \det(B) = \det(A) \det(B).
\end{aligned}$$

This proves the theorem.

Lemma 3.5.11 *Suppose a matrix is of the form*

$$M = \begin{pmatrix} A & * \\ \mathbf{0} & a \end{pmatrix} \tag{3.30}$$

or

$$M = \begin{pmatrix} A & \mathbf{0} \\ * & a \end{pmatrix} \tag{3.31}$$

where a is a number and A is an $(n-1)\times(n-1)$ matrix and $*$ denotes either a column or a row having length $n-1$ and the $\mathbf{0}$ denotes either a column or a row of length $n-1$ consisting entirely of zeros. Then

$$\det(M) = a\det(A).$$

Proof: Denote M by (m_{ij}). Thus in the first case, $m_{nn} = a$ and $m_{ni} = 0$ if $i \neq n$ while in the second case, $m_{nn} = a$ and $m_{in} = 0$ if $i \neq n$. From the definition of the determinant,

$$\det(M) \equiv \sum_{(k_1,\cdots,k_n)} \operatorname{sgn}_n(k_1,\cdots,k_n)\, m_{1k_1}\cdots m_{nk_n}$$

Letting θ denote the position of n in the ordered list, $(k_1,\cdots,k_n)$ then using the earlier conventions used to prove Lemma 3.5.1, $\det(M)$ equals

$$\sum_{(k_1,\cdots,k_n)} (-1)^{n-\theta} \operatorname{sgn}_{n-1}\left(k_1,\cdots,k_{\theta-1},\overset{\theta}{k_{\theta+1}},\cdots,\overset{n-1}{k_n}\right) m_{1k_1}\cdots m_{nk_n}$$

Now suppose 3.31. Then if $k_n \neq n$, the term involving m_{nk_n} in the above expression equals zero. Therefore, the only terms which survive are those for which $\theta = n$ or in other words, those for which $k_n = n$. Therefore, the above expression reduces to

$$a \sum_{(k_1,\cdots,k_{n-1})} \operatorname{sgn}_{n-1}(k_1,\cdots k_{n-1})\, m_{1k_1}\cdots m_{(n-1)k_{n-1}} = a\det(A).$$

To get the assertion in the situation of 3.30 use Corollary 3.5.6 and 3.31 to write

$$\det(M) = \det\left(M^T\right) = \det\left(\begin{pmatrix} A^T & \mathbf{0} \\ * & a \end{pmatrix}\right) = a\det\left(A^T\right) = a\det(A).$$

This proves the lemma.

In terms of the theory of determinants, arguably the most important idea is that of Laplace expansion along a row or a column. This will follow from the above definition of a determinant.

Definition 3.5.12 *Let $A = (a_{ij})$ be an $n\times n$ matrix. Then a new matrix called the cofactor matrix, $\operatorname{cof}(A)$ is defined by $\operatorname{cof}(A) = (c_{ij})$ where to obtain c_{ij} delete the i^{th} row and the j^{th} column of A, take the determinant of the $(n-1)\times(n-1)$ matrix which results, (This is called the ij^{th} minor of A.) and then multiply this number by $(-1)^{i+j}$. To make the formulas easier to remember, $\operatorname{cof}(A)_{ij}$ will denote the ij^{th} entry of the cofactor matrix.*

The following is the main result.

Theorem 3.5.13 *Let A be an $n\times n$ matrix where $n \geq 2$. Then*

$$\det(A) = \sum_{j=1}^{n} a_{ij}\operatorname{cof}(A)_{ij} = \sum_{i=1}^{n} a_{ij}\operatorname{cof}(A)_{ij}. \tag{3.32}$$

The first formula consists of expanding the determinant along the i^{th} row and the second expands the determinant along the j^{th} column.

Proof: Let $(a_{i1}, \cdots, a_{in})$ be the i^{th} row of A. Let B_j be the matrix obtained from A by leaving every row the same except the i^{th} row which in B_j equals $(0, \cdots, 0, a_{ij}, 0, \cdots, 0)$. Then by Corollary 3.5.7,

$$\det(A) = \sum_{j=1}^{n} \det(B_j)$$

Denote by A^{ij} the $(n-1) \times (n-1)$ matrix obtained by deleting the i^{th} row and the j^{th} column of A. Thus $\operatorname{cof}(A)_{ij} \equiv (-1)^{i+j} \det\left(A^{ij}\right)$. At this point, recall that from Proposition 3.5.4, when two rows or two columns in a matrix, M, are switched, this results in multiplying the determinant of the old matrix by -1 to get the determinant of the new matrix. Therefore, by Lemma 3.5.11,

$$\begin{aligned} \det(B_j) &= (-1)^{n-j}(-1)^{n-i} \det\left(\begin{pmatrix} A^{ij} & * \\ \mathbf{0} & a_{ij} \end{pmatrix}\right) \\ &= (-1)^{i+j} \det\left(\begin{pmatrix} A^{ij} & * \\ \mathbf{0} & a_{ij} \end{pmatrix}\right) = a_{ij} \operatorname{cof}(A)_{ij}. \end{aligned}$$

Therefore,

$$\det(A) = \sum_{j=1}^{n} a_{ij} \operatorname{cof}(A)_{ij}$$

which is the formula for expanding $\det(A)$ along the i^{th} row. Also,

$$\begin{aligned} \det(A) &= \det\left(A^T\right) = \sum_{j=1}^{n} a_{ij}^T \operatorname{cof}\left(A^T\right)_{ij} \\ &= \sum_{j=1}^{n} a_{ji} \operatorname{cof}(A)_{ji} \end{aligned}$$

which is the formula for expanding $\det(A)$ along the i^{th} column. This proves the theorem.

Note that this gives an easy way to write a formula for the inverse of an $n \times n$ matrix.

Theorem 3.5.14 *A^{-1} exists if and only if $\det(A) \neq 0$. If $\det(A) \neq 0$, then $A^{-1} = \left(a_{ij}^{-1}\right)$ where*

$$a_{ij}^{-1} = \det(A)^{-1} \operatorname{cof}(A)_{ji}$$

for $\operatorname{cof}(A)_{ij}$ the ij^{th} cofactor of A.

Proof: By Theorem 3.5.13 and letting $(a_{ir}) = A$, if $\det(A) \neq 0$,

$$\sum_{i=1}^{n} a_{ir} \operatorname{cof}(A)_{ir} \det(A)^{-1} = \det(A) \det(A)^{-1} = 1.$$

Now consider

$$\sum_{i=1}^{n} a_{ir} \operatorname{cof}(A)_{ik} \det(A)^{-1}$$

when $k \neq r$. Replace the k^{th} column with the r^{th} column to obtain a matrix, B_k whose determinant equals zero by Corollary 3.5.7. However, expanding this matrix along the k^{th} column yields

$$0 = \det(B_k) \det(A)^{-1} = \sum_{i=1}^{n} a_{ir} \operatorname{cof}(A)_{ik} \det(A)^{-1}$$

Summarizing,

$$\sum_{i=1}^{n} a_{ir} \operatorname{cof}(A)_{ik} \det(A)^{-1} = \delta_{rk}.$$

Using the other formula in Theorem 3.5.13, and similar reasoning,

$$\sum_{j=1}^{n} a_{rj} \operatorname{cof}(A)_{kj} \det(A)^{-1} = \delta_{rk}$$

This proves that if $\det(A) \neq 0$, then A^{-1} exists with $A^{-1} = \left(a_{ij}^{-1}\right)$, where

$$a_{ij}^{-1} = \operatorname{cof}(A)_{ji} \det(A)^{-1}.$$

Now suppose A^{-1} exists. Then by Theorem 3.5.10,

$$1 = \det(I) = \det\left(AA^{-1}\right) = \det(A) \det\left(A^{-1}\right)$$

so $\det(A) \neq 0$. This proves the theorem.

The next corollary points out that if an $n \times n$ matrix, A has a right or a left inverse, then it has an inverse.

Corollary 3.5.15 *Let A be an $n \times n$ matrix and suppose there exists an $n \times n$ matrix, B such that $BA = I$. Then A^{-1} exists and $A^{-1} = B$. Also, if there exists C an $n \times n$ matrix such that $AC = I$, then A^{-1} exists and $A^{-1} = C$.*

Proof: Since $BA = I$, Theorem 3.5.10 implies

$$\det B \det A = 1$$

and so $\det A \neq 0$. Therefore from Theorem 3.5.14, A^{-1} exists. Therefore,

$$A^{-1} = (BA) A^{-1} = B\left(AA^{-1}\right) = BI = B.$$

The case where $CA = I$ is handled similarly.

The conclusion of this corollary is that left inverses, right inverses and inverses are all the same in the context of $n \times n$ matrices.

Theorem 3.5.14 says that to find the inverse, take the transpose of the cofactor matrix and divide by the determinant. The transpose of the cofactor matrix is called the adjugate or sometimes the classical adjoint of the matrix A. It is an abomination to call it the adjoint although you do sometimes see it referred to in this way. In words, A^{-1} is equal to one over the determinant of A times the adjugate matrix of A.

In case you are solving a system of equations, $A\mathbf{x} = \mathbf{y}$ for $\mathbf{x}$, it follows that if A^{-1} exists,

$$\mathbf{x} = \left(A^{-1}A\right)\mathbf{x} = A^{-1}(A\mathbf{x}) = A^{-1}\mathbf{y}$$

thus solving the system. Now in the case that A^{-1} exists, there is a formula for A^{-1} given above. Using this formula,

$$x_i = \sum_{j=1}^{n} a_{ij}^{-1} y_j = \sum_{j=1}^{n} \frac{1}{\det(A)} \operatorname{cof}(A)_{ji} y_j.$$

By the formula for the expansion of a determinant along a column,

$$x_i = \frac{1}{\det(A)} \det \begin{pmatrix} * & \cdots & y_1 & \cdots & * \\ \vdots & & \vdots & & \vdots \\ * & \cdots & y_n & \cdots & * \end{pmatrix},$$

where here the i^{th} column of A is replaced with the column vector, $(y_1 \cdots, y_n)^T$, and the determinant of this modified matrix is taken and divided by $\det(A)$. This formula is known as Cramer's rule.

Definition 3.5.16 *A matrix M, is upper triangular if $M_{ij} = 0$ whenever $i > j$. Thus such a matrix equals zero below the main diagonal, the entries of the form M_{ii} as shown.*

$$\begin{pmatrix} * & * & \cdots & * \\ 0 & * & \ddots & \vdots \\ \vdots & \ddots & \ddots & * \\ 0 & \cdots & 0 & * \end{pmatrix}$$

A lower triangular matrix is defined similarly as a matrix for which all entries above the main diagonal are equal to zero.

With this definition, here is a simple corollary of Theorem 3.5.13.

Corollary 3.5.17 *Let M be an upper (lower) triangular matrix. Then $\det(M)$ is obtained by taking the product of the entries on the main diagonal.*

Definition 3.5.18 *A submatrix of a matrix A is the rectangular array of numbers obtained by deleting some rows and columns of A. Let A be an $m \times n$ matrix. The **determinant rank** of the matrix equals r where r is the largest number such that some $r \times r$ submatrix of A has a non zero determinant. The **row rank** is defined to be the dimension of the span of the rows. The **column rank** is defined to be the dimension of the span of the columns.*

Theorem 3.5.19 *If A has determinant rank, r, then there exist r rows of the matrix such that every other row is a linear combination of these r rows.*

Proof: Suppose the determinant rank of $A = (a_{ij})$ equals r. If rows and columns are interchanged, the determinant rank of the modified matrix is unchanged. Thus rows and columns can be interchanged to produce an $r \times r$ matrix in the upper left corner of the matrix which has non zero determinant. Now consider the $r+1 \times r+1$ matrix, M,

$$\begin{pmatrix} a_{11} & \cdots & a_{1r} & a_{1p} \\ \vdots & & \vdots & \vdots \\ a_{r1} & \cdots & a_{rr} & a_{rp} \\ a_{l1} & \cdots & a_{lr} & a_{lp} \end{pmatrix}$$

where C will denote the $r \times r$ matrix in the upper left corner which has non zero determinant. I claim $\det(M) = 0$.

There are two cases to consider in verifying this claim. First, suppose $p > r$. Then the claim follows from the assumption that A has determinant rank r. On the other hand, if $p < r$, then the determinant is zero because there are two identical columns. Expand the determinant along the last column and divide by $\det(C)$ to obtain

$$a_{lp} = -\sum_{i=1}^{r} \frac{\operatorname{cof}(M)_{ip}}{\det(C)} a_{ip}.$$

Now note that $\operatorname{cof}(M)_{ip}$ does not depend on p. Therefore the above sum is of the form

$$a_{lp} = \sum_{i=1}^{r} m_i a_{ip}$$

which shows the l^{th} row is a linear combination of the first r rows of A. Since l is arbitrary, this proves the theorem.

Corollary 3.5.20 *The determinant rank equals the row rank.*

Proof: From Theorem 3.5.19, the row rank is no larger than the determinant rank. Could the row rank be smaller than the determinant rank? If so, there exist p rows for $p < r$ such that the span of these p rows equals the row space. But this implies that the $r \times r$ submatrix whose determinant is nonzero also has row rank no larger than p which is impossible if its determinant is to be nonzero because at least one row is a linear combination of the others.

Corollary 3.5.21 *If A has determinant rank, r, then there exist r columns of the matrix such that every other column is a linear combination of these r columns. Also the column rank equals the determinant rank.*

Proof: This follows from the above by considering A^T. The rows of A^T are the columns of A and the determinant rank of A^T and A are the same. Therefore, from Corollary 3.5.20, column rank of A = row rank of A^T = determinant rank of A^T = determinant rank of A.

The following theorem is of fundamental importance and ties together many of the ideas presented above.

Theorem 3.5.22 *Let A be an $n \times n$ matrix. Then the following are equivalent.*

1. $\det(A) = 0$.
2. A, A^T *are not one to one.*
3. A *is not onto.*

Proof: Suppose $\det(A) = 0$. Then the determinant rank of $A = r < n$. Therefore, there exist r columns such that every other column is a linear combination of these columns by Theorem 3.5.19. In particular, it follows that for some m, the m^{th} column is a linear combination of all the others. Thus letting $A = \begin{pmatrix} \mathbf{a}_1 & \cdots & \mathbf{a}_m & \cdots & \mathbf{a}_n \end{pmatrix}$ where the columns are denoted by $\mathbf{a}_i$, there exists scalars, α_i such that

$$\mathbf{a}_m = \sum_{k \neq m} \alpha_k \mathbf{a}_k.$$

Now consider the column vector, $\mathbf{x} \equiv \begin{pmatrix} \alpha_1 & \cdots & -1 & \cdots & \boldsymbol{\alpha}_n \end{pmatrix}^T$. Then

$$A\mathbf{x} = -\mathbf{a}_m + \sum_{k \neq m} \alpha_k \mathbf{a}_k = \mathbf{0}.$$

Since also $A\mathbf{0} = \mathbf{0}$, it follows A is not one to one. Similarly, A^T is not one to one by the same argument applied to A^T. This verifies that 1.) implies 2.).

Now suppose 2.). Then since A^T is not one to one, it follows there exists $\mathbf{x} \neq \mathbf{0}$ such that

$$A^T\mathbf{x} = \mathbf{0}.$$

Taking the transpose of both sides yields

$$\mathbf{x}^T A = \mathbf{0}$$

where the $\mathbf{0}$ is a $1 \times n$ matrix or row vector. Now if $A\mathbf{y} = \mathbf{x}$, then

$$|\mathbf{x}|^2 = \mathbf{x}^T (A\mathbf{y}) = \left(\mathbf{x}^T A\right) \mathbf{y} = \mathbf{0}\mathbf{y} = 0$$

contrary to $\mathbf{x} \neq \mathbf{0}$. Consequently there can be no $\mathbf{y}$ such that $A\mathbf{y} = \mathbf{x}$ and so A is not onto. This shows that 2.) implies 3.).

Finally, suppose 3.). If 1.) does not hold, then $\det(A) \neq 0$ but then from Theorem 3.5.14 A^{-1} exists and so for every $\mathbf{y} \in \mathbb{F}^n$ there exists a unique $\mathbf{x} \in \mathbb{F}^n$ such that $A\mathbf{x} = \mathbf{y}$. In fact $\mathbf{x} = A^{-1}\mathbf{y}$. Thus A would be onto contrary to 3.). This shows 3.) implies 1.) and proves the theorem.

Corollary 3.5.23 *Let A be an $n \times n$ matrix. Then the following are equivalent.*

1. *$det(A) \neq 0$.*
2. *A and A^T are one to one.*
3. *A is onto.*

Proof: This follows immediately from the above theorem.

3.5.2 The Determinant Of A Linear Transformation

One can also define the determinant of a linear transformation.

Definition 3.5.24 *Let $L \in \mathcal{L}(V,V)$ and let $\{\mathbf{v}_1, \cdots, \mathbf{v}_n\}$ be a basis for V. Thus the matrix of L with respect to this basis is $(l_{ij}) \equiv M_L$ where*

$$L = \sum_{ij} l_{ij} \mathbf{v}_i \mathbf{v}_j$$

Then define

$$\det(L) \equiv \det((l_{ij})).$$

Proposition 3.5.25 *The above definition is well defined.*

Proof: Suppose $\{\mathbf{v}'_1, \cdots, \mathbf{v}'_n\}$ is another basis for V and $\left(l'_{ij}\right) \equiv M'_L$ is the matrix of L with respect to this basis. Then by Theorem 3.3.9,

$$M'_L = S^{-1} M_L S$$

for some matrix, S. Then by Theorem 3.5.10,

$$\begin{aligned} \det(M'_L) &= \det\left(S^{-1}\right) \det(M_L) \det(S) \\ &= \det\left(S^{-1}S\right) \det(M_L) = \det(M_L) \end{aligned}$$

because $S^{-1}S = I$ and $\det(I) = 1$. This shows the definition is well defined.

Also there is an equivalence just as in the case of matrices between various properties of L and the nonvanishing of the determinant.

Theorem 3.5.26 *Let $L \in \mathcal{L}(V,V)$ for V a finite dimensional vector space. Then the following are equivalent.*

1. *$\det(L) = 0$.*
2. *L is not one to one.*

3. L is not onto.

Proof: Suppose 1.). Let $\{\mathbf{v}_1, \cdots, \mathbf{v}_n\}$ be a basis for V and let (l_{ij}) be the matrix of L with respect to this basis. By definition, $\det((l_{ij})) = 0$ and so (l_{ij}) is not one to one. Thus there is a nonzero vector $\mathbf{x} \in \mathbb{F}^n$ such that $\sum_j l_{ij}x_j = 0$ for each i. Then letting $\mathbf{v} \equiv \sum_{j=1}^n x_j\mathbf{v}_j$,

$$\begin{aligned} L\mathbf{v} &= \sum_{rs} l_{rs}\mathbf{v}_r\mathbf{v}_s \left(\sum_{j=1}^n x_j\mathbf{v}_j\right) = \sum_j \sum_{rs} l_{rs}\mathbf{v}_r\delta_{sj}x_j \\ &= \sum_r \left(\sum_j l_{rj}x_j\right)\mathbf{v}_r = \mathbf{0} \end{aligned}$$

Thus L is not one to one because $L\mathbf{0} = \mathbf{0}$ and $L\mathbf{v} = \mathbf{0}$.

Suppose 2.). Thus there exists $\mathbf{v} \neq \mathbf{0}$ such that $L\mathbf{v} = \mathbf{0}$. Say

$$\mathbf{v} = \sum_i x_i\mathbf{v}_i.$$

Then if $\{L\mathbf{v}_i\}_{i=1}^n$ were linearly independent, it would follow that

$$\mathbf{0} = L\mathbf{v} = \sum_i x_i L\mathbf{v}_i$$

and so all the x_i would equal zero which is not the case. Hence these vectors cannot be linearly independent so they do not span V. Hence there exists

$$\mathbf{w} \in V \setminus \text{span}\,(L\mathbf{v}_1, \cdots, L\mathbf{v}_n)$$

and therefore, there is no $\mathbf{u} \in V$ such that $L\mathbf{u} = \mathbf{w}$ because if there were such a $\mathbf{u}$, then

$$\mathbf{u} = \sum_i x_i\mathbf{v}_i$$

and so $L\mathbf{u} = \sum_i x_i L\mathbf{v}_i \in \text{span}\,(L\mathbf{v}_1, \cdots, L\mathbf{v}_n)$.

Finally suppose L is not onto. Then (l_{ij}) also cannot be onto $\mathbb{F}^n$. Therefore, $\det((l_{ij})) \equiv \det(L) = 0$. Why can't (l_{ij}) be onto? If it were, then for any $\mathbf{y} \in \mathbb{F}^n$, there exists $\mathbf{x} \in \mathbb{F}^n$ such that $y_i = \sum_j l_{ij}x_j$. Thus

$$\sum_k y_k\mathbf{v}_k = \sum_{rs} l_{rs}\mathbf{v}_r\mathbf{v}_s \left(\sum_j x_j\mathbf{v}_j\right) = L\left(\sum_j x_j\mathbf{v}_j\right)$$

but the expression on the left in the above formula is that of a general element of V and so L would be onto. This proves the theorem.

3.6 Eigenvalues And Eigenvectors Of Linear Transformations

Let V be a finite dimensional vector space. For example, it could be a subspace of $\mathbb{C}^n$or $\mathbb{R}^n$. Also suppose $A \in \mathcal{L}(V, V)$.

Definition 3.6.1 *The characteristic polynomial of A is defined as $q(\lambda) \equiv \det(\lambda \mathrm{id} - A)$ where* id *is the identity map which takes every vector in V to itself. The zeros of $q(\lambda)$ in $\mathbb{C}$ are called the eigenvalues of A.*

Lemma 3.6.2 *When λ is an eigenvalue of A which is also in $\mathbb{F}$, the field of scalars, then there exists $\mathbf{v} \neq \mathbf{0}$ such that $A\mathbf{v} = \lambda\mathbf{v}$.*

Proof: This follows from Theorem 3.5.26. Since $\lambda \in \mathbb{F}$,

$$\lambda \mathrm{id} - A \in \mathcal{L}(V, V)$$

and since it has zero determinant, it is not one to one so there exists $\mathbf{v} \neq \mathbf{0}$ such that $(\lambda \mathrm{id} - A)\mathbf{v} = \mathbf{0}$.

The following lemma gives the existence of something called the minimal polynomial. It is an interesting application of the notion of the dimension of $\mathcal{L}(V, V)$.

Lemma 3.6.3 *Let $A \in \mathcal{L}(V, V)$ where V is either a real or a complex finite dimensional vector space of dimension n. Then there exists a polynomial of the form*

$$p(\lambda) = \lambda^m + c_{m-1}\lambda^{m-1} + \cdots + c_1\lambda + c_0$$

such that $p(A) = 0$ and m is as small as possible for this to occur.

Proof: Consider the linear transformations, $I, A, A^2, \cdots, A^{n^2}$. There are $n^2 + 1$ of these transformations and so by Theorem 3.3.5 the set is linearly dependent. Thus there exist constants, $c_i \in \mathbb{F}$ (either $\mathbb{R}$ or $\mathbb{C}$) such that

$$c_0 I + \sum_{k=1}^{n^2} c_k A^k = 0.$$

This implies there exists a polynomial, $q(\lambda)$ which has the property that $q(A) = 0$. In fact, one example is $q(\lambda) \equiv c_0 + \sum_{k=1}^{n^2} c_k \lambda^k$. Dividing by the leading term, it can be assumed this polynomial is of the form $\lambda^m + c_{m-1}\lambda^{m-1} + \cdots + c_1\lambda + c_0$, a monic polynomial. Now consider all such monic polynomials, q such that $q(A) = 0$ and pick the one which has the smallest degree. This is called the minimial polynomial and will be denoted here by $p(\lambda)$. This proves the lemma.

Theorem 3.6.4 *Let V be a nonzero finite dimensional vector space of dimension n with the field of scalars equal to $\mathbb{F}$ which is either $\mathbb{R}$ or $\mathbb{C}$. Suppose $A \in \mathcal{L}(V, V)$ and for $p(\lambda)$ the minimal polynomial defined above, let $\mu \in \mathbb{F}$ be a zero of this polynomial. Then there exists $\mathbf{v} \neq \mathbf{0}, \mathbf{v} \in V$ such that*

$$A\mathbf{v} = \mu\mathbf{v}.$$

If $\mathbb{F} = \mathbb{C}$, then A always has an eigenvector and eigenvalue. Furthermore, if $\{\lambda_1, \cdots, \lambda_m\}$ are the zeros of $p(\lambda)$ in $\mathbb{F}$, these are exactly the eigenvalues of A for which there exists an eigenvector in V.

Proof: Suppose first μ is a zero of $p(\lambda)$. Since $p(\mu) = 0$, it follows

$$p(\lambda) = (\lambda - \mu) k(\lambda)$$

where $k(\lambda)$ is a polynomial having coefficients in $\mathbb{F}$. Since p has minimal degree, $k(A) \neq 0$ and so there exists a vector, $\mathbf{u} \neq \mathbf{0}$ such that $k(A)\mathbf{u} \equiv \mathbf{v} \neq \mathbf{0}$. But then

$$(A - \mu I)\mathbf{v} = (A - \mu I) k(A)(\mathbf{u}) = \mathbf{0}.$$

The next claim about the existence of an eigenvalue follows from the fundamental theorem of algebra and what was just shown.

It has been shown that every zero of $p(\lambda)$ is an eigenvalue which has an eigenvector in V. Now suppose μ is an eigenvalue which has an eigenvector in V so that $A\mathbf{v} = \mu\mathbf{v}$ for some $\mathbf{v} \in V, \mathbf{v} \neq \mathbf{0}$. Does it follow μ is a zero of $p(\lambda)$?

$$\mathbf{0} = p(A)\,\mathbf{v} = p(\mu)\,\mathbf{v}$$

and so μ is indeed a zero of $p(\lambda)$. This proves the theorem.

In summary, the theorem says the eigenvalues which have eigenvectors in V are exactly the zeros of the minimal polynomial which are in the field of scalars, $\mathbb{F}$.

The idea of block multiplication turns out to be very useful later. For now here is an interesting and significant application which has to do with characteristic polynomials. In this theorem, $p_M(t)$ denotes the polynomial, $\det(tI - M)$. Thus the zeros of this polynomial are the eigenvalues of the matrix, M.

Theorem 3.6.5 *Let A be an $m \times n$ matrix and let B be an $n \times m$ matrix for $m \leq n$. Then*

$$p_{BA}(t) = t^{n-m} p_{AB}(t),$$

so the eigenvalues of BA and AB are the same including multiplicities except that BA has $n - m$ extra zero eigenvalues.

Proof: Use block multiplication to write

$$\begin{pmatrix} AB & 0 \\ B & 0 \end{pmatrix} \begin{pmatrix} I & A \\ 0 & I \end{pmatrix} = \begin{pmatrix} AB & ABA \\ B & BA \end{pmatrix}$$

$$\begin{pmatrix} I & A \\ 0 & I \end{pmatrix} \begin{pmatrix} 0 & 0 \\ B & BA \end{pmatrix} = \begin{pmatrix} AB & ABA \\ B & BA \end{pmatrix}.$$

Therefore,

$$\begin{pmatrix} I & A \\ 0 & I \end{pmatrix}^{-1} \begin{pmatrix} AB & 0 \\ B & 0 \end{pmatrix} \begin{pmatrix} I & A \\ 0 & I \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ B & BA \end{pmatrix}$$

Since the two matrices above are similar it follows that $\begin{pmatrix} 0 & 0 \\ B & BA \end{pmatrix}$ and $\begin{pmatrix} AB & 0 \\ B & 0 \end{pmatrix}$ have the same characteristic polynomials. Therefore, noting that BA is an $n \times n$ matrix and AB is an $m \times m$ matrix,

$$t^m \det(tI - BA) = t^n \det(tI - AB)$$

and so $\det(tI - BA) = p_{BA}(t) = t^{n-m} \det(tI - AB) = t^{n-m} p_{AB}(t)$. This proves the theorem.

3.7 Exercises

1. Let M be an $n \times n$ matrix. Thus letting $M\mathbf{x}$ be defined by ordinary matrix multiplication, it follows $M \in \mathcal{L}(\mathbb{C}^n, \mathbb{C}^n)$. Show that all the zeros of the minimal polynomial are also zeros of the characteristic polynomial. Explain why this requires the minimal polynomial to divide the characteristic polynomial. Thus $q(\lambda) = p(\lambda)\,k(\lambda)$ for some polynomial $k(\lambda)$ where $q(\lambda)$ is the characteristic polynomial. Now explain why $q(M) = 0$. That every $n \times n$ matrix satisfies its characteristic polynomial is the Cayley Hamilton theorem. Can you extend this to a result about $L \in \mathcal{L}(V, V)$ for V an n dimensional real or complex vector space?

2. Give examples of subspaces of $\mathbb{R}^n$ and examples of subsets of $\mathbb{R}^n$ which are not subspaces.

3. Let $L \in \mathcal{L}(V,W)$. Define $\ker L \equiv \{\mathbf{v} \in V : L\mathbf{v} = \mathbf{0}\}$. Determine whether $\ker L$ is a subspace.

4. Let $L \in \mathcal{L}(V,W)$. Then $L(V)$ denotes those vectors in W such that for some $\mathbf{v}$,$L\mathbf{v} = \mathbf{w}$. Show $L(V)$ is a subspace.

5. Let $L \in \mathcal{L}(V,W)$ and suppose $\{\mathbf{w}_1, \cdots, \mathbf{w}_k\}$ are linearly independent and that $L\mathbf{z}_i = \mathbf{w}_i$. Show $\{\mathbf{z}_1, \cdots, \mathbf{z}_k\}$ is also linearly independent.

6. If $L \in \mathcal{L}(V,W)$ and $\{\mathbf{z}_1, \cdots, \mathbf{z}_k\}$ is linearly independent, what is needed in order that $\{L\mathbf{z}_1, \cdots, L\mathbf{z}_k\}$ be linearly independent? Explain your answer.

7. Let $L \in \mathcal{L}(V,W)$. The rank of L is defined as the dimension of $L(V)$. The nullity of L is the dimension of $\ker(L)$. Show

$$\dim(V) = \text{rank} + \text{nullity}.$$

8. Let $L \in \mathcal{L}(V,W)$ and let $M \in \mathcal{L}(W,Y)$. Show $\text{rank}(ML) \leq \min(\text{rank}(L), \text{rank}(M))$.

9. Let $M(t) = (\mathbf{b}_1(t), \cdots, \mathbf{b}_n(t))$ where each $\mathbf{b}_k(t)$ is a column vector whose component functions are differentiable functions. For such a column vector,

$$\mathbf{b}(t) = (b_1(t), \cdots, b_n(t))^T,$$

define

$$\mathbf{b}'(t) \equiv (b_1'(t), \cdots, b_n'(t))^T$$

Show

$$\det(M(t))' = \sum_{i=1}^{n} \det M_i(t)$$

where $M_i(t)$ has all the same columns as $M(t)$ except the i^{th} column is replaced with $\mathbf{b}_i'(t)$.

10. Let $A = (a_{ij})$ be an $n \times n$ matrix. Consider this as a linear transformation using ordinary matrix multiplication. Show

$$A = \sum_{ij} a_{ij}\mathbf{e}_i\mathbf{e}_j$$

where $\mathbf{e}_i$ is the vector which has a 1 in the i^{th} place and zeros elsewhere.

11. Let $\{\mathbf{w}_1, \cdots, \mathbf{w}_n\}$ be a basis for the vector space, V. Show id, the identity map is given by

$$\text{id} = \sum_{ij} \delta_{ij}\mathbf{w}_i\mathbf{w}_j$$

3.8 Inner Product And Normed Linear Spaces

3.8.1 The Inner Product In $\mathbb{F}^n$

To do calculus, you must understand what you mean by distance. For functions of one variable, the distance was provided by the absolute value of the difference of two numbers. This must be generalized to $\mathbb{F}^n$ and to more general situations.

Definition 3.8.1 *Let $\mathbf{x}, \mathbf{y} \in \mathbb{F}^n$. Thus $\mathbf{x} = (x_1, \cdots, x_n)$ where each $x_k \in \mathbb{F}$ and a similar formula holding for $\mathbf{y}$. Then the dot product of these two vectors is defined to be*

$$\mathbf{x} \cdot \mathbf{y} \equiv \sum_j x_j \overline{y_j} \equiv x_1 \overline{y_1} + \cdots + x_n \overline{y_n}.$$

This is also often denoted by $(\mathbf{x}, \mathbf{y})$ and is called an inner product. I will use either notation.

Notice how you put the conjugate on the entries of the vector, $\mathbf{y}$. It makes no difference if the vectors happen to be real vectors but with complex vectors you must do it this way. The reason for this is that when you take the dot product of a vector with itself, you want to get the square of the length of the vector, a positive number. Placing the conjugate on the components of $\mathbf{y}$ in the above definition assures this will take place. Thus

$$\mathbf{x} \cdot \mathbf{x} = \sum_j x_j \overline{x_j} = \sum_j |x_j|^2 \geq 0.$$

If you didn't place a conjugate as in the above definition, things wouldn't work out correctly. For example,

$$(1+i)^2 + 2^2 = 4 + 2i$$

and this is not a positive number.

The following properties of the dot product follow immediately from the definition and you should verify each of them.

Properties of the dot product:

1. $\mathbf{u} \cdot \mathbf{v} = \overline{\mathbf{v} \cdot \mathbf{u}}$.
2. If a, b are numbers and $\mathbf{u}, \mathbf{v}, \mathbf{z}$ are vectors then $(a\mathbf{u} + b\mathbf{v}) \cdot \mathbf{z} = a(\mathbf{u} \cdot \mathbf{z}) + b(\mathbf{v} \cdot \mathbf{z})$.
3. $\mathbf{u} \cdot \mathbf{u} \geq 0$ and it equals 0 if and only if $\mathbf{u} = \mathbf{0}$.

Note this implies $(\mathbf{x} \cdot \alpha\mathbf{y}) = \overline{\alpha}(\mathbf{x} \cdot \mathbf{y})$ because

$$(\mathbf{x} \cdot \alpha\mathbf{y}) = \overline{(\alpha\mathbf{y} \cdot \mathbf{x})} = \overline{\alpha(\mathbf{y} \cdot \mathbf{x})} = \overline{\alpha}(\mathbf{x} \cdot \mathbf{y})$$

The norm is defined as follows.

Definition 3.8.2 *For $\mathbf{x} \in \mathbb{F}^n$,*

$$|\mathbf{x}| \equiv \left(\sum_{k=1}^n |x_k|^2 \right)^{1/2} = (\mathbf{x} \cdot \mathbf{x})^{1/2}$$

3.8.2 General Inner Product Spaces

Any time you have a vector space which possesses an inner product, something satisfying the properties 1 - 3 above, it is called an inner product space.

Here is a fundamental inequality called the **Cauchy Schwarz inequality** which holds in any inner product space. First here is a simple lemma.

Lemma 3.8.3 *If $z \in \mathbb{F}$ there exists $\theta \in \mathbb{F}$ such that $\theta z = |z|$ and $|\theta| = 1$.*

Proof: Let $\theta = 1$ if $z = 0$ and otherwise, let $\theta = \dfrac{\overline{z}}{|z|}$. Recall that for $z = x+iy, \overline{z} = x-iy$ and $\overline{z}z = |z|^2$. In case z is real, there is no change in the above.

Theorem 3.8.4 *(Cauchy Schwarz)Let H be an inner product space. The following inequality holds for $\mathbf{x}$ and $\mathbf{y} \in H$.*

$$|(\mathbf{x} \cdot \mathbf{y})| \leq (\mathbf{x} \cdot \mathbf{x})^{1/2} (\mathbf{y} \cdot \mathbf{y})^{1/2} \tag{3.33}$$

Equality holds in this inequality if and only if one vector is a multiple of the other.

Proof: Let $\theta \in \mathbb{F}$ such that $|\theta| = 1$ and

$$\theta (\mathbf{x} \cdot \mathbf{y}) = |(\mathbf{x} \cdot \mathbf{y})|$$

Consider $p(t) \equiv \left(\mathbf{x} + \overline{\theta} t\mathbf{y} \cdot \mathbf{x} + t\overline{\theta}\mathbf{y}\right)$ where $t \in \mathbb{R}$. Then from the above list of properties of the dot product,

$$\begin{aligned} 0 &\leq p(t) = (\mathbf{x} \cdot \mathbf{x}) + t\theta (\mathbf{x} \cdot \mathbf{y}) + t\overline{\theta} (\mathbf{y} \cdot \mathbf{x}) + t^2 (\mathbf{y} \cdot \mathbf{y}) \\ &= (\mathbf{x} \cdot \mathbf{x}) + t\theta (\mathbf{x} \cdot \mathbf{y}) + t\overline{\theta(\mathbf{x} \cdot \mathbf{y})} + t^2 (\mathbf{y} \cdot \mathbf{y}) \\ &= (\mathbf{x} \cdot \mathbf{x}) + 2t \operatorname{Re}(\theta (\mathbf{x} \cdot \mathbf{y})) + t^2 (\mathbf{y} \cdot \mathbf{y}) \\ &= (\mathbf{x} \cdot \mathbf{x}) + 2t |(\mathbf{x} \cdot \mathbf{y})| + t^2 (\mathbf{y} \cdot \mathbf{y}) \end{aligned} \tag{3.34}$$

and this must hold for all $t \in \mathbb{R}$. Therefore, if $(\mathbf{y} \cdot \mathbf{y}) = 0$ it must be the case that $|(\mathbf{x} \cdot \mathbf{y})| = 0$ also since otherwise the above inequality would be violated. Therefore, in this case,

$$|(\mathbf{x} \cdot \mathbf{y})| \leq (\mathbf{x} \cdot \mathbf{x})^{1/2} (\mathbf{y} \cdot \mathbf{y})^{1/2}.$$

On the other hand, if $(\mathbf{y} \cdot \mathbf{y}) \neq 0$, then $p(t) \geq 0$ for all t means the graph of $y = p(t)$ is a parabola which opens up and it either has exactly one real zero in the case its vertex touches the t axis or it has no real zeros. From the quadratic formula this happens exactly when

$$4 |(\mathbf{x} \cdot \mathbf{y})|^2 - 4 (\mathbf{x} \cdot \mathbf{x}) (\mathbf{y} \cdot \mathbf{y}) \leq 0$$

which is equivalent to 3.33.

It is clear from a computation that if one vector is a scalar multiple of the other that equality holds in 3.33. Conversely, suppose equality does hold. Then this is equivalent to saying $4 |(\mathbf{x} \cdot \mathbf{y})|^2 - 4 (\mathbf{x} \cdot \mathbf{x}) (\mathbf{y} \cdot \mathbf{y}) = 0$ and so from the quadratic formula, there exists one real zero to $p(t) = 0$. Call it t_0. Then

$$p(t_0) \equiv \left(\mathbf{x} + \overline{\theta} t_0 \mathbf{y}, \mathbf{x} + t_0 \overline{\theta}\mathbf{y}\right) = \left|\mathbf{x} + \overline{\theta} t\mathbf{y}\right|^2 = 0$$

and so $\mathbf{x} = -\overline{\theta} t_0 \mathbf{y}$. This proves the theorem.

Note that in establishing the inequality, I only used part of the above properties of the dot product. It was not necessary to use the one which says that if $(\mathbf{x} \cdot \mathbf{x}) = 0$ then $\mathbf{x} = \mathbf{0}$.

Now the length of a vector can be defined.

Definition 3.8.5 *Let* $\mathbf{z} \in H$. *Then* $|\mathbf{z}| \equiv (\mathbf{z} \cdot \mathbf{z})^{1/2}$.

Theorem 3.8.6 *For length defined in Definition 3.8.5, the following hold.*

$$|\mathbf{z}| \geq 0 \text{ and } |\mathbf{z}| = 0 \text{ if and only if } \mathbf{z} = \mathbf{0} \tag{3.35}$$

$$\text{If } \alpha \text{ is a scalar, } |\alpha\mathbf{z}| = |\alpha|\,|\mathbf{z}| \tag{3.36}$$

$$|\mathbf{z} + \mathbf{w}| \leq |\mathbf{z}| + |\mathbf{w}|\,. \tag{3.37}$$

Proof: The first two claims are left as exercises. To establish the third,

$$\begin{aligned} |\mathbf{z} + \mathbf{w}|^2 &\equiv (\mathbf{z} + \mathbf{w}, \mathbf{z} + \mathbf{w}) \\ &= \mathbf{z} \cdot \mathbf{z} + \mathbf{w} \cdot \mathbf{w} + \mathbf{w} \cdot \mathbf{z} + \mathbf{z} \cdot \mathbf{w} \\ &= |\mathbf{z}|^2 + |\mathbf{w}|^2 + 2\,\mathrm{Re}\,\mathbf{w} \cdot \mathbf{z} \\ &\leq |\mathbf{z}|^2 + |\mathbf{w}|^2 + 2\,|\mathbf{w} \cdot \mathbf{z}| \\ &\leq |\mathbf{z}|^2 + |\mathbf{w}|^2 + 2\,|\mathbf{w}|\,|\mathbf{z}| = (|\mathbf{z}| + |\mathbf{w}|)^2\,. \end{aligned}$$

3.8.3 Normed Vector Spaces

The best sort of a norm is one which comes from an inner product. However, any vector space, V which has a function, $||\cdot||$ which maps V to $[0, \infty)$ is called a normed vector space if $||\cdot||$ satisfies 3.35 - 3.37. That is

$$||\mathbf{z}|| \geq 0 \text{ and } ||\mathbf{z}|| = 0 \text{ if and only if } \mathbf{z} = \mathbf{0} \tag{3.38}$$

$$\text{If } \alpha \text{ is a scalar, } ||\alpha\mathbf{z}|| = |\alpha|\,||\mathbf{z}|| \tag{3.39}$$

$$||\mathbf{z} + \mathbf{w}|| \leq ||\mathbf{z}|| + ||\mathbf{w}||\,. \tag{3.40}$$

The last inequality above is called the triangle inequality. Another version of this is

$$|\,||\mathbf{z}|| - ||\mathbf{w}||\,| \leq ||\mathbf{z} - \mathbf{w}|| \tag{3.41}$$

To see that 3.41 holds, note

$$||\mathbf{z}|| = ||\mathbf{z} - \mathbf{w} + \mathbf{w}|| \leq ||\mathbf{z} - \mathbf{w}|| + ||\mathbf{w}||$$

which implies

$$||\mathbf{z}|| - ||\mathbf{w}|| \leq ||\mathbf{z} - \mathbf{w}||$$

and now switching $\mathbf{z}$ and $\mathbf{w}$, yields

$$||\mathbf{w}|| - ||\mathbf{z}|| \leq ||\mathbf{z} - \mathbf{w}||$$

which implies 3.41.

3.8.4 Orthonormal Bases

Not all bases for an inner product space H are created equal. The best bases are orthonormal.

Definition 3.8.7 *Suppose* $\{\mathbf{v}_1, \cdots, \mathbf{v}_k\}$ *is a set of vectors in an inner product space* H. *It is an orthonormal set if*

$$\mathbf{v}_i \cdot \mathbf{v}_j = \delta_{ij} = \begin{cases} 1 \text{ if } i = j \\ 0 \text{ if } i \neq j \end{cases}$$

Every orthonormal set of vectors is automatically linearly independent.

Proposition 3.8.8 *Suppose* $\{\mathbf{v}_1, \cdots, \mathbf{v}_k\}$ *is an orthonormal set of vectors. Then it is linearly independent.*

Proof: Suppose $\sum_{i=1}^{k} c_i \mathbf{v}_i = \mathbf{0}$. Then taking dot products with $\mathbf{v}_j$,

$$0 = \mathbf{0} \cdot \mathbf{v}_j = \sum_i c_i \mathbf{v}_i \cdot \mathbf{v}_j = \sum_i c_i \delta_{ij} = c_j.$$

Since j is arbitrary, this shows the set is linearly independent as claimed.

It turns out that if X is any subspace of H, then there exists an orthonormal basis for X.

Lemma 3.8.9 *Let* X *be a subspace of dimension* n *whose basis is* $\{\mathbf{x}_1, \cdots, \mathbf{x}_n\}$. *Then there exists an orthonormal basis for* X, $\{\mathbf{u}_1, \cdots, \mathbf{u}_n\}$ *which has the property that for each* $k \leq n$, *span*$(\mathbf{x}_1, \cdots, \mathbf{x}_k) = \text{span}(\mathbf{u}_1, \cdots, \mathbf{u}_k)$.

Proof: Let $\{\mathbf{x}_1, \cdots, \mathbf{x}_n\}$ be a basis for X. Let $\mathbf{u}_1 \equiv \mathbf{x}_1 / |\mathbf{x}_1|$. Thus for $k = 1$, $\text{span}(\mathbf{u}_1) = \text{span}(\mathbf{x}_1)$ and $\{\mathbf{u}_1\}$ is an orthonormal set. Now suppose for some $k < n$, $\mathbf{u}_1, \cdots, \mathbf{u}_k$ have been chosen such that $(\mathbf{u}_j, \mathbf{u}_l) = \delta_{jl}$ and $\text{span}(\mathbf{x}_1, \cdots, \mathbf{x}_k) = \text{span}(\mathbf{u}_1, \cdots, \mathbf{u}_k)$. Then define

$$\mathbf{u}_{k+1} \equiv \frac{\mathbf{x}_{k+1} - \sum_{j=1}^{k} (\mathbf{x}_{k+1} \cdot \mathbf{u}_j)\, \mathbf{u}_j}{\left| \mathbf{x}_{k+1} - \sum_{j=1}^{k} (\mathbf{x}_{k+1} \cdot \mathbf{u}_j)\, \mathbf{u}_j \right|}, \tag{3.42}$$

where the denominator is not equal to zero because the $\mathbf{x}_j$ form a basis and so

$$\mathbf{x}_{k+1} \notin \text{span}(\mathbf{x}_1, \cdots, \mathbf{x}_k) = \text{span}(\mathbf{u}_1, \cdots, \mathbf{u}_k)$$

Thus by induction,

$$\mathbf{u}_{k+1} \in \text{span}(\mathbf{u}_1, \cdots, \mathbf{u}_k, \mathbf{x}_{k+1}) = \text{span}(\mathbf{x}_1, \cdots, \mathbf{x}_k, \mathbf{x}_{k+1}).$$

Also, $\mathbf{x}_{k+1} \in \text{span}(\mathbf{u}_1, \cdots, \mathbf{u}_k, \mathbf{u}_{k+1})$ which is seen easily by solving 3.42 for $\mathbf{x}_{k+1}$ and it follows

$$\text{span}(\mathbf{x}_1, \cdots, \mathbf{x}_k, \mathbf{x}_{k+1}) = \text{span}(\mathbf{u}_1, \cdots, \mathbf{u}_k, \mathbf{u}_{k+1}).$$

If $l \leq k$, then denoting by C the scalar $\left| \mathbf{x}_{k+1} - \sum_{j=1}^{k} (\mathbf{x}_{k+1} \cdot \mathbf{u}_j)\, \mathbf{u}_j \right|^{-1}$,

$$\begin{aligned}
(\mathbf{u}_{k+1} \cdot \mathbf{u}_l) &= C \left((\mathbf{x}_{k+1} \cdot \mathbf{u}_l) - \sum_{j=1}^{k} (\mathbf{x}_{k+1} \cdot \mathbf{u}_j)\, (\mathbf{u}_j \cdot \mathbf{u}_l) \right) \\
&= C \left((\mathbf{x}_{k+1} \cdot \mathbf{u}_l) - \sum_{j=1}^{k} (\mathbf{x}_{k+1} \cdot \mathbf{u}_j)\, \delta_{lj} \right) \\
&= C \left((\mathbf{x}_{k+1} \cdot \mathbf{u}_l) - (\mathbf{x}_{k+1} \cdot \mathbf{u}_l) \right) = 0.
\end{aligned}$$

The vectors, $\{\mathbf{u}_j\}_{j=1}^{n}$, generated in this way are therefore an orthonormal basis because each vector has unit length.

The process by which these vectors were generated is called the Gram Schmidt process.

3.8.5 The Adjoint Of A Linear Transformation

There is a very important collection of ideas which relates a linear transformation to the inner product in an inner product space. In order to discuss these ideas, it is necessary to prove a simple and very interesting lemma about linear transformations which map an inner product space H to the field of scalars, $\mathbb{F}$. This is sometimes called the Riesz representation theorem.

Theorem 3.8.10 *Let H be a finite dimensional inner product space and let $L \in \mathcal{L}(H, \mathbb{F})$. Then there exists a unique $\mathbf{z} \in H$ such that for all $\mathbf{x} \in H$,*

$$L\mathbf{x} = (\mathbf{x} \cdot \mathbf{z}).$$

Proof: By the Gram Schmidt process, there exists an orthonormal basis for H, $\{\mathbf{e}_1, \cdots, \mathbf{e}_n\}$. First note that if $\mathbf{x}$ is arbitrary, there exist unique scalars, x_i such that

$$\mathbf{x} = \sum_{i=1}^{n} x_i \mathbf{e}_i$$

Taking the dot product of both sides with $\mathbf{e}_k$ yields

$$(\mathbf{x} \cdot \mathbf{e}_k) = \left(\sum_{i=1}^{n} x_i \mathbf{e}_i \cdot \mathbf{e}_k \right) = \sum_{i=1}^{n} x_i (\mathbf{e}_i \cdot \mathbf{e}_k) = \sum_{i=1}^{n} x_i \delta_{ik} = x_k$$

which shows that

$$\mathbf{x} = \sum_{i=1}^{n} (\mathbf{x} \cdot \mathbf{e}_i) \mathbf{e}_i$$

and so by the properties of the dot product,

$$L\mathbf{x} = \sum_{i=1}^{n} (\mathbf{x} \cdot \mathbf{e}_i) L\mathbf{e}_i = \left(\mathbf{x} \cdot \sum_{i=1}^{n} \mathbf{e}_i \overline{L\mathbf{e}_i} \right)$$

so let $\mathbf{z} = \sum_{i=1}^{n} \mathbf{e}_i \overline{L\mathbf{e}_i}$. It only remains to verify $\mathbf{z}$ is unique. However, this is obvious because if $(\mathbf{x} \cdot \mathbf{z}_1) = (\mathbf{x} \cdot \mathbf{z}_2) = L\mathbf{x}$ for all $\mathbf{x}$, then

$$(\mathbf{x} \cdot \mathbf{z}_1 - \mathbf{z}_2) = 0$$

for all $\mathbf{x}$ and in particular for $\mathbf{x} = \mathbf{z}_1 - \mathbf{z}_2$ which requires $\mathbf{z}_1 = \mathbf{z}_2$. This proves the theorem.

Now with this theorem, it becomes easy to define something called the adjoint of a linear operator. Let $L \in \mathcal{L}(H_1, H_2)$ where H_1 and H_2 are finite dimensional inner product spaces. Then letting $(\cdot)_i$ denote the inner product in H_i,

$$\mathbf{x} \to (L\mathbf{x} \cdot \mathbf{y})_2$$

is in $\mathcal{L}(H_1, \mathbb{F})$ and so from Theorem 3.8.10 there exists a unique element of H_1, denoted by $L^*\mathbf{y}$ such that for all $\mathbf{x} \in H_1$,

$$(L\mathbf{x} \cdot \mathbf{y})_2 = (\mathbf{x} \cdot L^*\mathbf{y})_1$$

Thus $L^*\mathbf{y} \in H_1$ when $\mathbf{y} \in H_2$. Also L^* is linear. This is because by the properties of the dot product,

$$\begin{aligned} (\mathbf{x} \cdot L^*(\alpha\mathbf{y} + \beta\mathbf{z}))_1 &\equiv (L\mathbf{x} \cdot \alpha\mathbf{y} + \beta\mathbf{z})_2 \\ &= \overline{\alpha}(L\mathbf{x} \cdot \mathbf{y})_2 + \overline{\beta}(L\mathbf{x} \cdot \mathbf{z})_2 \\ &= \overline{\alpha}(\mathbf{x} \cdot L^*\mathbf{y})_1 + \overline{\beta}(\mathbf{x} \cdot L^*\mathbf{z})_1 \\ &= \overline{\alpha}(\mathbf{x} \cdot L^*\mathbf{y})_1 + \overline{\beta}(\mathbf{x} \cdot L^*\mathbf{z})_1 \end{aligned}$$

and

$$(\mathbf{x}\cdot\alpha L^*\mathbf{y}+\beta L^*\mathbf{z})_1=\overline{\alpha}\,(\mathbf{x}\cdot L^*\mathbf{y})_1+\overline{\beta}\,(\mathbf{x}\cdot L^*\mathbf{z})_1$$

Since

$$(\mathbf{x}\cdot L^*(\alpha\mathbf{y}+\beta\mathbf{z}))_1=(\mathbf{x}\cdot\alpha L^*\mathbf{y}+\beta L^*\mathbf{z})_1$$

for all $\mathbf{x}$, this requires

$$L^*(\alpha\mathbf{y}+\beta\mathbf{z})=\alpha L^*\mathbf{y}+\beta L^*\mathbf{z}.$$

In simple words, when you take it across the dot, you put a star on it. More precisely, here is the definition.

Definition 3.8.11 *Let H_1 and H_2 be finite dimensional inner product spaces and let $L\in\mathcal{L}(H_1,H_2)$. Then $L^*\in\mathcal{L}(H_2,H_1)$ is defined by the formula*

$$(L\mathbf{x}\cdot\mathbf{y})_2=(\mathbf{x}\cdot L^*\mathbf{y})_1.$$

In the case where $H_1=H_2=H$, an operator $L\in\mathcal{L}(H,H)$ is said to be self adjoint if $L=L^*$. This is also called Hermitian.

The following diagram might help.

$$\begin{array}{ccc} H_1 & \overset{L^*}{\leftarrow} & H_2 \\ H_1 & \overset{L}{\rightarrow} & H_2 \end{array}$$

I will not bother to place subscripts on the symbol for the dot product in the future. I will be clear from context which inner product is meant.

Proposition 3.8.12 *The adjoint has the following properties.*

1. $(\mathbf{x}\cdot L\mathbf{y})=(L^*\mathbf{x}\cdot\mathbf{y}),\ (L\mathbf{x}\cdot\mathbf{y})=(\mathbf{x}\cdot L^*\mathbf{y})$
2. $(L^*)^*=L$
3. $(aL+bM)^*=\overline{a}L^*+\overline{b}M^*$
4. $(ML)^*=L^*M^*$

Proof: Consider the first claim.

$$(\mathbf{x}\cdot L\mathbf{y})=\overline{(L\mathbf{y}\cdot\mathbf{x})}=\overline{(\mathbf{y}\cdot L^*\mathbf{x})}=(L^*\mathbf{x}\cdot\mathbf{y})$$

This does the first claim. The second part was discussed earlier when the adjoint was defined.

Consider the second claim. From the first claim,

$$(L\mathbf{x}\cdot\mathbf{y})=(\mathbf{x}\cdot L^*\mathbf{y})=\left((L^*)^*\mathbf{x}\cdot\mathbf{y}\right)$$

and since this holds for all $\mathbf{y}$, it follows $L\mathbf{x}=(L^*)^*\mathbf{x}$.

Consider the third claim.

$$\left(\mathbf{x}\cdot(aL+bM)^*\mathbf{y}\right)=((aL+bM)\mathbf{x}\cdot\mathbf{y})=a(L\mathbf{x}\cdot\mathbf{y})+b(M\mathbf{x}\cdot\mathbf{y})$$

and

$$\left(\mathbf{x}\cdot\left(\overline{a}L^*+\overline{b}M^*\right)\mathbf{y}\right)=a(\mathbf{x}\cdot L^*\mathbf{y})+b(\mathbf{x}\cdot M^*\mathbf{y})=a(L\mathbf{x}\cdot\mathbf{y})+b(M\mathbf{x}\cdot\mathbf{y})$$

and since $\left(\mathbf{x}\cdot\left(aL+bM\right)^{*}\mathbf{y}\right)=\left(\mathbf{x}\cdot\left(\overline{a}L^{*}+\overline{b}M^{*}\right)\mathbf{y}\right)$ for all $\mathbf{x}$, it must be that

$$\left(aL+bM\right)^{*}\mathbf{y}=\left(\overline{a}L^{*}+\overline{b}M^{*}\right)\mathbf{y}$$

for all $\mathbf{y}$ which yields the third claim.

Consider the fourth.

$$\left(\mathbf{x}\cdot\left(ML\right)^{*}\mathbf{y}\right)=\left(\left(ML\right)\mathbf{x}\cdot\mathbf{y}\right)=\left(L\mathbf{x}\cdot M^{*}\mathbf{y}\right)=\left(\mathbf{x}\cdot L^{*}M^{*}\mathbf{y}\right)$$

Since this holds for all $\mathbf{x},\mathbf{y}$ the conclusion follows as above. This proves the theorem.

Here is a very important example.

Example 3.8.13 *Suppose $F\in\mathcal{L}\left(H_1,H_2\right)$. Then $FF^{*}\in\mathcal{L}\left(H_2,H_2\right)$ and is self adjoint.*

To see this is so, note it is the composition of linear transformations and is therefore linear as stated. To see it is self adjoint, Proposition 3.8.12 implies

$$\left(FF^{*}\right)^{*}=\left(F^{*}\right)^{*}F^{*}=FF^{*}$$

In the case where $A\in\mathcal{L}\left(\mathbb{F}^{n},\mathbb{F}^{m}\right)$, considering the matrix of A with respect to the usual bases, there is no loss of generality in considering A to be an $m\times n$ matrix,

$$\left(A\mathbf{x}\right)_{i}=\sum_{j}A_{ij}x_{j}.$$

Then in terms of components of the matrix, A,

$$\left(A^{*}\right)_{ij}=\overline{A_{ji}}.$$

You should verify this is so from the definition of the usual inner product on $\mathbb{F}^{k}$. The following little proposition is useful.

Proposition 3.8.14 *Suppose A is an $m\times n$ matrix where $m\leq n$. Also suppose*

$$\det\left(AA^{*}\right)\neq 0.$$

Then A has m linearly independent rows and m independent columns.

Proof: Since $\det\left(AA^{*}\right)\neq 0$, it follows the $m\times m$ matrix AA^{*} has m independent rows. If this is not true of A, then there exists $\mathbf{x}$ a $1\times m$ matrix such that

$$\mathbf{x}A=\mathbf{0}.$$

Hence

$$\mathbf{x}AA^{*}=\mathbf{0}$$

and this contradicts the independence of the rows of AA^{*}. Thus the row rank of A equals m and by Corollary 3.5.20 this implies the column rank of A also equals m. This proves the proposition.

3.8.6 Schur's Theorem

Recall that for a linear transformation, $L \in \mathcal{L}(V,V)$, it could be represented in the form

$$L = \sum_{ij} l_{ij} \mathbf{v}_i \mathbf{v}_j$$

where $\{\mathbf{v}_1, \cdots, \mathbf{v}_2\}$ is a basis. Of course different bases will yield different matrices, (l_{ij}). Schur's theorem gives the existence of a basis in an inner product space such that (l_{ij}) is particularly simple.

Definition 3.8.15 *Let $L \in \mathcal{L}(V,V)$ where V is vector space. Then a subspace U of V is L invariant if $L(U) \subseteq U$.*

Theorem 3.8.16 *Let $L \in \mathcal{L}(H,H)$ for H a finite dimensional inner product space such that the restriction of L^* to every L invariant subspace has its eigenvalues in $\mathbb{F}$. Then there exist constants, c_{ij} for $i \leq j$ and an orthonormal basis, $\{\mathbf{w}_i\}_{i=1}^n$ such that*

$$L = \sum_{j=1}^{n} \sum_{i=1}^{j} c_{ij} \mathbf{w}_i \mathbf{w}_j$$

The constants, c_{ii} are the eigenvalues of L.

Proof: If $\dim(H) = 1$ let $H = \operatorname{span}(\mathbf{w})$ where $|\mathbf{w}| = 1$. Then $L\mathbf{w} = k\mathbf{w}$ for some k. Then

$$L = k\mathbf{w}\mathbf{w}$$

because by definition, $\mathbf{w}\mathbf{w}(\mathbf{w}) = \mathbf{w}$. Therefore, the theorem holds if H is 1 dimensional.

Now suppose the theorem holds for $n-1 = \dim(H)$. By Theorem 3.6.4 and the assumption, there exists $\mathbf{w}_n$, an eigenvector for L^*. Dividing by its length, it can be assumed $|\mathbf{w}_n| = 1$. Say $L^*\mathbf{w}_n = \mu\mathbf{w}_n$. Using the Gram Schmidt process, there exists an orthonormal basis for H of the form $\{\mathbf{v}_1, \cdots, \mathbf{v}_{n-1}, \mathbf{w}_n\}$. Then

$$(L\mathbf{v}_k \cdot \mathbf{w}_n) = (\mathbf{v}_k \cdot L^*\mathbf{w}_n) = (\mathbf{v}_k \cdot \mu\mathbf{w}_n) = 0,$$

which shows

$$L : H_1 \equiv \operatorname{span}(\mathbf{v}_1, \cdots, \mathbf{v}_{n-1}) \to \operatorname{span}(\mathbf{v}_1, \cdots, \mathbf{v}_{n-1}).$$

Denote by L_1 the restriction of L to H_1. Since H_1 has dimension $n-1$, the induction hypothesis yields an orthonormal basis, $\{\mathbf{w}_1, \cdots, \mathbf{w}_{n-1}\}$ for H_1 such that

$$L_1 = \sum_{j=1}^{n-1} \sum_{i=1}^{j} c_{ij} \mathbf{w}_i \mathbf{w}_j. \tag{3.43}$$

Then $\{\mathbf{w}_1, \cdots, \mathbf{w}_n\}$ is an orthonormal basis for H because every vector in $\operatorname{span}(\mathbf{v}_1, \cdots, \mathbf{v}_{n-1})$. has the property that its dot product with $\mathbf{w}_n$ is 0 so in particular, this is true for the vectors $\{\mathbf{w}_1, \cdots, \mathbf{w}_{n-1}\}$. Now define c_{in} to be the scalars satisfying

$$L\mathbf{w}_n \equiv \sum_{i=1}^{n} c_{in} \mathbf{w}_i \tag{3.44}$$

and let

$$B \equiv \sum_{j=1}^{n} \sum_{i=1}^{j} c_{ij} \mathbf{w}_i \mathbf{w}_j.$$

Then by 3.44,

$$B\mathbf{w}_n = \sum_{j=1}^{n}\sum_{i=1}^{j} c_{ij}\mathbf{w}_i\delta_{nj} = \sum_{j=1}^{n} c_{in}\mathbf{w}_i = L\mathbf{w}_n.$$

If $1 \leq k \leq n-1$,

$$B\mathbf{w}_k = \sum_{j=1}^{n}\sum_{i=1}^{j} c_{ij}\mathbf{w}_i\delta_{kj} = \sum_{i=1}^{k} c_{ik}\mathbf{w}_i$$

while from 3.43,

$$L\mathbf{w}_k = L_1\mathbf{w}_k = \sum_{j=1}^{n-1}\sum_{i=1}^{j} c_{ij}\mathbf{w}_i\delta_{jk} = \sum_{i=1}^{k} c_{ik}\mathbf{w}_i.$$

Since $L = B$ on the basis $\{\mathbf{w}_1, \cdots, \mathbf{w}_n\}$, it follows $L = B$.

It remains to verify the constants, c_{kk} are the eigenvalues of L, solutions of the equation, $\det(\lambda I - L) = 0$. However, the definition of $\det(\lambda I - L)$ is the same as

$$\det(\lambda I - C)$$

where C is the upper triangular matrix which has c_{ij} for $i \leq j$ and zeros elsewhere. This equals 0 if and only if λ is one of the diagonal entries, one of the c_{kk}. This proves the theorem.

There is a technical assumption in the above theorem about the eigenvalues of restrictions of L^* being in $\mathbb{F}$, the field of scalars. If $\mathbb{F} = \mathbb{C}$ this is no restriction. There is also another situation in which $\mathbb{F} = \mathbb{R}$ for which this will hold.

Lemma 3.8.17 *Suppose H is a finite dimensional inner product space and $\{\mathbf{w}_1, \cdots, \mathbf{w}_n\}$ is an orthonormal basis for H. Then*

$$(\mathbf{w}_i\mathbf{w}_j)^* = \mathbf{w}_j\mathbf{w}_i$$

Proof: It suffices to verify the two linear transformations are equal on $\{\mathbf{w}_1, \cdots, \mathbf{w}_n\}$. Then

$$\begin{aligned} \left(\mathbf{w}_p \cdot (\mathbf{w}_i\mathbf{w}_j)^* \mathbf{w}_k\right) &\equiv ((\mathbf{w}_i\mathbf{w}_j)\mathbf{w}_p \cdot \mathbf{w}_k) = (\mathbf{w}_i\delta_{jp} \cdot \mathbf{w}_k) = \delta_{jp}\delta_{ik} \\ (\mathbf{w}_p \cdot (\mathbf{w}_j\mathbf{w}_i)\mathbf{w}_k) &= (\mathbf{w}_p \cdot \mathbf{w}_j\delta_{ik}) = \delta_{ik}\delta_{jp} \end{aligned}$$

Since $\mathbf{w}_p$ is arbitrary, it follows from the properties of the inner product that

$$\left(\mathbf{x} \cdot (\mathbf{w}_i\mathbf{w}_j)^* \mathbf{w}_k\right) = (\mathbf{x} \cdot (\mathbf{w}_j\mathbf{w}_i)\mathbf{w}_k)$$

for all $\mathbf{x} \in H$ and hence $(\mathbf{w}_i\mathbf{w}_j)^* \mathbf{w}_k = (\mathbf{w}_j\mathbf{w}_i)\mathbf{w}_k$. Since $\mathbf{w}_k$ is arbitrary, this proves the lemma.

Lemma 3.8.18 *Let $L \in \mathcal{L}(H, H)$ for H an inner product space. Then if $L = L^*$ so L is self adjoint, it follows all the eigenvalues of L are real.*

Proof: Let $\{\mathbf{w}_1, \cdots, \mathbf{w}_n\}$ be an orthonormal basis for H and let (l_{ij}) be the matrix of L with respect to this orthonormal basis. Thus

$$L = \sum_{ij} l_{ij}\mathbf{w}_i\mathbf{w}_j, \ \mathrm{id} = \sum_{ij} \delta_{ij}\mathbf{w}_i\mathbf{w}_j$$

Denote by M_L the matrix whose ij^{th} entry is l_{ij}. Then by definition of what is meant by the determinant of a linear transformation,

$$\det(\lambda \,\mathrm{id} - L) = \det(\lambda I - M_L)$$

and so the eigenvalues of L are the same as the eigenvalues of M_L. However, $M_L \in \mathcal{L}(\mathbb{C}^n, \mathbb{C}^n)$ with $M_L\mathbf{x}$ determined by ordinary matrix multiplication. Therefore, by the fundamental theorem of algebra and Theorem 3.6.4, if λ is an eigenvalue of L it follows there exists a nonzero $\mathbf{x} \in \mathbb{C}^n$ such that $M_L\mathbf{x} = \lambda\mathbf{x}$. Since L is self adjoint, it follows from Lemma 3.8.17

$$L = \sum_{ij} l_{ij}\mathbf{w}_i\mathbf{w}_j = L^* = \sum_{ij} \overline{l_{ij}}\mathbf{w}_j\mathbf{w}_i = \sum_{ij} \overline{l_{ji}}\mathbf{w}_i\mathbf{w}_j$$

which shows $l_{ij} = \overline{l_{ji}}$.

Then

$$\begin{aligned} \lambda|\mathbf{x}|^2 &= \lambda(\mathbf{x}\cdot\mathbf{x}) = (\lambda\mathbf{x}\cdot\mathbf{x}) = (M_L\mathbf{x}\cdot\mathbf{x}) = \sum_{ij} l_{ij}x_j\overline{x_i} \\ &= \overline{\sum_{ij} \overline{l_{ij}}\overline{x_j}x_i} = \overline{\sum_{ij} l_{ji}\overline{x_j}x_i} = \overline{(M_L\mathbf{x}\cdot\mathbf{x})} = (\mathbf{x}\cdot M_L\mathbf{x}) = \overline{\lambda}|\mathbf{x}|^2 \end{aligned}$$

showing $\lambda = \overline{\lambda}$. This proves the lemma.

If L is a self adjoint operator on H, either a real or complex inner product space, it follows the condition about the eigenvalues of the restrictions of L^* to L invariant subspaces of H must hold because these restrictions are self adjoint. Here is why. Let $\mathbf{x}, \mathbf{y}$ be in one of those invariant subspaces. Then since $L^* = L$,

$$(L^*\mathbf{x}\cdot\mathbf{y}) = (\mathbf{x}\cdot L\mathbf{y}) = (\mathbf{x}\cdot L^*\mathbf{y})$$

so by the above lemma, the eigenvalues are real and are therefore, in the field of scalars.

Now with this lemma, the following theorem is obtained. This is another major theorem. It is equivalent to the theorem in matrix theory which states every self adjoint matrix can be diagonalized.

Theorem 3.8.19 *Let H be a finite dimensional inner product space, real or complex, and let $L \in \mathcal{L}(H, H)$ be self adjoint. Then there exists an orthonormal basis $\{\mathbf{w}_1, \cdots, \mathbf{w}_n\}$ and real scalars, λ_k such that*

$$L = \sum_{k=1}^{n} \lambda_k \mathbf{w}_k\mathbf{w}_k.$$

The scalars are the eigenvalues and $\mathbf{w}_k$ is an eigenvector for λ_k for each k.

Proof: By Theorem 3.8.16, there exists an orthonormal basis, $\{\mathbf{w}_1, \cdots, \mathbf{w}_n\}$ such that

$$L = \sum_{j=1}^{n}\sum_{i=1}^{n} c_{ij}\mathbf{w}_i\mathbf{w}_j$$

where $c_{ij} = 0$ if $i > j$. Now using Lemma 3.8.17 and Proposition 3.8.12 along with the assumption that L is self adjoint,

$$L = \sum_{j=1}^{n}\sum_{i=1}^{n} c_{ij}\mathbf{w}_i\mathbf{w}_j = L^* = \sum_{j=1}^{n}\sum_{i=1}^{n} \overline{c_{ij}}\mathbf{w}_j\mathbf{w}_i = \sum_{i=1}^{n}\sum_{j=1}^{n} \overline{c_{ji}}\mathbf{w}_i\mathbf{w}_j$$

If $i < j$, then this shows $c_{ij} = \overline{c_{ji}}$ and the second number equals zero because $j > i$. Thus $c_{ij} = 0$ if $i < j$ and it is already known that $c_{ij} = 0$ if $i > j$. Therefore, let $\lambda_k = c_{kk}$ and the above reduces to

$$L = \sum_{j=1}^{n} \lambda_j \mathbf{w}_j \mathbf{w}_j = \sum_{j=1}^{n} \overline{\lambda_j} \mathbf{w}_j \mathbf{w}_j$$

showing that $\lambda_j = \overline{\lambda_j}$ so the eigenvalues are all real. Now

$$L\mathbf{w}_k = \sum_{j=1}^{n} \lambda_j \mathbf{w}_j \mathbf{w}_j (\mathbf{w}_k) = \sum_{j=1}^{n} \lambda_j \mathbf{w}_j \delta_{jk} = \lambda_k \mathbf{w}_k$$

which shows all the $\mathbf{w}_k$ are eigenvectors. This proves the theorem.

3.9 Polar Decompositions

An application of Theorem 3.8.19, is the following fundamental result, important in geometric measure theory and continuum mechanics. It is sometimes called the right polar decomposition. When the following theorem is applied in continuum mechanics, F is normally the deformation gradient, the derivative of a nonlinear map from some subset of three dimensional space to three dimensional space. In this context, U is called the right Cauchy Green strain tensor. It is a measure of how a body is stretched independent of rigid motions. First, here is a simple lemma.

Lemma 3.9.1 *Suppose $R \in \mathcal{L}(X, Y)$ where X, Y are finite dimensional inner product spaces and R preserves distances,*

$$|R\mathbf{x}|_Y = |\mathbf{x}|_X .$$

Then $R^*R = I$.

Proof: Since R preserves distances, $|R\mathbf{x}| = |\mathbf{x}|$ for every $\mathbf{x}$. Therefore from the axioms of the dot product,

$$\begin{aligned} & |\mathbf{x}|^2 + |\mathbf{y}|^2 + (\mathbf{x} \cdot \mathbf{y}) + (\mathbf{y} \cdot \mathbf{x}) \\ = \; & |\mathbf{x} + \mathbf{y}|^2 \\ = \; & (R(\mathbf{x} + \mathbf{y}) \cdot R(\mathbf{x} + \mathbf{y})) \\ = \; & (R\mathbf{x} \cdot R\mathbf{x}) + (R\mathbf{y} \cdot R\mathbf{y}) + (R\mathbf{x} \cdot R\mathbf{y}) + (R\mathbf{y} \cdot R\mathbf{x}) \\ = \; & |\mathbf{x}|^2 + |\mathbf{y}|^2 + (R^*R\mathbf{x} \cdot \mathbf{y}) + (\mathbf{y} \cdot R^*R\mathbf{x}) \end{aligned}$$

and so for all $\mathbf{x}, \mathbf{y}$,

$$(R^*R\mathbf{x} - \mathbf{x} \cdot \mathbf{y}) + (\mathbf{y} \cdot R^*R\mathbf{x} - \mathbf{x}) = 0$$

Hence for all $\mathbf{x}, \mathbf{y}$,

$$\operatorname{Re}(R^*R\mathbf{x} - \mathbf{x} \cdot \mathbf{y}) = 0$$

Now for $\mathbf{x}, \mathbf{y}$ given, choose $\alpha \in \mathbb{C}$ such that

$$\alpha (R^*R\mathbf{x} - \mathbf{x} \cdot \mathbf{y}) = |(R^*R\mathbf{x} - \mathbf{x} \cdot \mathbf{y})|$$

Then

$$\begin{aligned} 0 \; = \; & \operatorname{Re}(R^*R\mathbf{x} - \mathbf{x} \cdot \overline{\alpha}\mathbf{y}) = \operatorname{Re} \alpha (R^*R\mathbf{x} - \mathbf{x} \cdot \mathbf{y}) \\ = \; & |(R^*R\mathbf{x} - \mathbf{x} \cdot \mathbf{y})| \end{aligned}$$

Thus $|(R^*R\mathbf{x}-\mathbf{x}\cdot\mathbf{y})|=0$ for all $\mathbf{x},\mathbf{y}$ because the given $\mathbf{x},\mathbf{y}$ were arbitrary. Let $\mathbf{y}=R^*R\mathbf{x}-\mathbf{x}$ to conclude that for all $\mathbf{x}$,

$$R^*R\mathbf{x}-\mathbf{x}=\mathbf{0}$$

which says $R^*R=I$ since $\mathbf{x}$ is arbitrary. This proves the lemma.

Definition 3.9.2 *In case $R\in\mathcal{L}(X,X)$ for X a real or complex inner product space of dimension n, R is said to be unitary if it preserves distances. Thus, from the above lemma, unitary transformations are those which satisfy*

$$R^*R=RR^*=\mathrm{id}$$

where id *is the identity map on X.*

Theorem 3.9.3 *Let X be a real or complex inner product space of dimension n, let Y be a real or complex inner product space of dimension $m\geq n$ and let $F\in\mathcal{L}(X,Y)$. Then there exists $R\in\mathcal{L}(X,Y)$ and $U\in\mathcal{L}(X,X)$ such that*

$$F=RU,\ U=U^*,(U\ \textit{is Hermitian}),$$

all eigenvalues of U are non negative,

$$U^2=F^*F,R^*R=I,$$

and $|R\mathbf{x}|=|\mathbf{x}|$.

Proof: $(F^*F)^*=F^*F$ and so by Theorem 3.8.19, there is an orthonormal basis of eigenvectors for X, $\{\mathbf{v}_1,\cdots,\mathbf{v}_n\}$ such that

$$F^*F=\sum_{i=1}^{n}\lambda_i\mathbf{v}_i\mathbf{v}_i,\ F^*F\mathbf{v}_k=\lambda_k\mathbf{v}_k.$$

It is also clear that $\lambda_i\geq 0$ because

$$\lambda_i(\mathbf{v}_i\cdot\mathbf{v}_i)=(F^*F\mathbf{v}_i\cdot\mathbf{v}_i)=(F\mathbf{v}_i\cdot F\mathbf{v}_i)\geq 0.$$

Let

$$U\equiv\sum_{i=1}^{n}\lambda_i^{1/2}\mathbf{v}_i\mathbf{v}_i.$$

so U maps X to X and is self adjoint. Then from 3.13,

$$\begin{aligned}U^2&=\sum_{ij}\lambda_i^{1/2}\lambda_j^{1/2}(\mathbf{v}_i\mathbf{v}_i)(\mathbf{v}_j\mathbf{v}_j)\\&=\sum_{ij}\lambda_i^{1/2}\lambda_j^{1/2}\mathbf{v}_i\mathbf{v}_j\delta_{ij}=\sum_{i=1}^{n}\lambda_i\mathbf{v}_i\mathbf{v}_i=F^*F\end{aligned}$$

Let $\{U\mathbf{x}_1,\cdots,U\mathbf{x}_r\}$ be an orthonormal basis for $U(X)$. Extend this using the Gram Schmidt procedure to an orthonormal basis for X,

$$\{U\mathbf{x}_1,\cdots,U\mathbf{x}_r,\mathbf{y}_{r+1},\cdots,\mathbf{y}_n\}.$$

Next note that $\{F\mathbf{x}_1,\cdots,F\mathbf{x}_r\}$ is also an orthonormal set of vectors in Y because

$$(F\mathbf{x}_k\cdot F\mathbf{x}_j)=(F^*F\mathbf{x}_k\cdot\mathbf{x}_j)=\left(U^2\mathbf{x}_k\cdot\mathbf{x}_j\right)=(U\mathbf{x}_k\cdot U\mathbf{x}_j)=\delta_{jk}.$$

Now extend $\{F\mathbf{x}_1, \cdots, F\mathbf{x}_r\}$ to an orthonormal basis for Y,

$$\{F\mathbf{x}_1, \cdots, F\mathbf{x}_r, \mathbf{z}_{r+1}, \cdots, \mathbf{z}_m\}.$$

Since $m \geq n$, there are at least as many $\mathbf{z}_k$ as there are $\mathbf{y}_k$.

Now define R as follows. For $\mathbf{x} \in X$, there exist unique scalars, c_k and d_k such that

$$\mathbf{x} = \sum_{k=1}^{r} c_k U\mathbf{x}_k + \sum_{k=r+1}^{n} d_k \mathbf{y}_k.$$

Then

$$R\mathbf{x} \equiv \sum_{k=1}^{r} c_k F\mathbf{x}_k + \sum_{k=r+1}^{n} d_k \mathbf{z}_k. \tag{3.45}$$

Thus, since $\{F\mathbf{x}_1, \cdots, F\mathbf{x}_r, \mathbf{z}_{r+1}, \cdots, \mathbf{z}_m\}$ is orthonormal, a short computation shows

$$|R\mathbf{x}|^2 = \sum_{k=1}^{r} |c_k|^2 + \sum_{k=r+1}^{n} |d_k|^2 = |\mathbf{x}|^2.$$

Now I need to verify $RU\mathbf{x} = F\mathbf{x}$. Since $\{U\mathbf{x}_1, \cdots, U\mathbf{x}_r\}$ is an orthonormal basis for UX, there exist scalars, b_k such that

$$U\mathbf{x} = \sum_{k=1}^{r} b_k U\mathbf{x}_k \tag{3.46}$$

and so from the definition of R given in 3.45,

$$RU\mathbf{x} \equiv \sum_{k=1}^{r} b_k F\mathbf{x}_k = F\left(\sum_{k=1}^{r} b_k \mathbf{x}_k\right).$$

$RU = F$ is shown if $F\left(\sum_{k=1}^{r} b_k \mathbf{x}_k\right) = F(\mathbf{x})$.

$$\left(F\left(\sum_{k=1}^{r} b_k \mathbf{x}_k\right) - F(\mathbf{x}) \cdot F\left(\sum_{k=1}^{r} b_k \mathbf{x}_k\right) - F(\mathbf{x})\right)$$

$$\begin{aligned}
&= \left(F^*F\left(\sum_{k=1}^{r} b_k \mathbf{x}_k - \mathbf{x}\right) \cdot \sum_{k=1}^{r} b_k \mathbf{x}_k - \mathbf{x}\right) \\
&= \left(U^2\left(\sum_{k=1}^{r} b_k \mathbf{x}_k - \mathbf{x}\right) \cdot \left(\sum_{k=1}^{r} b_k \mathbf{x}_k - \mathbf{x}\right)\right) \\
&= \left(U\left(\sum_{k=1}^{r} b_k \mathbf{x}_k - \mathbf{x}\right) \cdot U\left(\sum_{k=1}^{r} b_k \mathbf{x}_k - \mathbf{x}\right)\right) \\
&= \left(\sum_{k=1}^{r} b_k U\mathbf{x}_k - U\mathbf{x} \cdot \sum_{k=1}^{r} b_k U\mathbf{x}_k - U\mathbf{x}\right) = 0
\end{aligned}$$

by 3.46.

Since $|R\mathbf{x}| = |\mathbf{x}|$, it follows $R^*R = I$ from Lemma 3.9.1. This proves the theorem.

The following corollary follows as a simple consequence of this theorem. It is called the left polar decomposition.

Corollary 3.9.4 *Let $F \in \mathcal{L}(X,Y)$ and suppose $n \geq m$ where X is a inner product space of dimension n and Y is a inner product space of dimension m. Then there exists a Hermitian $U \in \mathcal{L}(X,X)$, and an element of $\mathcal{L}(X,Y)$, R, such that*

$$F = UR,\ RR^* = I.$$

Proof: Recall that $L^{**} = L$ and $(ML)^* = L^*M^*$. Now apply Theorem 3.9.3 to $F^* \in \mathcal{L}(X,Y)$. Thus,

$$F^* = R^*U$$

where R^* and U satisfy the conditions of that theorem. Then

$$F = UR$$

and $RR^* = R^{**}R^* = I$. This proves the corollary.

This is a good place to consider a useful lemma.

Lemma 3.9.5 *Let X be a finite dimensional inner product space of dimension n and let $R \in \mathcal{L}(X,X)$ be unitary. Then $|\det(R)| = 1$.*

Proof: Let $\{\mathbf{w}_k\}$ be an orthonormal basis for X. Then to take the determinant it suffices to take the determinant of the matrix, (c_{ij}) where

$$R = \sum_{ij} c_{ij}\mathbf{w}_i\mathbf{w}_j.$$

$R\mathbf{w}_k = \sum_i c_{ik}\mathbf{w}_i$ and so

$$(R\mathbf{w}_k, \mathbf{w}_l) = c_{lk}.$$

and hence

$$R = \sum_{lk} (R\mathbf{w}_k, \mathbf{w}_l)\, \mathbf{w}_l\mathbf{w}_k$$

Similarly

$$R^* = \sum_{ij} (R^*\mathbf{w}_j, \mathbf{w}_i)\, \mathbf{w}_i\mathbf{w}_j.$$

Since R is given to be unitary,

$$\begin{aligned} RR^* &= \mathrm{id} = \sum_{lk}\sum_{ij} (R\mathbf{w}_k, \mathbf{w}_l)\,(R^*\mathbf{w}_j, \mathbf{w}_i)\,(\mathbf{w}_l\mathbf{w}_k)\,(\mathbf{w}_i\mathbf{w}_j) \\ &= \sum_{ijkl} (R\mathbf{w}_k, \mathbf{w}_l)\,(R^*\mathbf{w}_j, \mathbf{w}_i)\,\delta_{ki}\mathbf{w}_l\mathbf{w}_j \\ &= \sum_{jl} \left(\sum_i (R\mathbf{w}_i, \mathbf{w}_l)\,(R^*\mathbf{w}_j, \mathbf{w}_i) \right) \mathbf{w}_l\mathbf{w}_j \end{aligned}$$

Hence

$$\sum_i (R^*\mathbf{w}_j, \mathbf{w}_i)\,(R\mathbf{w}_i, \mathbf{w}_l) = \delta_{jl} = \sum_i \overline{(R\mathbf{w}_i, \mathbf{w}_j)}\,(R\mathbf{w}_i, \mathbf{w}_l) \tag{3.47}$$

because

$$\mathrm{id} = \sum_{jl} \delta_{jl}\mathbf{w}_l\mathbf{w}_j$$

Thus letting M be the matrix whose ij^{th} entry is $(R\mathbf{w}_i, \mathbf{w}_l)$, $\det(R)$ is defined as $\det(M)$ and 3.47 says

$$\sum_i \left(\overline{M^T}\right)_{ji} M_{il} = \delta_{jl}.$$

It follows

$$1 = \det(M)\det\left(\overline{M}^T\right) = \det(M)\det\left(\overline{M}\right) = \det(M)\overline{\det(M)} = |\det(M)|^2.$$

Thus $|\det(R)| = |\det(M)| = 1$ as claimed.

3.10 Exercises

1. For $\mathbf{u}, \mathbf{v}$ vectors in $\mathbb{F}^3$, define the product, $\mathbf{u} * \mathbf{v} \equiv u_1\overline{v_1} + 2u_2\overline{v_2} + 3u_3\overline{v_3}$. Show the axioms for a dot product all hold for this funny product. Prove

$$|\mathbf{u} * \mathbf{v}| \leq (\mathbf{u} * \mathbf{u})^{1/2} (\mathbf{v} * \mathbf{v})^{1/2}.$$

2. Suppose you have a real or complex vector space. Can it always be considered as an inner product space? What does this mean about Schur's theorem? **Hint:** Start with a basis and decree the basis is orthonormal. Then define an inner product accordingly.

3. Show that $(\mathbf{a} \cdot \mathbf{b}) = \frac{1}{4}\left[|\mathbf{a}+\mathbf{b}|^2 - |\mathbf{a}-\mathbf{b}|^2\right]$.

4. Prove from the axioms of the dot product the parallelogram identity, $|\mathbf{a}+\mathbf{b}|^2 + |\mathbf{a}-\mathbf{b}|^2 = 2|\mathbf{a}|^2 + 2|\mathbf{b}|^2$.

5. Suppose f, g are two Darboux Stieltjes integrable functions defined on $[0,1]$. Define

$$(f \cdot g) = \int_0^1 f(x)\overline{g(x)}dF.$$

   Show this dot product satisfies the axioms for the inner product. Explain why the Cauchy Schwarz inequality continues to hold in this context and state the Cauchy Schwarz inequality in terms of integrals. Does the Cauchy Schwarz inequality still hold if

$$(f \cdot g) = \int_0^1 f(x)\overline{g(x)}p(x)\,dF$$

   where $p(x)$ is a given nonnegative function? If so, what would it be in terms of integrals.

6. If A is an $n \times n$ matrix considered as an element of $\mathcal{L}(\mathbb{C}^n, \mathbb{C}^n)$ by ordinary matrix multiplication, use the inner product in $\mathbb{C}^n$ to show that $(A^*)_{ij} = \overline{A_{ji}}$. In words, the adjoint is the transpose of the conjugate.

7. A symmetric matrix is a real $n \times n$ matrix A which satisfies $A^T = A$. Show every symmetric matrix is self adjoint and that there exists an orthonormal set of real vectors $\{\mathbf{x}_1, \cdots, \mathbf{x}_n\}$ such that

$$A = \sum_k \lambda_k \mathbf{x}_k \mathbf{x}_k$$

8. A normal matrix is an $n \times n$ matrix, A such that $A^*A = AA^*$. Show that for a normal matrix there is an orthonormal basis of $\mathbb{C}^n$, $\{\mathbf{x}_1, \cdots, \mathbf{x}_n\}$ such that

$$A = \sum_i a_i \mathbf{x}_i \mathbf{x}_i$$

That is, with respect to this basis the matrix of A is diagonal. **Hint:** This is a harder version of what was done to prove Theorem 3.8.19. Use Schur's theorem to write $A = \sum_{j=1}^n \sum_{i=1}^n B_{ij} \mathbf{w}_i \mathbf{w}_j$ where B_{ij} is an upper triangular matrix. Then use the condition that A is normal and eventually get an equation

$$\sum_k B_{ik} \overline{B_{lk}} = \sum_k \overline{B_{ki}} B_{kl}$$

Next let $i = l$ and consider first $l = 1$, then $l = 2$, etc. If you are careful, you will find $B_{ij} = 0$ unless $i = j$.

9. Suppose $A \in \mathcal{L}(H, H)$ where H is an inner product space and

$$A = \sum_i a_i \mathbf{w}_i \mathbf{w}_i$$

where the vectors $\{\mathbf{w}_1, \cdots, \mathbf{w}_n\}$ are an orthonormal set. Show that A must be normal. In other words, you can't represent $A \in \mathcal{L}(H, H)$ in this very convenient way unless it is normal.

10. If L is a self adjoint operator defined on an inner product space, H such that L has all only nonnegative eigenvalues. Explain how to define $L^{1/n}$ and show why what you come up with is indeed the n^{th} root of the operator. For a self adjoint operator L on an inner product space, can you define $\sin(L) \equiv \sum_{k=0}^{\infty} (-1)^k L^{2k+1} / (2k+1)!$? What does the infinite series mean? Can you make some sense of this using the representation of L given in Theorem 3.8.19?

11. If L is a self adjoint linear transformation defined on $\mathcal{L}(H, H)$ for H an inner product space which has all eigenvalues nonnegative, show the square root is unique.

12. Using Problem 11 show $F \in \mathcal{L}(H, H)$ for H an inner product space is normal if and only if $RU = UR$ where $F = RU$ is the right polar decomposition defined above. Recall R preserves distances and U is self adjoint. What is the geometric significance of a linear transformation being normal?

13. Suppose you have a basis, $\{\mathbf{v}_1, \cdots, \mathbf{v}_n\}$ in an inner product space, X. The Grammian matrix is the $n \times n$ matrix whose ij^{th} entry is $(\mathbf{v}_i \cdot \mathbf{v}_j)$. Show this matrix is invertible. **Hint:** You might try to show that the inner product of two vectors, $\sum_k a_k \mathbf{v}_k$ and $\sum_k b_k \mathbf{v}_k$ has something to do with the Grammian.

14. Suppose you have a basis, $\{\mathbf{v}_1, \cdots, \mathbf{v}_n\}$ in an inner product space, X. Show there exists a "dual basis" $\{\mathbf{v}^1, \cdots, \mathbf{v}^n\}$ which satisfies $\mathbf{v}^k \cdot \mathbf{v}_j = \delta_j^k$, which equals 0 if $j \neq k$ and equals 1 if $j = k$.

Sequences

4.1 Vector Valued Sequences And Their Limits

Functions defined on the set of integers larger than a given integer which have values in a vector space are called vector valued sequences. I will always assume the vector space is a normed vector space. Actually, it will specialized even more to $\mathbb{F}^n$, although everything can be done for an arbitrary vector space and when it creates no difficulties, I will state certain definitions and easy theorems in the more general context and use the symbol $||\cdot||$ to refer to the norm. Other than this, the notation is almost the same as it was when the sequences had values in $\mathbb{C}$. The main difference is that certain variables are placed in bold face to indicate they are vectors. Even this is not really necessary but it is conventional to do it.The concept of subsequence is also the same as it was for sequences of numbers. To review,

Definition 4.1.1 *Let $\{\mathbf{a}_n\}$ be a sequence and let $n_1 < n_2 < n_3, \cdots$ be any strictly increasing list of integers such that n_1 is at least as large as the first number in the domain of the function. Then if $\mathbf{b}_k \equiv \mathbf{a}_{n_k}$, $\{\mathbf{b}_k\}$ is called a subsequence of $\{\mathbf{a}_n\}$.*

Example 4.1.2 *Let $\mathbf{a}_n = \left(n+1, \sin\left(\frac{1}{n}\right)\right)$. Then $\{\mathbf{a}_n\}_{n=1}^{\infty}$ is a vector valued sequence.*

The definition of a limit of a vector valued sequence is given next. It is just like the definition given for sequences of scalars. However, here the symbol $|\cdot|$ refers to the usual norm in $\mathbb{F}^n$. In a general normed vector space, it will be denoted by $||\cdot||$.

Definition 4.1.3 *A vector valued sequence $\{\mathbf{a}_n\}_{n=1}^{\infty}$ converges to $\mathbf{a}$ in a normed vector space V, written as*

$$\lim_{n\to\infty} \mathbf{a}_n = \mathbf{a} \text{ or } \mathbf{a}_n \to \mathbf{a}$$

if and only if for every $\varepsilon > 0$ there exists n_ε such that whenever $n \geq n_\varepsilon$,

$$||\mathbf{a}_n - \mathbf{a}|| < \varepsilon.$$

In words the definition says that given any measure of closeness ε, the terms of the sequence are eventually this close to $\mathbf{a}$. Here, the word "eventually" refers to n being sufficiently large.

Theorem 4.1.4 *If $\lim_{n\to\infty} \mathbf{a}_n = \mathbf{a}$ and $\lim_{n\to\infty} \mathbf{a}_n = \mathbf{a}_1$ then $\mathbf{a}_1 = \mathbf{a}$.*

Proof: Suppose $\mathbf{a}_1 \neq \mathbf{a}$. Then let $0 < \varepsilon < ||\mathbf{a}_1 - \mathbf{a}||/2$ in the definition of the limit. It follows there exists n_ε such that if $n \geq n_\varepsilon$, then $||\mathbf{a}_n - \mathbf{a}|| < \varepsilon$ and $|\mathbf{a}_n - \mathbf{a}_1| < \varepsilon$. Therefore,

for such n,

$$\begin{aligned} ||\mathbf{a}_1 - \mathbf{a}|| &\leq ||\mathbf{a}_1 - \mathbf{a}_n|| + ||\mathbf{a}_n - \mathbf{a}|| \\ &< \varepsilon + \varepsilon < ||\mathbf{a}_1 - \mathbf{a}|| /2 + ||\mathbf{a}_1 - \mathbf{a}|| /2 = ||\mathbf{a}_1 - \mathbf{a}|| , \end{aligned}$$

a contradiction.

Theorem 4.1.5 *Suppose $\{\mathbf{a}_n\}$ and $\{\mathbf{b}_n\}$ are vector valued sequences and that*

$$\lim_{n\to\infty} \mathbf{a}_n = \mathbf{a} \text{ and } \lim_{n\to\infty} \mathbf{b}_n = \mathbf{b}.$$

Also suppose x and y are scalars in $\mathbb{F}$. Then

$$\lim_{n\to\infty} x\mathbf{a}_n + y\mathbf{b}_n = x\mathbf{a} + y\mathbf{b} \tag{4.1}$$

Also,

$$\lim_{n\to\infty} (\mathbf{a}_n \cdot \mathbf{b}_n) = (\mathbf{a} \cdot \mathbf{b}) \tag{4.2}$$

If $\{x_n\}$ is a sequence of scalars in $\mathbb{F}$ converging to x and if $\{\mathbf{a}_n\}$ is a sequence of vectors in $\mathbb{F}^n$ converging to $\mathbf{a}$, then

$$\lim_{n\to\infty} x_n\mathbf{a}_n = x\mathbf{a}. \tag{4.3}$$

Also if $\{\mathbf{x}_k\}$ is a sequence of vectors in $\mathbb{F}^n$ then $\mathbf{x}_k \to \mathbf{x}$, if and only if for each j,

$$\lim_{k\to\infty} x_k^j = x^j. \tag{4.4}$$

where here

$$\mathbf{x}_k = \left(x_k^1, \cdots, x_k^n\right), \ \mathbf{x} = \left(x^1, \cdots, x^n\right).$$

Proof: Consider the first claim. By the triangle inequality

$$||x\mathbf{a} + y\mathbf{b} - (x\mathbf{a}_n + y\mathbf{b}_n)|| \leq |x| \, ||\mathbf{a} - \mathbf{a}_n|| + |y| \, ||\mathbf{b} - \mathbf{b}_n|| .$$

By definition, there exists n_ε such that if $n \geq n_\varepsilon$,

$$||\mathbf{a} - \mathbf{a}_n|| , ||\mathbf{b} - \mathbf{b}_n|| < \frac{\varepsilon}{2\left(1 + |x| + |y|\right)}$$

so for $n > n_\varepsilon$,

$$||x\mathbf{a} + y\mathbf{b} - (x\mathbf{a}_n + y\mathbf{b}_n)|| < |x| \frac{\varepsilon}{2\left(1 + |x| + |y|\right)} + |y| \frac{\varepsilon}{2\left(1 + |x| + |y|\right)} \leq \varepsilon.$$

Now consider the second. Let $\varepsilon > 0$ be given and choose n_1 such that if $n \geq n_1$ then

$$|\mathbf{a}_n - \mathbf{a}| < 1.$$

For such n, it follows from the Cauchy Schwarz inequality and properties of the inner product that

$$\begin{aligned} |\mathbf{a}_n \cdot \mathbf{b}_n - \mathbf{a} \cdot \mathbf{b}| &\leq |(\mathbf{a}_n \cdot \mathbf{b}_n) - (\mathbf{a}_n \cdot \mathbf{b})| + |(\mathbf{a}_n \cdot \mathbf{b}) - (\mathbf{a} \cdot \mathbf{b})| \\ &\leq |\mathbf{a}_n| \, |\mathbf{b}_n - \mathbf{b}| + |\mathbf{b}| \, |\mathbf{a}_n - \mathbf{a}| \\ &\leq (|\mathbf{a}| + 1) \, |\mathbf{b}_n - \mathbf{b}| + |\mathbf{b}| \, |\mathbf{a}_n - \mathbf{a}| . \end{aligned}$$

Now let n_2 be large enough that for $n \geq n_2$,

$$|\mathbf{b}_n - \mathbf{b}| < \frac{\varepsilon}{2(|\mathbf{a}|+1)}, \text{ and } |\mathbf{a}_n - \mathbf{a}| < \frac{\varepsilon}{2(|\mathbf{b}|+1)}.$$

Such a number exists because of the definition of limit. Therefore, let

$$n_\varepsilon > \max(n_1, n_2).$$

For $n \geq n_\varepsilon$,

$$\begin{aligned} |\mathbf{a}_n \cdot \mathbf{b}_n - \mathbf{a} \cdot \mathbf{b}| &\leq (|\mathbf{a}|+1)|\mathbf{b}_n - \mathbf{b}| + |\mathbf{b}||\mathbf{a}_n - \mathbf{a}| \\ &< (|\mathbf{a}|+1)\frac{\varepsilon}{2(|\mathbf{a}|+1)} + |\mathbf{b}|\frac{\varepsilon}{2(|\mathbf{b}|+1)} \leq \varepsilon. \end{aligned}$$

This proves 4.2. The claim, 4.3 is left for you to do.

Finally consider the last claim. If 4.4 holds, then from the definition of distance in $\mathbb{F}^n$,

$$\lim_{k\to\infty} |\mathbf{x} - \mathbf{x}_k| \equiv \lim_{k\to\infty} \sqrt{\sum_{j=1}^{n} \left(x^j - x_k^j\right)^2} = 0.$$

On the other hand, if $\lim_{k\to\infty} |\mathbf{x} - \mathbf{x}_k| = 0$, then since $\left|x_k^j - x^j\right| \leq |\mathbf{x} - \mathbf{x}_k|$, it follows from the squeezing theorem that

$$\lim_{k\to\infty} \left|x_k^j - x^j\right| = 0.$$

This proves the theorem.

An important theorem is the one which states that if a sequence converges, so does every subsequence. You should review Definition 4.1.1 at this point. The proof is identical to the one involving sequences of numbers.

Theorem 4.1.6 *Let $\{\mathbf{x}_n\}$ be a vector valued sequence with $\lim_{n\to\infty} \mathbf{x}_n = \mathbf{x}$ and let $\{\mathbf{x}_{n_k}\}$ be a subsequence. Then $\lim_{k\to\infty} \mathbf{x}_{n_k} = \mathbf{x}$.*

Proof: Let $\varepsilon > 0$ be given. Then there exists n_ε such that if $n > n_\varepsilon$, then $||\mathbf{x}_n - \mathbf{x}|| < \varepsilon$. Suppose $k > n_\varepsilon$. Then $n_k \geq k > n_\varepsilon$ and so

$$||\mathbf{x}_{n_k} - \mathbf{x}|| < \varepsilon$$

showing $\lim_{k\to\infty} \mathbf{x}_{n_k} = \mathbf{x}$ as claimed.

Theorem 4.1.7 *Let $\{x_n\}$ be a sequence of real numbers and suppose each $x_n \leq l$ $(\geq l)$ and $\lim_{n\to\infty} x_n = x$. Then $x \leq l$ $(\geq l)$. More generally, suppose $\{x_n\}$ and $\{y_n\}$ are two sequences such that $\lim_{n\to\infty} x_n = x$ and $\lim_{n\to\infty} y_n = y$. Then if $x_n \leq y_n$ for all n sufficiently large, then $x \leq y$.*

Proof: Let $\varepsilon > 0$ be given. Then for n large enough,

$$l \geq x_n > x - \varepsilon$$

and so

$$l + \varepsilon \geq x.$$

Since $\varepsilon > 0$ is arbitrary, this requires $l \geq x$. The other case is entirely similar or else you could consider $-l$ and $\{-x_n\}$ and apply the case just considered.

Consider the last claim. There exists N such that if $n \geq N$ then $x_n \leq y_n$ and

$$|x - x_n| + |y - y_n| < \varepsilon/2.$$

Then considering $n > N$ in what follows,

$$x - y \leq x_n + \varepsilon/2 - (y_n - \varepsilon/2) = x_n - y_n + \varepsilon \leq \varepsilon.$$

Since ε was arbitrary, it follows

$$x - y \leq 0.$$

This proves the theorem.

Theorem 4.1.8 *Let $\{\mathbf{x}_n\}$ be a sequence vectors and suppose each $||\mathbf{x}_n|| \leq l$ $(\geq l)$ and $\lim_{n\to\infty} \mathbf{x}_n = \mathbf{x}$. Then $\mathbf{x} \leq l$ $(\geq l)$. More generally, suppose $\{\mathbf{x}_n\}$ and $\{\mathbf{y}_n\}$ are two sequences such that $\lim_{n\to\infty} \mathbf{x}_n = \mathbf{x}$ and $\lim_{n\to\infty} \mathbf{y}_n = \mathbf{y}$. Then if $||\mathbf{x}_n|| \leq ||\mathbf{y}_n||$ for all n sufficiently large, then $||\mathbf{x}|| \leq ||\mathbf{y}||$.*

Proof: It suffices to just prove the second part since the first part is similar. By the triangle inequality,

$$|||\mathbf{x}_n|| - ||\mathbf{x}||| \leq ||\mathbf{x}_n - \mathbf{x}||$$

and for large n this is given to be small. Thus $\{||\mathbf{x}_n||\}$ converges to $||\mathbf{x}||$. Similarly $\{||\mathbf{y}_n||\}$ converges to $||\mathbf{y}||$. Now the desired result follows from Theorem 4.1.7. This proves the theorem.

4.2 Sequential Compactness

The following is the definition of sequential compactness. It is a very useful notion which can be used to prove existence theorems.

Definition 4.2.1 *A set, $K \subseteq V$, a normed vector space is sequentially compact if whenever $\{\mathbf{a}_n\} \subseteq K$ is a sequence, there exists a subsequence, $\{\mathbf{a}_{n_k}\}$ such that this subsequence converges to a point of K.*

First of all, it is convenient to consider the sequentially compact sets in $\mathbb{F}$.

Lemma 4.2.2 *Let $I_k = \left[a^k, b^k\right]$ and suppose that for all $k = 1, 2, \cdots$,*

$$I_k \supseteq I_{k+1}.$$

Then there exists a point, $c \in \mathbb{R}$ which is an element of every I_k.

Proof: Since $I_k \supseteq I_{k+1}$, this implies

$$a^k \leq a^{k+1},\ b^k \geq b^{k+1}. \tag{4.5}$$

Consequently, if $k \leq l$,

$$a^l \leq a^l \leq b^l \leq b^k. \tag{4.6}$$

Now define

$$c \equiv \sup\left\{a^l : l = 1, 2, \cdots\right\}$$

By the first inequality in 4.5, and 4.6

$$a^k \leq c = \sup\left\{a^l : l = k, k+1, \cdots\right\} \leq b^k \tag{4.7}$$

for each $k = 1, 2 \cdots$. Thus $c \in I_k$ for every k and this proves the lemma. If this went too fast, the reason for the last inequality in 4.7 is that from 4.6, b^k is an upper bound to $\left\{a^l : l = k, k+1, \cdots\right\}$. Therefore, it is at least as large as the least upper bound.

Theorem 4.2.3 *Every closed interval, $[a,b]$ is sequentially compact.*

Proof: Let $\{x_n\} \subseteq [a,b] \equiv I_0$. Consider the two intervals $\left[a, \frac{a+b}{2}\right]$ and $\left[\frac{a+b}{2}, b\right]$ each of which has length $(b-a)/2$. At least one of these intervals contains x_n for infinitely many values of n. Call this interval I_1. Now do for I_1 what was done for I_0. Split it in half and let I_2 be the interval which contains x_n for infinitely many values of n. Continue this way obtaining a sequence of nested intervals $I_0 \supseteq I_1 \supseteq I_2 \supseteq I_3 \cdots$ where the length of I_n is $(b-a)/2^n$. Now pick n_1 such that $x_{n_1} \in I_1$, n_2 such that $n_2 > n_1$ and $x_{n_2} \in I_2, n_3$ such that $n_3 > n_2$ and $x_{n_3} \in I_3$, etc. (This can be done because in each case the intervals contained x_n for infinitely many values of n.) By the nested interval lemma there exists a point, c contained in all these intervals. Furthermore,

$$|x_{n_k} - c| < (b-a)\, 2^{-k}$$

and so $\lim_{k\to\infty} x_{n_k} = c \in [a,b]$. This proves the theorem.

Theorem 4.2.4 *Let*

$$I = \prod_{k=1}^{n} K_k$$

where K_k is a sequentially compact set in $\mathbb{F}$. Then I is a sequentially compact set in $\mathbb{F}^n$.

Proof: Let $\{\mathbf{x}_k\}_{k=1}^{\infty}$ be a sequence of points in I. Let

$$\mathbf{x}_k = \left(x_k^1, \cdots, x_k^n\right)$$

Thus $\left\{x_k^i\right\}_{k=1}^{\infty}$ is a sequence of points in K_i. Since K_i is sequentially compact, there exists a subsequence of $\{\mathbf{x}_k\}_{k=1}^{\infty}$ denoted by $\{\mathbf{x}_{1k}\}$ such that $\left\{x_{1k}^1\right\}$ converges to x^1 for some $x^1 \in K_1$. Now there exists a further subsequence, $\{\mathbf{x}_{2k}\}$ such that $\left\{x_{2k}^1\right\}$ converges to x^1, because by Theorem 4.1.6, subsequences of convergent sequences converge to the same limit as the convergent sequence, and in addition, $\left\{x_{2k}^2\right\}$ converges to some $x^2 \in K_2$. Continue taking subsequences such that for $\{\mathbf{x}_{jk}\}_{k=1}^{\infty}$, it follows $\left\{x_{jk}^r\right\}$ converges to some $x^r \in K_r$ for all $r \leq j$. Then $\{\mathbf{x}_{nk}\}_{k=1}^{\infty}$ is the desired subsequence such that the sequence of numbers in $\mathbb{F}$ obtained by taking the j^{th} component of this subsequence converges to some $x^j \in K_j$. It follows from Theorem 4.1.5 that $\mathbf{x} \equiv \left(x^1, \cdots, x^n\right) \in I$ and is the limit of $\{\mathbf{x}_{nk}\}_{k=1}^{\infty}$. This proves the theorem.

Corollary 4.2.5 *Any box of the form*

$$[a,b] + i\,[c,d] \equiv \{x + iy : x \in [a,b], y \in [c,d]\}$$

is sequentially compact in $\mathbb{C}$.

Proof: The given box is essentially $[a,b] \times [c,d]$.

$$\{x_k + iy_k\}_{k=1}^{\infty} \subseteq [a,b] + i\,[c,d]$$

is the same as saying $(x_k, y_k) \in [a,b] \times [c,d]$. Therefore, there exists $(x,y) \in [a,b] \times [c,d]$ such that $x_k \to x$ and $y_k \to y$. In other words $x_k + iy_k \to x + iy$ and $x + iy \in [a,b] + i\,[c,d]$. This proves the corollary.

4.3 Closed And Open Sets

The definition of open and closed sets is next.

Definition 4.3.1 *Let U be a set of points in a normed vector space, V. A point, $\mathbf{p} \in U$ is said to be an interior point if whenever $||\mathbf{x}-\mathbf{p}||$ is sufficiently small, it follows $\mathbf{x} \in U$ also. The set of points, $\mathbf{x}$ which are closer to $\mathbf{p}$ than δ is denoted by*

$$B(\mathbf{p},\delta) \equiv \{\mathbf{x} \in V : ||\mathbf{x}-\mathbf{p}|| < \delta\}.$$

This symbol, $B(\mathbf{p},\delta)$ is called an open ball of radius δ. Thus a point, $\mathbf{p}$ is an interior point of U if there exists $\delta > 0$ such that $\mathbf{p} \in B(\mathbf{p},\delta) \subseteq U$. An open set is one for which every point of the set is an interior point. Closed sets are those which are complements of open sets. Thus H is closed means H^C is open.

Theorem 4.3.2 *The intersection of any finite collection of open sets is open. The union of any collection of open sets is open. The intersection of any collection of closed sets is closed and the union of any finite collection of closed sets is closed.*

Proof: To see that any union of open sets is open, note that every point of the union is in at least one of the open sets. Therefore, it is an interior point of that set and hence an interior point of the entire union.

Now let $\{U_1,\cdots,U_m\}$ be some open sets and suppose $\mathbf{p} \in \cap_{k=1}^m U_k$. Then there exists $r_k > 0$ such that $B(\mathbf{p},r_k) \subseteq U_k$. Let $0 < r \leq \min(r_1,r_2,\cdots,r_m)$. Then $B(\mathbf{p},r) \subseteq \cap_{k=1}^m U_k$ and so the finite intersection is open. Note that if the finite intersection is empty, there is nothing to prove because it is certainly true in this case that every point in the intersection is an interior point because there aren't any such points.

Suppose $\{H_1,\cdots,H_m\}$ is a finite set of closed sets. Then $\cup_{k=1}^m H_k$ is closed if its complement is open. However, from DeMorgan's laws,

$$(\cup_{k=1}^m H_k)^C = \cap_{k=1}^m H_k^C,$$

a finite intersection of open sets which is open by what was just shown.

Next let $\mathcal{C}$ be some collection of closed sets. Then

$$(\cap\mathcal{C})^C = \cup\left\{H^C : H \in \mathcal{C}\right\},$$

a union of open sets which is therefore open by the first part of the proof. Thus $\cap\mathcal{C}$ is closed. This proves the theorem.

Next there is the concept of a limit point which gives another way of characterizing closed sets.

Definition 4.3.3 *Let A be any nonempty set and let $\mathbf{x}$ be a point. Then $\mathbf{x}$ is said to be a limit point of A if for every $r > 0, B(\mathbf{x},r)$ contains a point of A which is not equal to $\mathbf{x}$.*

Example 4.3.4 *Consider $A = B(\mathbf{x},\delta)$, an open ball in a normed vector space. Then every point of $B(\mathbf{x},\delta)$ is a limit point. There are more general situations than normed vector spaces in which this assertion is false.*

If $\mathbf{z} \in B(\mathbf{x},\delta)$, consider $\mathbf{z}+\frac{1}{k}(\mathbf{x}-\mathbf{z}) \equiv \mathbf{w}_k$ for $k \in \mathbb{N}$. Then

$$\begin{aligned} ||\mathbf{w}_k - \mathbf{x}|| &= \left|\left|\mathbf{z}+\frac{1}{k}(\mathbf{x}-\mathbf{z})-\mathbf{x}\right|\right| \\ &= \left|\left|\left(1-\frac{1}{k}\right)\mathbf{z}-\left(1-\frac{1}{k}\right)\mathbf{x}\right|\right| \\ &= \frac{k-1}{k}||\mathbf{z}-\mathbf{x}|| < \delta \end{aligned}$$

and also

$$||\mathbf{w}_k - \mathbf{z}|| \leq \frac{1}{k} ||\mathbf{x} - \mathbf{z}|| < \delta/k$$

so $\mathbf{w}_k \to \mathbf{z}$. Furthermore, the $\mathbf{w}_k$ are distinct. Thus $\mathbf{z}$ is a limit point of A as claimed. This is because every ball containing $\mathbf{z}$ contains infinitely many of the $\mathbf{w}_k$ and since they are all distinct, they can't all be equal to $\mathbf{z}$.

Similarly, the following holds in any normed vector space.

Theorem 4.3.5 *Let A be a nonempty set in V, a normed vector space. A point $\mathbf{a}$ is a limit point of A if and only if there exists a sequence of distinct points of $A, \{\mathbf{a}_n\}$ which converges to $\mathbf{a}$. Also a nonempty set, A is closed if and only if it contains all its limit points.*

Proof: Suppose first $\mathbf{a}$ is a limit point of A. There exists $\mathbf{a}_1 \in B(\mathbf{a}, 1) \cap A$ such that $\mathbf{a}_1 \neq \mathbf{a}$. Now supposing distinct points, $\mathbf{a}_1, \cdots, \mathbf{a}_n$ have been chosen such that none are equal to $\mathbf{a}$ and for each $k \leq n$, $\mathbf{a}_k \in B(\mathbf{a}, 1/k)$, let

$$0 < r_{n+1} < \min\left\{\frac{1}{n+1}, ||\mathbf{a} - \mathbf{a}_1||, \cdots, ||\mathbf{a} - \mathbf{a}_n||\right\}.$$

Then there exists $\mathbf{a}_{n+1} \in B(\mathbf{a}, r_{n+1}) \cap A$ with $\mathbf{a}_{n+1} \neq \mathbf{a}$. Because of the definition of r_{n+1}, $\mathbf{a}_{n+1}$ is not equal to any of the other $\mathbf{a}_k$ for $k < n+1$. Also since $||\mathbf{a} - \mathbf{a}_m|| < 1/m$, it follows $\lim_{m\to\infty} \mathbf{a}_m = \mathbf{a}$. Conversely, if there exists a sequence of distinct points of A converging to $\mathbf{a}$, then $B(\mathbf{a}, r)$ contains all $\mathbf{a}_n$ for n large enough. Thus $B(\mathbf{a}, r)$ contains infinitely many points of A since all are distinct. Thus at least one of them is not equal to $\mathbf{a}$. This establishes the first part of the theorem.

Now consider the second claim. If A is closed then it is the complement of an open set. Since A^C is open, it follows that if $\mathbf{a} \in A^C$, then there exists $\delta > 0$ such that $B(\mathbf{a}, \delta) \subseteq A^C$ and so no point of A^C can be a limit point of A. In other words, every limit point of A must be in A. Conversely, suppose A contains all its limit points. Then A^C does not contain any limit points of A. It also contains no points of A. Therefore, if $\mathbf{a} \in A^C$, since it is not a limit point of A, there exists $\delta > 0$ such that $B(\mathbf{a}, \delta)$ contains no points of A different than $\mathbf{a}$. However, $\mathbf{a}$ itself is not in A because $\mathbf{a} \in A^C$. Therefore, $B(\mathbf{a}, \delta)$ is entirely contained in A^C. Since $\mathbf{a} \in A^C$ was arbitrary, this shows every point of A^C is an interior point and so A^C is open. This proves the theorem.

Closed subsets of sequentially compact sets are sequentially compact.

Theorem 4.3.6 *If K is a sequentially compact set in a normed vector space and if H is a closed subset of K then H is sequentially compact.*

Proof: Let $\{\mathbf{x}_n\} \subseteq H$. Then since K is sequentially compact, there is a subsequence, $\{\mathbf{x}_{n_k}\}$ which converges to a point, $\mathbf{x} \in K$. If $\mathbf{x} \notin H$, then since H^C is open, it follows there exists $B(\mathbf{x}, r)$ such that this open ball contains no points of H. However, this is a contradiction to having $\mathbf{x}_{n_k} \to \mathbf{x}$ which requires $\mathbf{x}_{n_k} \in B(\mathbf{x}, r)$ for all k large enough. Thus $\mathbf{x} \in H$ and this has shown H is sequentially compact.

Definition 4.3.7 *A set $S \subseteq V$, a normed vector space is bounded if there is some $r > 0$ such that $S \subseteq B(\mathbf{0}, r)$.*

Theorem 4.3.8 *Every closed and bounded set in $\mathbb{F}^n$ is sequentially compact. Conversely, every sequentially compact set in $\mathbb{F}^n$ is closed and bounded.*

Proof: Let H be a closed and bounded set in $\mathbb{F}^n$. Then $H \subseteq B(\mathbf{0},r)$ for some r. Therefore, if $\mathbf{x} \in H, \mathbf{x} = (x_1, \cdots, x_n)$, it must be that

$$\sqrt{\sum_{i=1}^{n} |x_i|^2} < r$$

and so each $x_i \in [-r,r] + i[-r,r] \equiv R_r$, a sequentially compact set by Corollary 4.2.5. Thus H is a closed subset of

$$\prod_{i=1}^{n} R_r$$

which is a sequentially compact set by Theorem 4.2.4. Therefore, by Theorem 4.3.6 it follows H is sequentially compact.

Conversely, suppose K is a sequentially compact set in $\mathbb{F}^n$. If it is not bounded, then there exists a sequence, $\{\mathbf{k}_m\}$ such that $\mathbf{k}_m \in K$ but $\mathbf{k}_m \notin B(\mathbf{0},m)$ for $m = 1, 2, \cdots$. However, this sequence cannot have any convergent subsequence because if $\mathbf{k}_{m_k} \to \mathbf{k}$, then for large enough m, $\mathbf{k} \in B(\mathbf{0},m) \subseteq D(\mathbf{0},m)$ and $\mathbf{k}_{m_k} \in B(\mathbf{0},m)^C$ for all k large enough and this is a contradiction because there can only be finitely many points of the sequence in $B(\mathbf{0},m)$. If K is not closed, then it is missing a limit point. Say $\mathbf{k}_\infty$ is a limit point of K which is not in K. Pick $\mathbf{k}_m \in B\left(\mathbf{k}_\infty, \frac{1}{m}\right)$. Then $\{\mathbf{k}_m\}$ converges to $\mathbf{k}_\infty$ and so every subsequence also converges to $\mathbf{k}_\infty$ by Theorem 4.1.6. Thus there is no point of K which is a limit of some subsequence of $\{\mathbf{k}_m\}$, a contradiction. This proves the theorem.

What are some examples of closed and bounded sets in a general normed vector space and more specifically $\mathbb{F}^n$?

Proposition 4.3.9 *Let $D(\mathbf{z},r)$ denote the set of points,*

$$\{\mathbf{w} \in V : ||\mathbf{w} - \mathbf{z}|| \leq r\}$$

Then $D(\mathbf{z},r)$ is closed and bounded. Also, let $S(\mathbf{z},r)$ denote the set of points

$$\{\mathbf{w} \in V : ||\mathbf{w} - \mathbf{z}|| = r\}$$

Then $S(\mathbf{z},r)$ is closed and bounded. It follows that if $V = \mathbb{F}^n$, then these sets are sequentially compact.

Proof: First note $D(\mathbf{z},r)$ is bounded because

$$D(\mathbf{z},r) \subseteq B(\mathbf{0}, ||\mathbf{z}|| + 2r)$$

Here is why. Let $\mathbf{x} \in D(\mathbf{z},r)$. Then $||\mathbf{x} - \mathbf{z}|| \leq r$ and so

$$||\mathbf{x}|| \leq ||\mathbf{x} - \mathbf{z}|| + ||\mathbf{z}|| \leq r + ||\mathbf{z}|| < 2r + ||\mathbf{z}||.$$

It remains to verify it is closed. Suppose then that $\mathbf{y} \notin D(\mathbf{z},r)$. This means $||\mathbf{y} - \mathbf{z}|| > r$. Consider the open ball $B(\mathbf{y}, ||\mathbf{y} - \mathbf{z}|| - r)$. If $\mathbf{x} \in B(\mathbf{y}, ||\mathbf{y} - \mathbf{z}|| - r)$, then

$$||\mathbf{x} - \mathbf{y}|| < ||\mathbf{y} - \mathbf{z}|| - r$$

and so by the triangle inequality,

$$||\mathbf{z} - \mathbf{x}|| \geq ||\mathbf{z} - \mathbf{y}|| - ||\mathbf{y} - \mathbf{x}|| > ||\mathbf{x} - \mathbf{y}|| + r - ||\mathbf{x} - \mathbf{y}|| = r$$

Thus the complement of $D(\mathbf{z},r)$ is open and so $D(\mathbf{z},r)$ is closed.

For the second type of set, note $S(\mathbf{z},r)^C = B(\mathbf{z},r) \cup D(\mathbf{z},r)^C$, the union of two open sets which by Theorem 4.3.2 is open. Therefore, $S(\mathbf{z},r)$ is a closed set which is clearly bounded because $S(\mathbf{z},r) \subseteq D(\mathbf{z},r)$.

4.4 Cauchy Sequences And Completeness

The concept of completeness is that every Cauchy sequence converges. Cauchy sequences are those sequences which have the property that ultimately the terms of the sequence are bunching up. More precisely,

Definition 4.4.1 *$\{\mathbf{a}_n\}$ is a Cauchy sequence in a normed vector space, V if for all $\varepsilon > 0$, there exists n_ε such that whenever $n, m \geq n_\varepsilon$,*

$$||\mathbf{a}_n - \mathbf{a}_m|| < \varepsilon.$$

Theorem 4.4.2 *The set of terms (values) of a Cauchy sequence in a normed vector space V is bounded.*

Proof: Let $\varepsilon = 1$ in the definition of a Cauchy sequence and let $n > n_1$. Then from the definition,

$$||\mathbf{a}_n - \mathbf{a}_{n_1}|| < 1.$$

It follows that for all $n > n_1$,

$$||\mathbf{a}_n|| < 1 + ||\mathbf{a}_{n_1}||.$$

Therefore, for all n,

$$||\mathbf{a}_n|| \leq 1 + ||\mathbf{a}_{n_1}|| + \sum_{k=1}^{n_1} ||\mathbf{a}_k||.$$

This proves the theorem.

Theorem 4.4.3 *If a sequence $\{\mathbf{a}_n\}$ in V, a normed vector space converges, then the sequence is a Cauchy sequence.*

Proof: Let $\varepsilon > 0$ be given and suppose $\mathbf{a}_n \to \mathbf{a}$. Then from the definition of convergence, there exists n_ε such that if $n > n_\varepsilon$, it follows that

$$||\mathbf{a}_n - \mathbf{a}|| < \frac{\varepsilon}{2}$$

Therefore, if $m, n \geq n_\varepsilon + 1$, it follows that

$$||\mathbf{a}_n - \mathbf{a}_m|| \leq ||\mathbf{a}_n - \mathbf{a}|| + ||\mathbf{a} - \mathbf{a}_m|| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

showing that, since $\varepsilon > 0$ is arbitrary, $\{\mathbf{a}_n\}$ is a Cauchy sequence.

The following theorem is very useful. It is identical to an earlier theorem. All that is required is to put things in bold face to indicate they are vectors.

Theorem 4.4.4 *Suppose $\{\mathbf{a}_n\}$ is a Cauchy sequence in any normed vector space and there exists a subsequence, $\{\mathbf{a}_{n_k}\}$ which converges to $\mathbf{a}$. Then $\{\mathbf{a}_n\}$ also converges to $\mathbf{a}$.*

Proof: Let $\varepsilon > 0$ be given. There exists N such that if $m, n > N$, then

$$||\mathbf{a}_m - \mathbf{a}_n|| < \varepsilon/2.$$

Also there exists K such that if $k > K$, then

$$||\mathbf{a} - \mathbf{a}_{n_k}|| < \varepsilon/2.$$

Then let $k > \max(K, N)$. Then for such k,

$$\begin{aligned} ||\mathbf{a}_k - \mathbf{a}|| &\leq ||\mathbf{a}_k - \mathbf{a}_{n_k}|| + ||\mathbf{a}_{n_k} - \mathbf{a}|| \\ &< \varepsilon/2 + \varepsilon/2 = \varepsilon. \end{aligned}$$

This proves the theorem.

Definition 4.4.5 *If V is a normed vector space having the property that every Cauchy sequence converges, then V is called complete. It is also referred to as a Banach space.*

Example 4.4.6 $\mathbb{R}$ *is given to be complete. This is a fundamental axiom on which calculus is developed.*

Given $\mathbb{R}$ is complete, the following lemma is easily obtained.

Lemma 4.4.7 $\mathbb{C}$ *is complete.*

Proof: Let $\{x_k + iy_k\}_{k=1}^{\infty}$ be a Cauchy sequence in $\mathbb{C}$. This requires $\{x_k\}$ and $\{y_k\}$ are both Cauchy sequences in $\mathbb{R}$. This follows from the obvious estimates

$$|x_k - x_m|, |y_k - y_m| \leq |(x_k + iy_k) - (x_m + iy_m)|.$$

By completeness of $\mathbb{R}$ there exists $x \in \mathbb{R}$ such that $x_k \to x$ and similarly there exists $y \in \mathbb{R}$ such that $y_k \to y$. Therefore, since

$$\begin{aligned} |(x_k + iy_k) - (x + iy)| &\leq \sqrt{(x_k - x)^2 + (y_k - y)^2} \\ &\leq |x_k - x| + |y_k - y| \end{aligned}$$

it follows $(x_k + iy_k) \to (x + iy)$.

A simple generalization of this idea yields the following theorem.

Theorem 4.4.8 $\mathbb{F}^n$ *is complete.*

Proof: By 4.4.7, $\mathbb{F}$ is complete. Now let $\{\mathbf{a}_m\}$ be a Cauchy sequence in $\mathbb{F}^n$. Then by the definition of the norm

$$\left|a_m^j - a_k^j\right| \leq |\mathbf{a}_m - \mathbf{a}_k|$$

where a_m^j denotes the j^{th} component of $\mathbf{a}_m$. Thus for each $j = 1, 2, \cdots, n$, $\left\{a_m^j\right\}_{m=1}^{\infty}$ is a Cauchy sequence. It follows from Theorem 4.4.7, the completeness of $\mathbb{F}$, there exists a^j such that

$$\lim_{m \to \infty} a_m^j = a^j$$

Theorem 4.1.5 implies that $\lim_{m \to \infty} \mathbf{a}_m = \mathbf{a}$ where

$$\mathbf{a} = \left(a^1, \cdots, a^n\right).$$

This proves the theorem.

4.5 Shrinking Diameters

It is useful to consider another version of the nested interval lemma. This involves a sequence of sets such that set $(n+1)$ is contained in set n and such that their diameters converge to 0. It turns out that if the sets are also closed, then often there exists a unique point in all of them.

Definition 4.5.1 *Let S be a nonempty set in a normed vector space, V. Then* $\operatorname{diam}(S)$ *is defined as*

$$\operatorname{diam}(S) \equiv \sup\{||\mathbf{x} - \mathbf{y}|| : \mathbf{x}, \mathbf{y} \in S\}.$$

This is called the diameter of S.

Theorem 4.5.2 *Let $\{F_n\}_{n=1}^{\infty}$ be a sequence of closed sets in $\mathbb{F}^n$ such that*

$$\lim_{n\to\infty} \operatorname{diam}(F_n) = 0$$

and $F_n \supseteq F_{n+1}$ for each n. Then there exists a unique $\mathbf{p} \in \cap_{k=1}^{\infty} F_k$.

Proof: Pick $\mathbf{p}_k \in F_k$. This is always possible because by assumption each set is nonempty. Then $\{\mathbf{p}_k\}_{k=m}^{\infty} \subseteq F_m$ and since the diameters converge to 0, it follows $\{\mathbf{p}_k\}$ is a Cauchy sequence. Therefore, it converges to a point, $\mathbf{p}$ by completeness of $\mathbb{F}^n$discussed in Theorem 4.4.8. Since each F_k is closed, $\mathbf{p} \in F_k$ for all k. This is because it is a limit of a sequence of points only finitely many of which are not in the closed set F_k. Therefore, $\mathbf{p} \in \cap_{k=1}^{\infty} F_k$. If $\mathbf{q} \in \cap_{k=1}^{\infty} F_k$, then since both $\mathbf{p}, \mathbf{q} \in F_k$,

$$|\mathbf{p} - \mathbf{q}| \leq \operatorname{diam}(F_k).$$

It follows since these diameters converge to 0, $|\mathbf{p} - \mathbf{q}| \leq \varepsilon$ for every ε. Hence $p = q$. This proves the theorem.

A sequence of sets, $\{G_n\}$ which satisfies $G_n \supseteq G_{n+1}$ for all n is called a nested sequence of sets.

4.6 Exercises

1. For a nonempty set, S in a normed vector space, V, define a function

$$\mathbf{x} \to \operatorname{dist}(\mathbf{x},S) \equiv \inf\{||\mathbf{x} - \mathbf{y}|| : \mathbf{y} \in S\}.$$

   Show

$$||\operatorname{dist}(\mathbf{x}, S) - \operatorname{dist}(\mathbf{y}, S)|| \leq ||\mathbf{x} - \mathbf{y}||.$$

2. Let A be a nonempty set in $\mathbb{F}^n$ or more generally in a normed vector space. Define the closure of A to equal the intersection of all closed sets which contain A. This is usually denoted by $\overline{A}$. Show $\overline{A} = A \cup A'$ where A' consists of the set of limit points of A. Also explain why $\overline{A}$ is closed.

3. The interior of a set was defined above. Tell why the interior of a set is always an open set. The interior of a set A is sometimes denoted by A^0.

4. Given an example of a set A whose interior is empty but whose closure is all of $\mathbb{R}^n$.

5. A point, p is said to be in the boundary of a nonempty set, A if for every $r > 0$, $B(p, r)$ contains points of A as well as points of A^C. Sometimes this is denoted as ∂A. In a normed vector space, is it always the case that $A \cup \partial A = \overline{A}$? Prove or disprove.

6. Give an example of a finite dimensional normed vector space where the field of scalars is the rational numbers which is not complete.

7. Explain why as far as the theorems of this chapter are concerned, $\mathbb{C}^n$ is essentially the same as $\mathbb{R}^{2n}$.

8. A set, $A \subseteq \mathbb{R}^n$ is said to be convex if whenever $\mathbf{x}, \mathbf{y} \in A$ it follows $t\mathbf{x} + (1-t)\mathbf{y} \in A$ whenever $t \in [0, 1]$. Show $B(\mathbf{z}, r)$ is convex. Also show $D(\mathbf{z},r)$ is convex. If A is convex, does it follow $\overline{A}$ is convex? Explain why or why not.

9. Let A be any nonempty subset of $\mathbb{R}^n$. The convex hull of A, usually denoted by $\mathrm{co}(A)$ is defined as the set of all convex combinations of points in A. A convex combination is of the form $\sum_{k=1}^{p} t_k \mathbf{a}_k$ where each $t_k \geq 0$ and $\sum_k t_k = 1$. Note that p can be any finite number. Show $\mathrm{co}(A)$ is convex.

10. Suppose $A \subseteq \mathbb{R}^n$ and $\mathbf{z} \in \mathrm{co}(A)$. Thus $\mathbf{z} = \sum_{k=1}^{p} t_k \mathbf{a}_k$ for $t_k \geq 0$ and $\sum_k t_k = 1$. Show there exists $n+1$ of the points $\{\mathbf{a}_1, \cdots, \mathbf{a}_p\}$ such that $\mathbf{z}$ is a convex combination of these $n+1$ points. **Hint:** Show that if $p > n+1$ then the vectors $\{\mathbf{a}_k - \mathbf{a}_1\}_{k=2}^{p}$ must be linearly dependent. Conclude from this the existence of scalars $\{\alpha_i\}$ such that $\sum_{i=1}^{p} \alpha_i \mathbf{a}_i = \mathbf{0}$. Now for $s \in \mathbb{R}$, $\mathbf{z} = \sum_{k=1}^{p} (t_k + s\alpha_i)\, \mathbf{a}_k$. Consider small s and adjust till one or more of the $t_k + s\alpha_k$ vanish. Now you are in the same situation as before but with only $p-1$ of the $\mathbf{a}_k$. Repeat the argument till you end up with only $n+1$ at which time you can't repeat again.

11. Show that any uncountable set of points in $\mathbb{F}^n$ must have a limit point.

12. Let V be any finite dimensional vector space having a basis $\{\mathbf{v}_1, \cdots, \mathbf{v}_n\}$. For $\mathbf{x} \in V$, let
$$\mathbf{x} = \sum_{k=1}^{n} x_k \mathbf{v}_k$$
so that the scalars, x_k are the components of $\mathbf{x}$ with respect to the given basis. Define for $\mathbf{x}, \mathbf{y} \in V$
$$(\mathbf{x} \cdot \mathbf{y}) \equiv \sum_{i=1}^{n} x_i \overline{y_i}$$
Show this is a dot product for V satisfying all the axioms of a dot product presented earlier.

13. In the context of Problem 12 let $|\mathbf{x}|$ denote the norm of $\mathbf{x}$ which is produced by this inner product and suppose $||\cdot||$ is some other norm on V. Thus
$$|\mathbf{x}| \equiv \left(\sum_i |x_i|^2 \right)^{1/2}$$
where
$$\mathbf{x} = \sum_k x_k \mathbf{v}_k. \tag{4.8}$$
Show there exist positive numbers $\delta < \Delta$ independent of $\mathbf{x}$ such that
$$\delta\, |\mathbf{x}| \leq ||\mathbf{x}|| \leq \Delta\, |\mathbf{x}|$$
This is referred to by saying the two norms are equivalent. **Hint:** The top half is easy using the Cauchy Schwarz inequality. The bottom half is somewhat harder. Argue that if it is not so, there exists a sequence $\{\mathbf{x}_k\}$ such that $|\mathbf{x}_k| = 1$ but $k^{-1} |\mathbf{x}_k| = k^{-1} \geq ||\mathbf{x}_k||$ and then note the vector of components of $\mathbf{x}_k$ is on $S(\mathbf{0}, 1)$ which was shown to be sequentially compact. Pass to a limit in 4.8 and use the assumed inequality to get a contradiction to $\{\mathbf{v}_1, \cdots, \mathbf{v}_n\}$ being a basis.

14. It was shown above that in $\mathbb{F}^n$, the sequentially compact sets are exactly those which are closed and bounded. Show that in any finite dimensional normed vector space, V the closed and bounded sets are those which are sequentially compact.

15. Two norms on a finite dimensional vector space, $||\cdot||_1$ and $||\cdot||_2$ are said to be equivalent if there exist positive numbers $\delta < \Delta$ such that

$$\delta ||\mathbf{x}||_1 \leq ||\mathbf{x}||_2 \leq \Delta ||\mathbf{x}_1||_1 .$$

Show the statement that two norms are equivalent is an equivalence relation. Explain using the result of Problem 13 why any two norms on a finite dimensional vector space are equivalent.

16. A normed vector space, V is separable if there is a countable set $\{\mathbf{w}_k\}_{k=1}^{\infty}$ such that whenever $B(\mathbf{x}, \delta)$ is an open ball in V, there exists some $\mathbf{w}_k$ in this open ball. Show that $\mathbb{F}^n$ is separable. This set of points is called a countable dense set.

17. Let V be any normed vector space with norm $||\cdot||$. Using Problem 13 show that V is separable.

18. Suppose V is a normed vector space. Show there exists a countable set of open balls $\mathcal{B} \equiv \{B(\mathbf{x}_k, r_k)\}_{k=1}^{\infty}$ having the remarkable property that any open set, U is the union of some subset of $\mathcal{B}$. This collection of balls is called a countable basis. **Hint:** Use Problem 17 to get a countable dense dense set of points, $\{\mathbf{x}_k\}_{k=1}^{\infty}$ and then consider balls of the form $B\left(\mathbf{x}_k, \frac{1}{r}\right)$ where $r \in \mathbb{N}$. Show this collection of balls is countable and then show it has the remarkable property mentioned.

19. Suppose S is any nonempty set in V a finite dimensional normed vector space. Suppose $\mathcal{C}$ is a set of open sets such that $\cup\mathcal{C} \supseteq S$. (Such a collection of sets is called an open cover.) Show using Problem 18 that there are countably many sets from $\mathcal{C}$, $\{U_k\}_{k=1}^{\infty}$ such that $S \subseteq \cup_{k=1}^{\infty} U_k$. This is called the Lindeloff property when every open cover can be reduced to a countable sub cover.

20. A set, H in a normed vector space is said to be compact if whenever $\mathcal{C}$ is a set of open sets such that $\cup\mathcal{C} \supseteq H$, there are finitely many sets of $\mathcal{C}$, $\{U_1, \cdots, U_p\}$ such that

$$H \subseteq \cup_{i=1}^{p} U_i.$$

Show using Problem 19 that if a set in a normed vector space is sequentially compact, then it must be compact. Next show using Problem 14 that a set in a normed vector space is compact if and only if it is closed and bounded. Explain why the sets which are compact, closed and bounded, and sequentially compact are the same sets in any finite dimensional normed vector space

Continuous Functions

Continuous functions are defined as they are for a function of one variable.

Definition 5.0.1 *Let V, W be normed vector spaces. A function $f\colon D(\mathbf{f}) \subseteq V \to W$ is continuous at $\mathbf{x} \in D(\mathbf{f})$ if for each $\varepsilon > 0$ there exists $\delta > 0$ such that whenever $\mathbf{y} \in D(\mathbf{f})$ and*

$$||\mathbf{y} - \mathbf{x}||_V < \delta$$

it follows that

$$||\mathbf{f}(\mathbf{x}) - \mathbf{f}(\mathbf{y})||_W < \varepsilon.$$

A function, f is continuous if it is continuous at every point of $D(\mathbf{f})$.

There is a theorem which makes it easier to verify certain functions are continuous without having to always go to the above definition. The statement of this theorem is purposely just a little vague. Some of these things tend to hold in almost any context, certainly for any normed vector space.

Theorem 5.0.2 *The following assertions are valid*

1. *The function, $af+bg$ is continuous at x when f, g are continuous at $x \in D(\mathbf{f}) \cap D(\mathbf{g})$ and $a, b \in \mathbb{F}$.*

2. *If and $\mathbf{f}$ and $\mathbf{g}$ have values in $\mathbb{F}^n$ and they are each continuous at $\mathbf{x}$, then $\mathbf{f} \cdot \mathbf{g}$ is continuous at $\mathbf{x}$. If g has values in $\mathbb{F}$ and $g(\mathbf{x}) \neq 0$ with g continuous, then $\mathbf{f}/g$ is continuous at $\mathbf{x}$.*

3. *If $\mathbf{f}$ is continuous at $\mathbf{x}$, $\mathbf{f}(\mathbf{x}) \in D(\mathbf{g})$, and $\mathbf{g}$ is continuous at $\mathbf{f}(\mathbf{x})$, then $\mathbf{g} \circ \mathbf{f}$ is continuous at $\mathbf{x}$.*

4. *If V is any normed vector space, the function $\mathbf{f} : V \to \mathbb{R}$, given by $\mathbf{f}(\mathbf{x}) = ||\mathbf{x}||$ is continuous.*

5. *$\mathbf{f}$ is continuous at every point of V if and only if whenever U is an open set in $W, \mathbf{f}^{-1}(W)$ is open.*

Proof: First consider 1.) Let $\varepsilon > 0$ be given. By assumption, there exist $\delta_1 > 0$ such that whenever $|\mathbf{x} - \mathbf{y}| < \delta_1$, it follows $|\mathbf{f}(\mathbf{x}) - \mathbf{f}(\mathbf{y})| < \frac{\varepsilon}{2(|a|+|b|+1)}$ and there exists $\delta_2 > 0$ such that whenever $|\mathbf{x} - \mathbf{y}| < \delta_2$, it follows that $|\mathbf{g}(\mathbf{x}) - \mathbf{g}(\mathbf{y})| < \frac{\varepsilon}{2(|a|+|b|+1)}$. Then let $0 < \delta \leq \min(\delta_1, \delta_2)$. If $|\mathbf{x} - \mathbf{y}| < \delta$, then everything happens at once. Therefore, using the triangle inequality

$$|a\mathbf{f}(\mathbf{x}) + b\mathbf{f}(\mathbf{x}) - (a\mathbf{g}(\mathbf{y}) + b\mathbf{g}(\mathbf{y}))|$$

$$\leq |a| |\mathbf{f}(\mathbf{x}) - \mathbf{f}(\mathbf{y})| + |b| |\mathbf{g}(\mathbf{x}) - \mathbf{g}(\mathbf{y})|$$
$$< |a| \left(\frac{\varepsilon}{2(|a| + |b| + 1)} \right) + |b| \left(\frac{\varepsilon}{2(|a| + |b| + 1)} \right) < \varepsilon.$$

Now consider 2.) There exists $\delta_1 > 0$ such that if $|\mathbf{y} - \mathbf{x}| < \delta_1$, then $|\mathbf{f}(\mathbf{x}) - \mathbf{f}(\mathbf{y})| < 1$. Therefore, for such $\mathbf{y}$,

$$|\mathbf{f}(\mathbf{y})| < 1 + |\mathbf{f}(\mathbf{x})|.$$

It follows that for such $\mathbf{y}$,

$$|\mathbf{f} \cdot \mathbf{g}(\mathbf{x}) - \mathbf{f} \cdot \mathbf{g}(\mathbf{y})| \leq |\mathbf{f}(\mathbf{x}) \cdot \mathbf{g}(\mathbf{x}) - \mathbf{g}(\mathbf{x}) \cdot \mathbf{f}(\mathbf{y})| + |\mathbf{g}(\mathbf{x}) \cdot \mathbf{f}(\mathbf{y}) - \mathbf{f}(\mathbf{y}) \cdot \mathbf{g}(\mathbf{y})|$$

$$\leq |\mathbf{g}(\mathbf{x})| |\mathbf{f}(\mathbf{x}) - \mathbf{f}(\mathbf{y})| + |\mathbf{f}(\mathbf{y})| |\mathbf{g}(\mathbf{x}) - \mathbf{g}(\mathbf{y})|$$
$$\leq (1 + |\mathbf{g}(\mathbf{x})| + |\mathbf{f}(\mathbf{y})|) [|\mathbf{g}(\mathbf{x}) - \mathbf{g}(\mathbf{y})| + |\mathbf{f}(\mathbf{x}) - \mathbf{f}(\mathbf{y})|]$$
$$\leq (2 + |\mathbf{g}(\mathbf{x})| + |\mathbf{f}(\mathbf{x})|) [|\mathbf{g}(\mathbf{x}) - \mathbf{g}(\mathbf{y})| + |\mathbf{f}(\mathbf{x}) - \mathbf{f}(\mathbf{y})|]$$

Now let $\varepsilon > 0$ be given. There exists δ_2 such that if $|\mathbf{x} - \mathbf{y}| < \delta_2$, then

$$|\mathbf{g}(\mathbf{x}) - \mathbf{g}(\mathbf{y})| < \frac{\varepsilon}{2(2 + |\mathbf{g}(\mathbf{x})| + |\mathbf{f}(\mathbf{x})|)},$$

and there exists δ_3 such that if $|\mathbf{x} - \mathbf{y}| < \delta_3$, then

$$|\mathbf{f}(\mathbf{x}) - \mathbf{f}(\mathbf{y})| < \frac{\varepsilon}{2(2 + |\mathbf{g}(\mathbf{x})| + |\mathbf{f}(\mathbf{x})|)}$$

Now let $0 < \delta \leq \min(\delta_1, \delta_2, \delta_3)$. Then if $|\mathbf{x} - \mathbf{y}| < \delta$, all the above hold at once and so

$$|\mathbf{f} \cdot \mathbf{g}(\mathbf{x}) - \mathbf{f} \cdot \mathbf{g}(\mathbf{y})| \leq$$

$$(2 + |\mathbf{g}(\mathbf{x})| + |\mathbf{f}(\mathbf{x})|) [|\mathbf{g}(\mathbf{x}) - \mathbf{g}(\mathbf{y})| + |\mathbf{f}(\mathbf{x}) - \mathbf{f}(\mathbf{y})|]$$
$$< (2 + |\mathbf{g}(\mathbf{x})| + |\mathbf{f}(\mathbf{x})|) \left(\frac{\varepsilon}{2(2 + |\mathbf{g}(\mathbf{x})| + |\mathbf{f}(\mathbf{x})|)} + \frac{\varepsilon}{2(2 + |\mathbf{g}(\mathbf{x})| + |\mathbf{f}(\mathbf{x})|)} \right) = \varepsilon.$$

This proves the first part of 2.) To obtain the second part, let δ_1 be as described above and let $\delta_0 > 0$ be such that for $|\mathbf{x} - \mathbf{y}| < \delta_0$,

$$|g(\mathbf{x}) - g(\mathbf{y})| < |g(\mathbf{x})| / 2$$

and so by the triangle inequality,

$$-|g(\mathbf{x})| / 2 \leq |g(\mathbf{y})| - |g(\mathbf{x})| \leq |g(\mathbf{x})| / 2$$

which implies $|g(\mathbf{y})| \geq |g(\mathbf{x})| / 2$, and $|g(\mathbf{y})| < 3 |g(\mathbf{x})| / 2$.

Then if $|\mathbf{x} - \mathbf{y}| < \min(\delta_0, \delta_1)$,

$$\left| \frac{\mathbf{f}(\mathbf{x})}{g(\mathbf{x})} - \frac{\mathbf{f}(\mathbf{y})}{g(\mathbf{y})} \right| = \left| \frac{\mathbf{f}(\mathbf{x}) g(\mathbf{y}) - \mathbf{f}(\mathbf{y}) g(\mathbf{x})}{g(\mathbf{x}) g(\mathbf{y})} \right|$$
$$\leq \frac{|\mathbf{f}(\mathbf{x}) g(\mathbf{y}) - \mathbf{f}(\mathbf{y}) g(\mathbf{x})|}{\left(\frac{|g(\mathbf{x})|^2}{2} \right)}$$
$$= \frac{2 |\mathbf{f}(\mathbf{x}) g(\mathbf{y}) - \mathbf{f}(\mathbf{y}) g(\mathbf{x})|}{|g(\mathbf{x})|^2}$$

$$\begin{aligned}
&\leq \frac{2}{|g(\mathbf{x})|^2}\left[|\mathbf{f}(\mathbf{x})\, g(\mathbf{y}) - \mathbf{f}(\mathbf{y})\, g(\mathbf{y}) + \mathbf{f}(\mathbf{y})\, g(\mathbf{y}) - \mathbf{f}(\mathbf{y})\, g(\mathbf{x})|\right] \\
&\leq \frac{2}{|g(\mathbf{x})|^2}\left[|g(\mathbf{y})|\,|\mathbf{f}(\mathbf{x}) - \mathbf{f}(\mathbf{y})| + |\mathbf{f}(\mathbf{y})|\,|g(\mathbf{y}) - g(\mathbf{x})|\right] \\
&\leq \frac{2}{|g(\mathbf{x})|^2}\left[\frac{3}{2}|g(\mathbf{x})|\,|\mathbf{f}(\mathbf{x}) - \mathbf{f}(\mathbf{y})| + (1 + |\mathbf{f}(\mathbf{x})|)\,|g(\mathbf{y}) - g(\mathbf{x})|\right] \\
&\leq \frac{2}{|g(\mathbf{x})|^2}\left(1 + 2|\mathbf{f}(\mathbf{x})| + 2|g(\mathbf{x})|\right)\left[|\mathbf{f}(\mathbf{x}) - \mathbf{f}(\mathbf{y})| + |g(\mathbf{y}) - g(\mathbf{x})|\right] \\
&\equiv M\left[|\mathbf{f}(\mathbf{x}) - \mathbf{f}(\mathbf{y})| + |g(\mathbf{y}) - g(\mathbf{x})|\right]
\end{aligned}$$

where M is defined by

$$M \equiv \frac{2}{|g(\mathbf{x})|^2}\left(1 + 2|\mathbf{f}(\mathbf{x})| + 2|g(\mathbf{x})|\right)$$

Now let δ_2 be such that if $|\mathbf{x} - \mathbf{y}| < \delta_2$, then

$$|\mathbf{f}(\mathbf{x}) - \mathbf{f}(\mathbf{y})| < \frac{\varepsilon}{2}M^{-1}$$

and let δ_3 be such that if $|\mathbf{x} - \mathbf{y}| < \delta_3$, then

$$|g(\mathbf{y}) - g(\mathbf{x})| < \frac{\varepsilon}{2}M^{-1}.$$

Then if $0 < \delta \leq \min(\delta_0, \delta_1, \delta_2, \delta_3)$, and $|\mathbf{x} - \mathbf{y}| < \delta$, everything holds and

$$\begin{aligned}
\left|\frac{\mathbf{f}(\mathbf{x})}{g(\mathbf{x})} - \frac{\mathbf{f}(\mathbf{y})}{g(\mathbf{y})}\right| &\leq M\left[|\mathbf{f}(\mathbf{x}) - \mathbf{f}(\mathbf{y})| + |g(\mathbf{y}) - g(\mathbf{x})|\right] \\
&< M\left[\frac{\varepsilon}{2}M^{-1} + \frac{\varepsilon}{2}M^{-1}\right] = \varepsilon.
\end{aligned}$$

This completes the proof of the second part of 2.)

Note that in these proofs no effort is made to find some sort of "best" δ. The problem is one which has a yes or a no answer. Either it is or it is not continuous.

Now consider 3.). If $\mathbf{f}$ is continuous at $\mathbf{x}$, $\mathbf{f}(\mathbf{x}) \in D(\mathbf{g})$, and $\mathbf{g}$ is continuous at $\mathbf{f}(\mathbf{x})$,then $\mathbf{g}\circ\mathbf{f}$ is continuous at $\mathbf{x}$. Let $\varepsilon > 0$ be given. Then there exists $\eta > 0$ such that if $|\mathbf{y} - \mathbf{f}(\mathbf{x})| < \eta$ and $\mathbf{y} \in D(\mathbf{g})$, it follows that $|\mathbf{g}(\mathbf{y}) - \mathbf{g}(\mathbf{f}(\mathbf{x}))| < \varepsilon$. From continuity of $\mathbf{f}$ at $\mathbf{x}$, there exists $\delta > 0$ such that if $|\mathbf{x} - \mathbf{z}| < \delta$ and $z \in D(\mathbf{f})$, then $|\mathbf{f}(\mathbf{z}) - \mathbf{f}(\mathbf{x})| < \eta$. Then if $|\mathbf{x} - \mathbf{z}| < \delta$ and $\mathbf{z} \in D(\mathbf{g} \circ \mathbf{f}) \subseteq D(\mathbf{f})$, all the above hold and so

$$|\mathbf{g}(\mathbf{f}(\mathbf{z})) - \mathbf{g}(\mathbf{f}(\mathbf{x}))| < \varepsilon.$$

This proves part 3.)

To verify part 4.), let $\varepsilon > 0$ be given and let $\delta = \varepsilon$. Then if $||\mathbf{x} - \mathbf{y}|| < \delta$, the triangle inequality implies

$$\begin{aligned}
|\mathbf{f}(\mathbf{x}) - \mathbf{f}(\mathbf{y})| &= |\,||\mathbf{x}|| - ||\mathbf{y}||\,| \\
&\leq ||\mathbf{x} - \mathbf{y}|| < \delta = \varepsilon.
\end{aligned}$$

This proves part 4.)

Next consider 5.) Suppose first $\mathbf{f}$ is continuous. Let U be open and let $\mathbf{x} \in \mathbf{f}^{-1}(U)$. This means $\mathbf{f}(\mathbf{x}) \in U$. Since U is open, there exists $\varepsilon > 0$ such that $B(\mathbf{f}(\mathbf{x}), \varepsilon) \subseteq U$. By

continuity, there exists $\delta > 0$ such that if $\mathbf{y} \in B(\mathbf{x}, \delta)$, then $\mathbf{f}(\mathbf{y}) \in B(\mathbf{f}(\mathbf{x}), \varepsilon)$ and so this shows $B(\mathbf{x},\delta) \subseteq \mathbf{f}^{-1}(U)$ which implies $\mathbf{f}^{-1}(U)$ is open since $\mathbf{x}$ is an arbitrary point of $\mathbf{f}^{-1}(U)$. Next suppose the condition about inverse images of open sets are open. Then apply this condition to the open set $B(\mathbf{f}(\mathbf{x}), \varepsilon)$. The condition says $\mathbf{f}^{-1}(B(\mathbf{f}(\mathbf{x}), \varepsilon))$ is open and since $\mathbf{x} \in \mathbf{f}^{-1}(B(\mathbf{f}(\mathbf{x}), \varepsilon))$, it follows $\mathbf{x}$ is an interior point of $\mathbf{f}^{-1}(B(\mathbf{f}(\mathbf{x}), \varepsilon))$ so there exists $\delta > 0$ such that $B(\mathbf{x}, \delta) \subseteq \mathbf{f}^{-1}(B(\mathbf{f}(\mathbf{x}), \varepsilon))$. This says $\mathbf{f}(B(\mathbf{x}, \delta)) \subseteq B(\mathbf{f}(\mathbf{x}), \varepsilon)$. In other words, whenever $||\mathbf{y} - \mathbf{x}|| < \delta, ||\mathbf{f}(\mathbf{y}) - \mathbf{f}(\mathbf{x})|| < \varepsilon$ which is the condition for continuity at the point $\mathbf{x}$. Since $\mathbf{x}$ is arbitrary, this proves the theorem.

5.1 Continuity And The Limit Of A Sequence

There is a very useful way of thinking of continuity in terms of limits of sequences found in the following theorem. In words, it says a function is continuous if it takes convergent sequences to convergent sequences whenever possible.

Theorem 5.1.1 *A function $\mathbf{f} : D(\mathbf{f}) \to W$ is continuous at $\mathbf{x} \in D(\mathbf{f})$ if and only if, whenever $\mathbf{x}_n \to \mathbf{x}$ with $\mathbf{x}_n \in D(\mathbf{f})$, it follows $\mathbf{f}(\mathbf{x}_n) \to \mathbf{f}(\mathbf{x})$.*

Proof: Suppose first that $\mathbf{f}$ is continuous at $\mathbf{x}$ and let $\mathbf{x}_n \to \mathbf{x}$. Let $\varepsilon > 0$ be given. By continuity, there exists $\delta > 0$ such that if $||\mathbf{y} - \mathbf{x}|| < \delta$, then $||\mathbf{f}(\mathbf{x}) - \mathbf{f}(\mathbf{y})|| < \varepsilon$. However, there exists n_δ such that if $n \geq n_\delta$, then $||\mathbf{x}_n - \mathbf{x}|| < \delta$ and so for all n this large,

$$||\mathbf{f}(\mathbf{x}) - \mathbf{f}(\mathbf{x}_n)|| < \varepsilon$$

which shows $\mathbf{f}(\mathbf{x}_n) \to \mathbf{f}(\mathbf{x})$.

Now suppose the condition about taking convergent sequences to convergent sequences holds at $\mathbf{x}$. Suppose $\mathbf{f}$ fails to be continuous at $\mathbf{x}$. Then there exists $\varepsilon > 0$ and $\mathbf{x}_n \in D(\mathbf{f})$ such that $||\mathbf{x} - \mathbf{x}_n|| < \frac{1}{n}$, yet

$$||\mathbf{f}(\mathbf{x}) - \mathbf{f}(\mathbf{x}_n)|| \geq \varepsilon.$$

But this is clearly a contradiction because, although $\mathbf{x}_n \to \mathbf{x}$, $\mathbf{f}(\mathbf{x}_n)$ fails to converge to $\mathbf{f}(\mathbf{x})$. It follows $\mathbf{f}$ must be continuous after all. This proves the theorem.

Theorem 5.1.2 *Suppose $\mathbf{f} : D(\mathbf{f}) \to \mathbb{R}$ is continuous at $\mathbf{x} \in D(\mathbf{f})$ and suppose $||\mathbf{f}(\mathbf{x}_n)|| \leq l$ $(\geq l)$ where $\{\mathbf{x}_n\}$ is a sequence of points of $D(\mathbf{f})$ which converges to $\mathbf{x}$. Then $||\mathbf{f}(\mathbf{x})|| \leq l$ $(\geq l)$.*

Proof: Since $||\mathbf{f}(\mathbf{x}_n)|| \leq l$ and $\mathbf{f}$ is continuous at $\mathbf{x}$, it follows from the triangle inequality, Theorem 4.1.8 and Theorem 5.1.1,

$$||\mathbf{f}(\mathbf{x})|| = \lim_{n\to\infty} ||\mathbf{f}(\mathbf{x}_n)|| \leq l.$$

The other case is entirely similar. This proves the theorem.

Another very useful idea involves the automatic continuity of the inverse function under certain conditions.

Theorem 5.1.3 *Let K be a sequentially compact set and suppose $\mathbf{f} : K \to \mathbf{f}(K)$ is continuous and one to one. Then $\mathbf{f}^{-1}$ must also be continuous.*

Proof: Suppose $\mathbf{f}(\mathbf{k}_n) \to \mathbf{f}(\mathbf{k})$. Does it follow $\mathbf{k}_n \to \mathbf{k}$? If this does not happen, then there exists $\varepsilon > 0$ and a subsequence still denoted as $\{\mathbf{k}_n\}$ such that

$$|\mathbf{k}_n - \mathbf{k}| \geq \varepsilon \tag{5.1}$$

Now since K is compact, there exists a further subsequence, still denoted as $\{\mathbf{k}_n\}$ such that

$$\mathbf{k}_n \to \mathbf{k}' \in K$$

However, the continuity of $\mathbf{f}$ requires

$$\mathbf{f}(\mathbf{k}_n) \to \mathbf{f}(\mathbf{k}')$$

and so $\mathbf{f}(\mathbf{k}') = \mathbf{f}(\mathbf{k})$. Since $\mathbf{f}$ is one to one, this requires $\mathbf{k}' = \mathbf{k}$, a contradiction to 5.1. This proves the theorem.

5.2 The Extreme Values Theorem

The extreme values theorem says continuous functions achieve their maximum and minimum provided they are defined on a sequentially compact set.

The next theorem is known as the max min theorem or extreme value theorem.

Theorem 5.2.1 *Let $K \subseteq \mathbb{F}^n$ be sequentially compact. Thus K is closed and bounded, and let $f : K \to \mathbb{R}$ be continuous. Then f achieves its maximum and its minimum on K. This means there exist, $\mathbf{x}_1, \mathbf{x}_2 \in K$ such that for all $\mathbf{x} \in K$,*

$$f(\mathbf{x}_1) \leq f(\mathbf{x}) \leq f(\mathbf{x}_2).$$

Proof: Let $\lambda = \sup\{f(\mathbf{x}) : \mathbf{x} \in K\}$. Next let $\{\lambda_k\}$ be an increasing sequence which converges to λ but each $\lambda_k < \lambda$. Therefore, for each k, there exists $\mathbf{x}_k \in K$ such that

$$f(\mathbf{x}_k) > \lambda_k.$$

Since K is sequentially compact, there exists a subsequence, $\{\mathbf{x}_{k_l}\}$ such that $\lim_{l\to\infty} \mathbf{x}_{k_l} = \mathbf{x} \in K$. Then by continuity of f,

$$f(\mathbf{x}) = \lim_{l\to\infty} f(\mathbf{x}_{k_l}) \geq \lim_{l\to\infty} \lambda_{k_l} = \lambda$$

which shows f achieves its maximum on K. To see it achieves its minimum, you could repeat the argument with a minimizing sequence or else you could consider $-f$ and apply what was just shown to $-f$, $-f$ having its minimum when f has its maximum. This proves the theorem.

5.3 Connected Sets

Stated informally, connected sets are those which are in one piece. In order to define what is meant by this, I will first consider what it means for a set to not be in one piece.

Definition 5.3.1 *Let A be a nonempty subset of V a normed vector space. Then $\overline{A}$ is defined to be the intersection of all closed sets which contain A. Note the whole space, V is one such closed set which contains A.*

Lemma 5.3.2 *Let A be a nonempty set in a normed vector space V. Then $\overline{A}$ is a closed set and*

$$\overline{A} = A \cup A'$$

where A' denotes the set of limit points of A.

Proof: First of all, denote by $\mathcal{C}$ the set of closed sets which contain A. Then

$$\overline{A} = \cap\mathcal{C}$$

and this will be closed if its complement is open. However,

$$\overline{A}^C = \cup\left\{H^C : H \in \mathcal{C}\right\}.$$

Each H^C is open and so the union of all these open sets must also be open. This is because if $\mathbf{x}$ is in this union, then it is in at least one of them. Hence it is an interior point of that one. But this implies it is an interior point of the union of them all which is an even larger set. Thus $\overline{A}$ is closed.

The interesting part is the next claim. First note that from the definition, $A \subseteq \overline{A}$ so if $\mathbf{x} \in A$, then $\mathbf{x} \in \overline{A}$. Now consider $\mathbf{y} \in A'$ but $\mathbf{y} \notin A$. If $\mathbf{y} \notin \overline{A}$, a closed set, then there exists $B(\mathbf{y}, r) \subseteq \overline{A}^C$. Thus $\mathbf{y}$ cannot be a limit point of A, a contradiction. Therefore,

$$A \cup A' \subseteq \overline{A}$$

Next suppose $\mathbf{x} \in \overline{A}$ and suppose $\mathbf{x} \notin A$. Then if $B(\mathbf{x}, r)$ contains no points of A different than $\mathbf{x}$, since $\mathbf{x}$ itself is not in A, it would follow that $B(\mathbf{x},r) \cap A = \emptyset$ and so recalling that open balls are open, $B(\mathbf{x}, r)^C$ is a closed set containing A so from the definition, it also contains $\overline{A}$ which is contrary to the assertion that $\mathbf{x} \in \overline{A}$. Hence if $\mathbf{x} \notin A$, then $\mathbf{x} \in A'$ and so

$$A \cup A' \supseteq \overline{A}$$

This proves the lemma.

Now that the closure of a set has been defined it is possible to define what is meant by a set being separated.

Definition 5.3.3 *A set, S in a normed vector space is separated if there exist sets, A, B such that*

$$S = A \cup B,\ A, B \neq \emptyset,\ \text{and } \overline{A} \cap B = \overline{B} \cap A = \emptyset.$$

In this case, the sets A and B are said to separate S. A set is connected if it is not separated. Remember $\overline{A}$ denotes the closure of the set A.

Note that the concept of connected sets is defined in terms of what it is not. This makes it somewhat difficult to understand. One of the most important theorems about connected sets is the following.

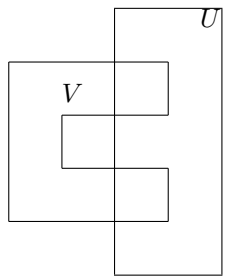
Theorem 5.3.4 *Suppose U and V are connected sets having nonempty intersection. Then $U \cup V$ is also connected.*

Proof: Suppose $U \cup V = A \cup B$ where $\overline{A} \cap B = \overline{B} \cap A = \emptyset$. Consider the sets, $A \cap U$ and $B \cap U$. Since

$$\overline{(A \cap U)} \cap (B \cap U) = (A \cap U) \cap \left(\overline{B \cap U}\right) = \emptyset,$$

It follows one of these sets must be empty since otherwise, U would be separated. It follows that U is contained in either A or B. Similarly, V must be contained in either A or B. Since U and V have nonempty intersection, it follows that both V and U are contained in one of the sets, A, B. Therefore, the other must be empty and this shows $U \cup V$ cannot be separated and is therefore, connected.

The intersection of connected sets is not necessarily connected as is shown by the following picture.



Theorem 5.3.5 *Let $\mathbf{f}: X \to Y$ be continuous where Y is a normed vector space and X is connected. Then $\mathbf{f}(X)$ is also connected.*

Proof: To do this you show $\mathbf{f}(X)$ is not separated. Suppose to the contrary that $\mathbf{f}(X) = A \cup B$ where A and B separate $\mathbf{f}(X)$. Then consider the sets, $\mathbf{f}^{-1}(A)$ and $\mathbf{f}^{-1}(B)$. If $\mathbf{z} \in \mathbf{f}^{-1}(B)$, then $\mathbf{f}(\mathbf{z}) \in B$ and so $\mathbf{f}(\mathbf{z})$ is not a limit point of A. Therefore, there exists an open set, U containing $\mathbf{f}(\mathbf{z})$ such that $U \cap A = \emptyset$. But then, the continuity of $\mathbf{f}$ and Theorem 5.0.2 implies that $\mathbf{f}^{-1}(U)$ is an open set containing $\mathbf{z}$ such that $\mathbf{f}^{-1}(U) \cap \mathbf{f}^{-1}(A) = \emptyset$. Therefore, $\mathbf{f}^{-1}(B)$ contains no limit points of $\mathbf{f}^{-1}(A)$. Similar reasoning implies $\mathbf{f}^{-1}(A)$ contains no limit points of $\mathbf{f}^{-1}(B)$. It follows that X is separated by $\mathbf{f}^{-1}(A)$ and $\mathbf{f}^{-1}(B)$, contradicting the assumption that X was connected.

An arbitrary set can be written as a union of maximal connected sets called connected components. This is the concept of the next definition.

Definition 5.3.6 *Let S be a set and let $\mathbf{p} \in S$. Denote by $C_{\mathbf{p}}$ the union of all connected subsets of S which contain $\mathbf{p}$. This is called the connected component determined by $\mathbf{p}$.*

Theorem 5.3.7 *Let $C_{\mathbf{p}}$ be a connected component of a set S in a normed vector space. Then $C_{\mathbf{p}}$ is a connected set and if $C_{\mathbf{p}} \cap C_{\mathbf{q}} \neq \emptyset$, then $C_{\mathbf{p}} = C_{\mathbf{q}}$.*

Proof: Let $\mathcal{C}$ denote the connected subsets of S which contain $\mathbf{p}$. If $C_{\mathbf{p}} = A \cup B$ where

$$\overline{A} \cap B = \overline{B} \cap A = \emptyset,$$

then $\mathbf{p}$ is in one of A or B. Suppose without loss of generality $\mathbf{p} \in A$. Then every set of $\mathcal{C}$ must also be contained in A since otherwise, as in Theorem 5.3.4, the set would be separated. But this implies B is empty. Therefore, $C_{\mathbf{p}}$ is connected. From this, and Theorem 5.3.4, the second assertion of the theorem is proved.

This shows the connected components of a set are equivalence classes and partition the set.

A set, I is an interval in $\mathbb{R}$ if and only if whenever $x, y \in I$ then $(x, y) \subseteq I$. The following theorem is about the connected sets in $\mathbb{R}$.

Theorem 5.3.8 *A set, C in $\mathbb{R}$ is connected if and only if C is an interval.*

Proof: Let C be connected. If C consists of a single point, p, there is nothing to prove. The interval is just $[p, p]$. Suppose $p < q$ and $p, q \in C$. You need to show $(p, q) \subseteq C$. If

$$x \in (p, q) \setminus C$$

let $C \cap (-\infty, x) \equiv A$, and $C \cap (x, \infty) \equiv B$. Then $C = A \cup B$ and the sets, A and B separate C contrary to the assumption that C is connected.

Conversely, let I be an interval. Suppose I is separated by A and B. Pick $x \in A$ and $y \in B$. Suppose without loss of generality that $x < y$. Now define the set,

$$S \equiv \{t \in [x, y] : [x, t] \subseteq A\}$$

and let l be the least upper bound of S. Then $l \in \overline{A}$ so $l \notin B$ which implies $l \in A$. But if $l \notin \overline{B}$, then for some $\delta > 0$,

$$(l, l + \delta) \cap B = \emptyset$$

contradicting the definition of l as an upper bound for S. Therefore, $l \in \overline{B}$ which implies $l \notin A$ after all, a contradiction. It follows I must be connected.

This yields a generalization of the intermediate value theorem from one variable calculus.

Corollary 5.3.9 *Let E be a connected set in a normed vector space and suppose $f : E \to \mathbb{R}$ and that $y \in (f(e_1), f(e_2))$ where $e_i \in E$. Then there exists $e \in E$ such that $f(e) = y$.*

Proof: From Theorem 5.3.5, $f(E)$ is a connected subset of $\mathbb{R}$. By Theorem 5.3.8 $f(E)$ must be an interval. In particular, it must contain y. This proves the corollary.

The following theorem is a very useful description of the open sets in $\mathbb{R}$.

Theorem 5.3.10 *Let U be an open set in $\mathbb{R}$. Then there exist countably many disjoint open sets, $\{(a_i, b_i)\}_{i=1}^{\infty}$ such that $U = \cup_{i=1}^{\infty} (a_i, b_i)$.*

Proof: Let $p \in U$ and let $z \in C_p$, the connected component determined by p. Since U is open, there exists, $\delta > 0$ such that $(z - \delta, z + \delta) \subseteq U$. It follows from Theorem 5.3.4 that

$$(z - \delta, z + \delta) \subseteq C_p.$$

This shows C_p is open. By Theorem 5.3.8, this shows C_p is an open interval, (a, b) where $a, b \in [-\infty, \infty]$. There are therefore at most countably many of these connected components because each must contain a rational number and the rational numbers are countable. Denote by $\{(a_i, b_i)\}_{i=1}^{\infty}$ the set of these connected components. This proves the theorem.

Definition 5.3.11 *A set E in a normed vector space is arcwise connected if for any two points, $\mathbf{p}, \mathbf{q} \in E$, there exists a closed interval, $[a, b]$ and a continuous function, $\gamma : [a, b] \to E$ such that $\boldsymbol{\gamma}(a) = p$ and $\boldsymbol{\gamma}(b) = q$.*

An example of an arcwise connected topological space would be any subset of $\mathbb{R}^n$ which is the continuous image of an interval. Arcwise connected is not the same as connected. A well known example is the following.

$$\left\{\left(x, \sin\frac{1}{x}\right) : x \in (0, 1]\right\} \cup \{(0, y) : y \in [-1, 1]\} \tag{5.2}$$

You can verify that this set of points in the normed vector space $\mathbb{R}^2$ is not arcwise connected but is connected.

Lemma 5.3.12 *In a normed vector space, $B(\mathbf{z}, r)$ is arcwise connected.*

Proof: This is easy from the convexity of the set. If $\mathbf{x}, \mathbf{y} \in B(\mathbf{z}, r)$, then let $\boldsymbol{\gamma}(t) = \mathbf{x} + t(\mathbf{y} - \mathbf{x})$ for $t \in [0, 1]$.

$$\begin{aligned} ||\mathbf{x} + t(\mathbf{y} - \mathbf{x}) - \mathbf{z}|| &= ||(1 - t)(\mathbf{x} - \mathbf{z}) + t(\mathbf{y} - \mathbf{z})|| \\ &\leq (1 - t)||\mathbf{x} - \mathbf{z}|| + t||\mathbf{y} - \mathbf{z}|| \\ &< (1 - t)r + tr = r \end{aligned}$$

showing $\boldsymbol{\gamma}(t)$ stays in $B(\mathbf{z}, r)$.

Proposition 5.3.13 *If X is arcwise connected, then it is connected.*

Proof: Let X be an arcwise connected set and suppose it is separated. Then $X = A \cup B$ where A, B are two separated sets. Pick $\mathbf{p} \in A$ and $\mathbf{q} \in B$. Since X is given to be arcwise connected, there must exist a continuous function $\gamma : [a,b] \to X$ such that $\gamma(a) = \mathbf{p}$ and $\gamma(b) = \mathbf{q}$. But then $\gamma([a,b]) = (\gamma([a,b]) \cap A) \cup (\gamma([a,b]) \cap B)$ and the two sets, $\gamma([a,b]) \cap A$ and $\gamma([a,b]) \cap B$ are separated thus showing that $\gamma([a,b])$ is separated and contradicting Theorem 5.3.8 and Theorem 5.3.5. It follows that X must be connected as claimed.

Theorem 5.3.14 *Let U be an open subset of a normed vector space. Then U is arcwise connected if and only if U is connected. Also the connected components of an open set are open sets.*

Proof: By Proposition 5.3.13 it is only necessary to verify that if U is connected and open in the context of this theorem, then U is arcwise connected. Pick $\mathbf{p} \in U$. Say $\mathbf{x} \in U$ satisfies $\mathcal{P}$ if there exists a continuous function, $\boldsymbol{\gamma} : [a,b] \to U$ such that $\gamma(a) = \mathbf{p}$ and $\boldsymbol{\gamma}(\mathbf{b}) = \mathbf{x}$.

$$A \equiv \{\mathbf{x} \in U \text{ such that } \mathbf{x} \text{ satisfies } \mathcal{P}.\}$$

If $\mathbf{x} \in A$, then Lemma 5.3.12 implies $B(\mathbf{x}, r) \subseteq U$ is arcwise connected for small enough r. Thus letting $\mathbf{y} \in B(\mathbf{x}, r)$, there exist intervals, $[a,b]$ and $[c,d]$ and continuous functions having values in U, $\boldsymbol{\gamma}, \boldsymbol{\eta}$ such that $\boldsymbol{\gamma}(a) = \mathbf{p}, \boldsymbol{\gamma}(b) = \mathbf{x}, \boldsymbol{\eta}(c) = \mathbf{x}$, and $\eta(d) = \mathbf{y}$. Then let $\boldsymbol{\gamma}_1 : [a, b+d-c] \to U$ be defined as

$$\boldsymbol{\gamma}_1(t) \equiv \begin{cases} \boldsymbol{\gamma}(t) & \text{if } t \in [a,b] \\ \boldsymbol{\eta}(t+c-b) & \text{if } t \in [b, b+d-c] \end{cases}$$

Then it is clear that $\boldsymbol{\gamma}_1$ is a continuous function mapping $\mathbf{p}$ to $\mathbf{y}$ and showing that $B(\mathbf{x}, r) \subseteq A$. Therefore, A is open. $A \neq \emptyset$ because since U is open there is an open set, $B(\mathbf{p}, \delta)$ containing $\mathbf{p}$ which is contained in U and is arcwise connected.

Now consider $B \equiv U \setminus A$. I claim this is also open. If B is not open, there exists a point $\mathbf{z} \in B$ such that every open set containing $\mathbf{z}$ is not contained in B. Therefore, letting $B(\mathbf{z}, \delta)$ be such that $\mathbf{z} \in B(\mathbf{z}, \delta) \subseteq U$, there exist points of A contained in $B(\mathbf{z}, \delta)$. But then, a repeat of the above argument shows $z \in A$ also. Hence B is open and so if $B \neq \emptyset$, then $U = B \cup A$ and so U is separated by the two sets, B and A contradicting the assumption that U is connected.

It remains to verify the connected components are open. Let $\mathbf{z} \in C_{\mathbf{p}}$ where $C_{\mathbf{p}}$ is the connected component determined by $\mathbf{p}$. Then picking $B(\mathbf{z}, \delta) \subseteq U$, $C_{\mathbf{p}} \cup B(\mathbf{z}, \delta)$ is connected and contained in U and so it must also be contained in $C_{\mathbf{p}}$. Thus $\mathbf{z}$ is an interior point of $C_{\mathbf{p}}$. This proves the theorem.

As an application, consider the following corollary.

Corollary 5.3.15 *Let $f : \Omega \to \mathbb{Z}$ be continuous where Ω is a connected open set in a normed vector space. Then f must be a constant.*

Proof: Suppose not. Then it achieves two different values, k and $l \neq k$. Then $\Omega = f^{-1}(l) \cup f^{-1}(\{m \in \mathbb{Z} : m \neq l\})$ and these are disjoint nonempty open sets which separate Ω. To see they are open, note

$$f^{-1}(\{m \in \mathbb{Z} : m \neq l\}) = f^{-1}\left(\cup_{m \neq l}\left(m - \frac{1}{6}, m + \frac{1}{6}\right)\right)$$

which is the inverse image of an open set while $f^{-1}(l) = f^{-1}\left(\left(l - \frac{1}{6}, l + \frac{1}{6}\right)\right)$ also an open set.

5.4 Uniform Continuity

The concept of uniform continuity is also similar to the one dimensional concept.

Definition 5.4.1 *Let* $\mathbf{f}$ *be a function. Then* $\mathbf{f}$ *is uniformly continuous if for every* $\varepsilon > 0$, *there exists a* δ ***depending only on*** ε *such that if* $||\mathbf{x} - \mathbf{y}|| < \delta$ *then* $||\mathbf{f}(\mathbf{x}) - \mathbf{f}(\mathbf{y})|| < \varepsilon$.

Theorem 5.4.2 *Let* $\mathbf{f} : K \to \mathbb{F}$ *be continuous where* K *is a sequentially compact set in* $\mathbb{F}^n$ *or more generally a normed vector space. Then* $\mathbf{f}$ *is uniformly continuous on* K.

Proof: If this is not true, there exists $\varepsilon > 0$ such that for every $\delta > 0$ there exists a pair of points, $\mathbf{x}_\delta$ and $\mathbf{y}_\delta$ such that even though $||\mathbf{x}_\delta - \mathbf{y}_\delta|| < \delta$, $||\mathbf{f}(\mathbf{x}_\delta) - \mathbf{f}(\mathbf{y}_\delta)|| \geq \varepsilon$. Taking a succession of values for δ equal to $1, 1/2, 1/3, \cdots$, and letting the exceptional pair of points for $\delta = 1/n$ be denoted by $\mathbf{x}_n$ and $\mathbf{y}_n$,

$$||\mathbf{x}_n - \mathbf{y}_n|| < \frac{1}{n}, ||\mathbf{f}(\mathbf{x}_n) - \mathbf{f}(\mathbf{y}_n)|| \geq \varepsilon.$$

Now since K is sequentially compact, there exists a subsequence, $\{\mathbf{x}_{n_k}\}$ such that $\mathbf{x}_{n_k} \to \mathbf{z} \in K$. Now $n_k \geq k$ and so

$$||\mathbf{x}_{n_k} - \mathbf{y}_{n_k}|| < \frac{1}{k}.$$

Hence

$$\begin{aligned} ||\mathbf{y}_{n_k} - \mathbf{z}|| &\leq ||\mathbf{y}_{n_k} - \mathbf{x}_{n_k}|| + ||\mathbf{x}_{n_k} - \mathbf{z}|| \\ &< \frac{1}{k} + ||\mathbf{x}_{n_k} - \mathbf{z}|| \end{aligned}$$

Consequently, $\mathbf{y}_{n_k} \to \mathbf{z}$ also. By continuity of $\mathbf{f}$ and Theorem 5.1.2,

$$0 = ||\mathbf{f}(\mathbf{z}) - \mathbf{f}(\mathbf{z})|| = \lim_{k \to \infty} ||\mathbf{f}(\mathbf{x}_{n_k}) - \mathbf{f}(\mathbf{y}_{n_k})|| \geq \varepsilon,$$

an obvious contradiction. Therefore, the theorem must be true.

Recall the closed and bounded subsets of $\mathbb{F}^n$ are those which are sequentially compact.

5.5 Sequences And Series Of Functions

Now it is an easy matter to consider sequences of vector valued functions.

Definition 5.5.1 *A sequence of functions is a map defined on* $\mathbb{N}$ *or some set of integers larger than or equal to a given integer,* m *which has values which are functions. It is written in the form* $\{\mathbf{f}_n\}_{n=m}^{\infty}$ *where* $\mathbf{f}_n$ *is a function. It is assumed also that the domain of all these functions is the same.*

Here the functions have values in some normed vector space.

The definition of uniform convergence is exactly the same as earlier only now it is not possible to draw representative pictures so easily.

Definition 5.5.2 *Let* $\{\mathbf{f}_n\}$ *be a sequence of functions. Then the sequence converges pointwise to a function* $\mathbf{f}$ *if for all* $\mathbf{x} \in D$, *the domain of the functions in the sequence,*

$$\mathbf{f}(\mathbf{x}) = \lim_{n \to \infty} \mathbf{f}_n(\mathbf{x})$$

Thus you consider for each $\mathbf{x} \in D$ the sequence of numbers $\{\mathbf{f}_n(\mathbf{x})\}$ and if this sequence converges for each $\mathbf{x} \in D$, the thing it converges to is called $\mathbf{f}(\mathbf{x})$.

Definition 5.5.3 *Let $\{\mathbf{f}_n\}$ be a sequence of functions defined on D. Then $\{\mathbf{f}_n\}$ is said to converge uniformly to $\mathbf{f}$ if it converges pointwise to $\mathbf{f}$ and for every $\varepsilon > 0$ there exists N such that for all $n \geq N$*

$$||\mathbf{f}(\mathbf{x}) - \mathbf{f}_n(\mathbf{x})|| < \varepsilon$$

for all $\mathbf{x} \in D$.

Theorem 5.5.4 *Let $\{\mathbf{f}_n\}$ be a sequence of continuous functions defined on D and suppose this sequence converges uniformly to $\mathbf{f}$. Then $\mathbf{f}$ is also continuous on D. If each $\mathbf{f}_n$ is uniformly continuous on D, then $\mathbf{f}$ is also uniformly continuous on D.*

Proof: Let $\varepsilon > 0$ be given and pick $\mathbf{z} \in D$. By uniform convergence, there exists N such that if $n > N$, then for all $\mathbf{x} \in D$,

$$||\mathbf{f}(\mathbf{x}) - \mathbf{f}_n(\mathbf{x})|| < \varepsilon/3. \tag{5.3}$$

Pick such an n. By assumption, $\mathbf{f}_n$ is continuous at $\mathbf{z}$. Therefore, there exists $\delta > 0$ such that if $||\mathbf{z} - \mathbf{x}|| < \delta$ then

$$||\mathbf{f}_n(\mathbf{x}) - \mathbf{f}_n(\mathbf{z})|| < \varepsilon/3.$$

It follows that for $||\mathbf{x} - \mathbf{z}|| < \delta$,

$$\begin{aligned} ||\mathbf{f}(\mathbf{x}) - \mathbf{f}(\mathbf{z})|| &\leq ||\mathbf{f}(\mathbf{x}) - \mathbf{f}_n(\mathbf{x})|| + ||\mathbf{f}_n(\mathbf{x}) - \mathbf{f}_n(\mathbf{z})|| + ||\mathbf{f}_n(\mathbf{z}) - \mathbf{f}(\mathbf{z})|| \\ &< \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon \end{aligned}$$

which shows that since ε was arbitrary, $\mathbf{f}$ is continuous at $\mathbf{z}$.

In the case where each $\mathbf{f}_n$ is uniformly continuous, and using the same $\mathbf{f}_n$ for which 5.3 holds, there exists a $\delta > 0$ such that if $||\mathbf{y} - \mathbf{z}|| < \delta$, then

$$||\mathbf{f}_n(\mathbf{z}) - \mathbf{f}_n(\mathbf{y})|| < \varepsilon/3.$$

Then for $||\mathbf{y} - \mathbf{z}|| < \delta$,

$$\begin{aligned} ||\mathbf{f}(\mathbf{y}) - \mathbf{f}(\mathbf{z})|| &\leq ||\mathbf{f}(\mathbf{y}) - \mathbf{f}_n(\mathbf{y})|| + ||\mathbf{f}_n(\mathbf{y}) - \mathbf{f}_n(\mathbf{z})|| + ||\mathbf{f}_n(\mathbf{z}) - \mathbf{f}(\mathbf{z})|| \\ &< \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon \end{aligned}$$

This shows uniform continuity of $\mathbf{f}$. This proves the theorem.

Definition 5.5.5 *Let $\{\mathbf{f}_n\}$ be a sequence of functions defined on D. Then the sequence is said to be uniformly Cauchy if for every $\varepsilon > 0$ there exists N such that whenever $m, n \geq N$,*

$$||\mathbf{f}_m(\mathbf{x}) - \mathbf{f}_n(\mathbf{x})|| < \varepsilon$$

for all $\mathbf{x} \in D$.

Then the following theorem follows easily.

Theorem 5.5.6 *Let $\{\mathbf{f}_n\}$ be a uniformly Cauchy sequence of functions defined on D having values in a complete normed vector space such as $\mathbb{F}^n$ for example. Then there exists $\mathbf{f}$ defined on D such that $\{\mathbf{f}_n\}$ converges uniformly to $\mathbf{f}$.*

Proof: For each $\mathbf{x} \in D, \{\mathbf{f}_n(\mathbf{x})\}$ is a Cauchy sequence. Therefore, it converges to some vector $\mathbf{f}(\mathbf{x})$. Let $\varepsilon > 0$ be given and let N be such that if $n, m \geq N$,

$$||\mathbf{f}_m(\mathbf{x}) - \mathbf{f}_n(\mathbf{x})|| < \varepsilon/2$$

for all $\mathbf{x} \in D$. Then for any $\mathbf{x} \in D$, pick $n \geq N$ and it follows from Theorem 4.1.8

$$||\mathbf{f}(\mathbf{x}) - \mathbf{f}_n(\mathbf{x})|| = \lim_{m\to\infty} ||\mathbf{f}_m(\mathbf{x}) - \mathbf{f}_n(\mathbf{x})|| \leq \varepsilon/2 < \varepsilon.$$

This proves the theorem.

Corollary 5.5.7 *Let $\{\mathbf{f}_n\}$ be a uniformly Cauchy sequence of functions continuous on D having values in a complete normed vector space like $\mathbb{F}^n$. Then there exists $\mathbf{f}$ defined on D such that $\{\mathbf{f}_n\}$ converges uniformly to $\mathbf{f}$ and $\mathbf{f}$ is continuous. Also, if each $\mathbf{f}_n$ is uniformly continuous, then so is $\mathbf{f}$.*

Proof: This follows from Theorem 5.5.6 and Theorem 5.5.4. This proves the Corollary.

Here is one more fairly obvious theorem.

Theorem 5.5.8 *Let $\{\mathbf{f}_n\}$ be a sequence of functions defined on D having values in a complete normed vector space like $\mathbb{F}^n$. Then it converges pointwise if and only if the sequence $\{\mathbf{f}_n(\mathbf{x})\}$ is a Cauchy sequence for every $\mathbf{x} \in D$. It converges uniformly if and only if $\{\mathbf{f}_n\}$ is a uniformly Cauchy sequence.*

Proof: If the sequence converges pointwise, then by Theorem 4.4.3 the sequence $\{\mathbf{f}_n(\mathbf{x})\}$ is a Cauchy sequence for each $\mathbf{x} \in D$. Conversely, if $\{\mathbf{f}_n(\mathbf{x})\}$ is a Cauchy sequence for each $\mathbf{x} \in D$, then $\{\mathbf{f}_n(\mathbf{x})\}$ converges for each $\mathbf{x} \in D$ because of completeness.

Now suppose $\{\mathbf{f}_n\}$ is uniformly Cauchy. Then from Theorem 5.5.6 there exists $\mathbf{f}$ such that $\{\mathbf{f}_n\}$ converges uniformly on D to $\mathbf{f}$. Conversely, if $\{\mathbf{f}_n\}$ converges uniformly to $\mathbf{f}$ on D, then if $\varepsilon > 0$ is given, there exists N such that if $n \geq N$,

$$|\mathbf{f}(\mathbf{x}) - \mathbf{f}_n(\mathbf{x})| < \varepsilon/2$$

for every $\mathbf{x} \in D$. Then if $m, n \geq N$ and $\mathbf{x} \in D$,

$$|\mathbf{f}_n(\mathbf{x}) - \mathbf{f}_m(\mathbf{x})| \leq |\mathbf{f}_n(\mathbf{x}) - \mathbf{f}(\mathbf{x})| + |\mathbf{f}(\mathbf{x}) - \mathbf{f}_m(\mathbf{x})| < \varepsilon/2 + \varepsilon/2 = \varepsilon.$$

Thus $\{\mathbf{f}_n\}$ is uniformly Cauchy.

Once you understand sequences, it is no problem to consider series.

Definition 5.5.9 *Let $\{\mathbf{f}_n\}$ be a sequence of functions defined on D. Then*

$$\left(\sum_{k=1}^{\infty} \mathbf{f}_k\right)(\mathbf{x}) \equiv \lim_{n\to\infty} \sum_{k=1}^{n} \mathbf{f}_k(\mathbf{x}) \tag{5.4}$$

whenever the limit exists. Thus there is a new function denoted by

$$\sum_{k=1}^{\infty} \mathbf{f}_k \tag{5.5}$$

and its value at $\mathbf{x}$ is given by the limit of the sequence of partial sums in 5.4. If for all $\mathbf{x} \in D$, the limit in 5.4 exists, then 5.5 is said to converge pointwise. $\sum_{k=1}^{\infty} \mathbf{f}_k$ is said to converge uniformly on D if the sequence of partial sums,

$$\left\{\sum_{k=1}^{n} \mathbf{f}_k\right\}$$

converges uniformly.If the indices for the functions start at some other value than 1, you make the obvious modification to the above definition.

Theorem 5.5.10 *Let $\{\mathbf{f}_n\}$ be a sequence of functions defined on D which have values in a complete normed vector space like $\mathbb{F}^n$. The series $\sum_{k=1}^{\infty}\mathbf{f}_k$ converges pointwise if and only if for each $\varepsilon>0$ and $\mathbf{x}\in D$, there exists $N_{\varepsilon,\mathbf{x}}$ which may depend on $\mathbf{x}$ as well as ε such that when $q>p\geq N_{\varepsilon,\mathbf{x}}$,*

$$\left|\left|\sum_{k=p}^{q}\mathbf{f}_k(\mathbf{x})\right|\right|<\varepsilon$$

The series $\sum_{k=1}^{\infty}\mathbf{f}_k$ converges uniformly on D if for every $\varepsilon>0$ there exists N_ε such that if $q>p\geq N$ then

$$\left|\left|\sum_{k=p}^{q}\mathbf{f}_k(\mathbf{x})\right|\right|<\varepsilon \tag{5.6}$$

for all $\mathbf{x}\in D$.

Proof: The first part follows from Theorem 5.5.8. The second part follows from observing the condition is equivalent to the sequence of partial sums forming a uniformly Cauchy sequence and then by Theorem 5.5.6, these partial sums converge uniformly to a function which is the definition of $\sum_{k=1}^{\infty}\mathbf{f}_k$. This proves the theorem.

Is there an easy way to recognize when 5.6 happens? Yes, there is. It is called the Weierstrass M test.

Theorem 5.5.11 *Let $\{\mathbf{f}_n\}$ be a sequence of functions defined on D having values in a complete normed vector space like $\mathbb{F}^n$. Suppose there exists M_n such that $\sup\{|\mathbf{f}_n(\mathbf{x})|:\mathbf{x}\in D\}<M_n$ and $\sum_{n=1}^{\infty}M_n$ converges. Then $\sum_{n=1}^{\infty}\mathbf{f}_n$ converges uniformly on D.*

Proof: Let $\mathbf{z}\in D$. Then letting $m<n$ and using the triangle inequality

$$\left|\left|\sum_{k=1}^{n}\mathbf{f}_k(\mathbf{z})-\sum_{k=1}^{m}\mathbf{f}_k(\mathbf{z})\right|\right|\leq\sum_{k=m+1}^{n}||\mathbf{f}_k(\mathbf{z})||\leq\sum_{k=m+1}^{\infty}M_k<\varepsilon$$

whenever m is large enough because of the assumption that $\sum_{n=1}^{\infty}M_n$ converges. Therefore, the sequence of partial sums is uniformly Cauchy on D and therefore, converges uniformly to $\sum_{k=1}^{\infty}\mathbf{f}_k$ on D. This proves the theorem.

Theorem 5.5.12 *If $\{\mathbf{f}_n\}$ is a sequence of continuous functions defined on D and $\sum_{k=1}^{\infty}\mathbf{f}_k$ converges uniformly, then the function, $\sum_{k=1}^{\infty}\mathbf{f}_k$ must also be continuous.*

Proof: This follows from Theorem 5.5.4 applied to the sequence of partial sums of the above series which is assumed to converge uniformly to the function, $\sum_{k=1}^{\infty}\mathbf{f}_k$.

5.6 Polynomials

General considerations about what a function is have already been considered earlier. For functions of one variable, the special kind of functions known as a polynomial has a corresponding version when one considers a function of many variables. This is found in the next definition.

Definition 5.6.1 *Let α be an n dimensional multi-index. This means*

$$\boldsymbol{\alpha}=(\alpha_1,\cdots,\alpha_n)$$

where each α_i is a positive integer or zero. Also, let

$$|\boldsymbol{\alpha}| \equiv \sum_{i=1}^{n} |\alpha_i|$$

Then $\mathbf{x}^{\alpha}$ means

$$\mathbf{x}^{\alpha} \equiv x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_3^{\alpha_n}$$

where each $x_j \in \mathbb{F}$. An n dimensional polynomial of degree m is a function of the form

$$p(\mathbf{x}) = \sum_{|\alpha| \leq m} d_{\alpha} \mathbf{x}^{\alpha}.$$

where the d_{α} are complex or real numbers. Rational functions are defined as the quotient of two polynomials. Thus these functions are defined on $\mathbb{F}^n$.

For example, $f(\mathbf{x}) = x_1 x_2^2 + 7x_3^4 x_1$ is a polynomial of degree 5 and

$$\frac{x_1 x_2^2 + 7x_3^4 x_1 + x_2^3}{4x_1^3 x_2^2 + 7x_3^2 x_1 - x_2^3}$$

is a rational function.

Note that in the case of a rational function, the domain of the function might not be all of $\mathbb{F}^n$. For example, if

$$f(\mathbf{x}) = \frac{x_1 x_2^2 + 7x_3^4 x_1 + x_2^3}{x_2^2 + 3x_1^2 - 4},$$

the domain of f would be all complex numbers such that $x_2^2 + 3x_1^2 \neq 4$.

By Theorem 5.0.2 all polynomials are continuous. To see this, note that the function,

$$\pi_k(\mathbf{x}) \equiv x_k$$

is a continuous function because of the inequality

$$|\pi_k(\mathbf{x}) - \pi_k(\mathbf{y})| = |x_k - y_k| \leq |\mathbf{x} - \mathbf{y}|.$$

Polynomials are simple sums of scalars times products of these functions. Similarly, by this theorem, rational functions, quotients of polynomials, are continuous at points where the denominator is non zero. More generally, if V is a normed vector space, consider a V valued function of the form

$$\mathbf{f}(\mathbf{x}) \equiv \sum_{|\alpha| \leq m} \mathbf{d}_{\alpha} \mathbf{x}^{\alpha}$$

where $\mathbf{d}_{\alpha} \in V$, sort of a V valued polynomial. Then such a function is continuous by application of Theorem 5.0.2 and the above observation about the continuity of the functions π_k.

Thus there are lots of examples of continuous functions. However, it is even better than the above discussion indicates. As in the case of a function of one variable, an arbitrary continuous function can typically be approximated uniformly by a polynomial. This is the n dimensional version of the Weierstrass approximation theorem.

5.7 Sequences Of Polynomials, Weierstrass Approximation

Just as an arbitrary continuous function defined on an interval can be approximated uniformly by a polynomial, there exists a similar theorem which is just a generalization of the earlier one which will hold for continuous functions defined on a box or more generally a closed and bounded set. The proof is based on the following lemma.

Lemma 5.7.1 *The following estimate holds for $x \in [0,1]$ and $m \geq 2$.*

$$\sum_{k=0}^{m} \binom{m}{k} (k - mx)^2 x^k (1-x)^{m-k} \leq \frac{1}{4} m$$

Proof: First of all, from the binomial theorem

$$\begin{aligned}
&\sum_{k=0}^{m} \binom{m}{k} k x^k (1-x)^{m-k} \\
= &\sum_{k=1}^{m} \binom{m}{k} k x^k (1-x)^{m-k} \\
= &\, mx \sum_{k=1}^{m} \frac{(m-1)!}{(k-1)!\,(m-k)!} x^{k-1} (1-x)^{m-k} \\
= &\, mx \sum_{k=1}^{m} \frac{(m-1)!}{(k-1)!\,(m-1-k)!} x^{k} (1-x)^{m-1-k} \\
= &\, mx (1+1-x)^{m-1} = mx.
\end{aligned}$$

Next, using what was just shown and the binomial theorem again,

$$\begin{aligned}
\sum_{k=0}^{m} \binom{m}{k} k^2 x^k (1-x)^{m-k} &= \sum_{k=1}^{m} \binom{m}{k} k (k-1) x^k (1-x)^{m-k} \\
&\quad + \sum_{k=0}^{m} \binom{m}{k} k x^k (1-x)^{m-k}
\end{aligned}$$

$$= \sum_{k=2}^{m} \binom{m}{k} k (k-1) x^k (1-x)^{m-k} + mx$$

$$= \sum_{k=0}^{m-2} \binom{m}{k+2} (k+2)(k+1) x^{k+2} (1-x)^{m-2-k} + mx$$

$$\begin{aligned}
&= m(m-1) \sum_{k=0}^{m-2} \binom{m-2}{k} x^{k+2} (1-x)^{m-2-k} + mx \\
&= x^2 m (m-1) \sum_{k=0}^{m-2} \binom{m-2}{k} x^{k} (1-x)^{m-2-k} + mx \\
&= x^2 m (m-1) + mx = x^2 m^2 - x^2 m + mx
\end{aligned}$$

It follows

$$\sum_{k=0}^{m}\binom{m}{k}(k-mx)^2 x^k (1-x)^{m-k}$$

$$=\sum_{k=0}^{m}\binom{m}{k}\left(k^2-2kmx+x^2m^2\right)x^k(1-x)^{m-k}$$

and from what was just shown along with the binomial theorem again, this equals

$$x^2m^2-x^2m+mx-2mx(mx)+x^2m^2=-x^2m+mx=\frac{m}{4}-m\left(x-\frac{1}{2}\right)^2.$$

Thus the expression is maximized when $x=1/2$ and yields $m/4$ in this case. This proves the lemma.

Now let f be a continuous function defined on $[0,1]$. Let p_n be the polynomial defined by

$$p_n(x)\equiv\sum_{k=0}^{n}\binom{n}{k}f\left(\frac{k}{n}\right)x^k(1-x)^{n-k}. \tag{5.7}$$

Now for f a continuous function defined on $[0,1]^n$ and for $\mathbf{x}=(x_1,\cdots,x_n)$,consider the polynomial,

$$p_m(\mathbf{x})\equiv\sum_{k_1=1}^{m}\cdots\sum_{k_n=1}^{m}\binom{m}{k_1}\binom{m}{k_2}\cdots\binom{m}{k_n}x_1^{k_1}(1-x_1)^{m-k_1}x_2^{k_2}(1-x_2)^{m-k_2}$$

$$\cdots x_n^{k_n}(1-x_n)^{m-k_n}f\left(\frac{k_1}{m},\cdots,\frac{k_n}{m}\right). \tag{5.8}$$

Also define if I is a set in $\mathbb{R}^n$

$$||h||_I\equiv\sup\{|h(\mathbf{x})|:\mathbf{x}\in I\}.$$

Thus p_m converges uniformly to f on a set, I if

$$\lim_{m\to\infty}||p_m-f||_I=0.$$

To simplify the notation, let $\mathbf{k}=(k_1,\cdots,k_n)$ where each $k_i\in[0,m]$, $\frac{\mathbf{k}}{\mathbf{m}}\equiv\left(\frac{k_1}{m},\cdots,\frac{k_n}{m}\right)$, and let

$$\binom{\mathbf{m}}{\mathbf{k}}\equiv\binom{m}{k_1}\binom{m}{k_2}\cdots\binom{m}{k_n}.$$

Also define

$$||\mathbf{k}||_\infty\equiv\max\{k_i,i=1,2,\cdots,n\}$$

$$\mathbf{x^k(1-x)^{m-k}}\equiv x_1^{k_1}(1-x_1)^{m-k_1}x_2^{k_2}(1-x_2)^{m-k_2}\cdots x_n^{k_n}(1-x_n)^{m-k_n}.$$

Thus in terms of this notation,

$$p_m(\mathbf{x})=\sum_{||\mathbf{k}||_\infty\le m}\binom{\mathbf{m}}{\mathbf{k}}\mathbf{x^k(1-x)^{m-k}}f\left(\frac{\mathbf{k}}{\mathbf{m}}\right)$$

This is the n dimensional version of the Bernstein polynomials presented earlier.

Lemma 5.7.2 *For $\mathbf{x}\in[0,1]^n$, f a continuous $\mathbb{F}$ valued function defined on $[0,1]^n$, and p_m given in 5.8, p_m converges uniformly to f on $[0,1]^n$ as $m\to\infty$.*

Proof: The function, f is uniformly continuous because it is continuous on a sequentially compact set, $[0,1]^n$. Therefore, there exists $\delta > 0$ such that if $|\mathbf{x}-\mathbf{y}| < \delta$, then

$$|f(\mathbf{x}) - f(\mathbf{y})| < \varepsilon.$$

Denote by G the set of $\mathbf{k}$ such that $(k_i - mx_i)^2 < \eta^2 m^2$ for each i where $\eta = \delta/\sqrt{n}$. Note this condition is equivalent to saying that for each i, $\left|\frac{k_i}{m} - x_i\right| < \eta$. A short computation shows that by the binomial theorem,

$$\sum_{||\mathbf{k}||_\infty \leq m} \binom{\mathbf{m}}{\mathbf{k}} \mathbf{x}^{\mathbf{k}} (\mathbf{1}-\mathbf{x})^{\mathbf{m}-\mathbf{k}} = 1$$

and so for $\mathbf{x} \in [0,1]^n$,

$$\begin{aligned} |p_m(\mathbf{x}) - f(\mathbf{x})| &\leq \sum_{||\mathbf{k}||_\infty \leq m} \binom{\mathbf{m}}{\mathbf{k}} \mathbf{x}^{\mathbf{k}} (\mathbf{1}-\mathbf{x})^{\mathbf{m}-\mathbf{k}} \left| f\left(\frac{\mathbf{k}}{\mathbf{m}}\right) - f(\mathbf{x}) \right| \\ &\leq \sum_{\mathbf{k} \in G} \binom{\mathbf{m}}{\mathbf{k}} \mathbf{x}^{\mathbf{k}} (\mathbf{1}-\mathbf{x})^{\mathbf{m}-\mathbf{k}} \left| f\left(\frac{\mathbf{k}}{\mathbf{m}}\right) - f(\mathbf{x}) \right| \\ &+ \sum_{\mathbf{k} \in G^C} \binom{\mathbf{m}}{\mathbf{k}} \mathbf{x}^{\mathbf{k}} (\mathbf{1}-\mathbf{x})^{\mathbf{m}-\mathbf{k}} \left| f\left(\frac{\mathbf{k}}{\mathbf{m}}\right) - f(\mathbf{x}) \right| \end{aligned} \tag{5.9}$$

Now for $\mathbf{k} \in G$ it follows that for each i

$$\left| \frac{k_i}{m} - x_i \right| < \frac{\delta}{\sqrt{n}} \tag{5.10}$$

and so $\left|f\left(\frac{\mathbf{k}}{\mathbf{m}}\right) - f(\mathbf{x})\right| < \varepsilon$ because the above implies $\left|\frac{\mathbf{k}}{\mathbf{m}} - \mathbf{x}\right| < \delta$. Therefore, the first sum on the right in 5.9 is no larger than

$$\sum_{\mathbf{k} \in G} \binom{\mathbf{m}}{\mathbf{k}} \mathbf{x}^{\mathbf{k}} (\mathbf{1}-\mathbf{x})^{\mathbf{m}-\mathbf{k}} \varepsilon \leq \sum_{||\mathbf{k}||_\infty \leq m} \binom{\mathbf{m}}{\mathbf{k}} \mathbf{x}^{\mathbf{k}} (\mathbf{1}-\mathbf{x})^{\mathbf{m}-\mathbf{k}} \varepsilon = \varepsilon.$$

Letting $M \geq \max\{|f(\mathbf{x})| : \mathbf{x} \in [0,1]^n\}$ it follows

$$\begin{aligned} & |p_m(\mathbf{x}) - f(\mathbf{x})| \\ \leq\ & \varepsilon + 2M \sum_{\mathbf{k} \in G^C} \binom{\mathbf{m}}{\mathbf{k}} \mathbf{x}^{\mathbf{k}} (\mathbf{1}-\mathbf{x})^{\mathbf{m}-\mathbf{k}} \\ \leq\ & \varepsilon + 2M \left(\frac{1}{\eta^2 m^2}\right)^n \sum_{\mathbf{k} \in G^C} \binom{\mathbf{m}}{\mathbf{k}} \prod_{j=1}^n (k_j - mx_j)^2 \mathbf{x}^{\mathbf{k}} (\mathbf{1}-\mathbf{x})^{\mathbf{m}-\mathbf{k}} \\ \leq\ & \varepsilon + 2M \left(\frac{1}{\eta^2 m^2}\right)^n \sum_{||\mathbf{k}||_\infty \leq m} \binom{\mathbf{m}}{\mathbf{k}} \prod_{j=1}^n (k_j - mx_j)^2 \mathbf{x}^{\mathbf{k}} (\mathbf{1}-\mathbf{x})^{\mathbf{m}-\mathbf{k}} \end{aligned}$$

because on G^C,

$$\frac{(k_j - mx_j)^2}{\eta^2 m^2} < 1,\ j = 1, \cdots, n.$$

Now by Lemma 5.7.1,

$$|p_m(\mathbf{x}) - f(\mathbf{x})| \leq \varepsilon + 2M \left(\frac{1}{\eta^2 m^2}\right)^n \left(\frac{m}{4}\right)^n.$$

Therefore, since the right side does not depend on $\mathbf{x}$, it follows that for all m sufficiently large,

$$||p_m - f||_{[0,1]^n} \leq 2\varepsilon$$

and since ε is arbitrary, this shows $\lim_{m\to\infty} ||p_m - f||_{[0,1]^n} = 0$. This proves the lemma.

Theorem 5.7.3 *Let f be a continuous function defined on*

$$R \equiv \prod_{k=1}^{n} [a_k, b_k].$$

Then there exists a sequence of polynomials $\{p_m\}$ converging uniformly to f on R.

Proof: Let $g_k : [0,1] \to [a_k, b_k]$ be linear, one to one, and onto and let

$$\mathbf{x} = \mathbf{g}(\mathbf{y}) \equiv (g_1(y_1), g_2(y_2), \cdots, g_n(y_n)).$$

Thus $\mathbf{g} : [0,1]^n \to \prod_{k=1}^{n} [a_k, b_k]$ is one to one, onto, and each component function is linear. Then $f \circ \mathbf{g}$ is a continuous function defined on $[0,1]^n$. It follows from Lemma 5.7.2 there exists a sequence of polynomials, $\{p_m(\mathbf{y})\}$ each defined on $[0,1]^n$ which converges uniformly to $f \circ \mathbf{g}$ on $[0,1]^n$. Therefore, $\left\{p_m\left(\mathbf{g}^{-1}(\mathbf{x})\right)\right\}$ converges uniformly to $f(\mathbf{x})$ on R. But

$$\mathbf{y} = (y_1, \cdots, y_n) = \left(g_1^{-1}(x_1), \cdots, g_n^{-1}(x_n)\right)$$

and each g_k^{-1} is linear. Therefore, $\left\{p_m\left(\mathbf{g}^{-1}(\mathbf{x})\right)\right\}$ is a sequence of polynomials. This proves the theorem.

There is a more general version of this theorem which is easy to get. It depends on the Tietze extension theorem, a wonderful little result which is interesting for its own sake.

5.7.1 The Tietze Extension Theorem

To generalize the Weierstrass approximation theorem I will give a special case of the Tietze extension theorem, a very useful result in topology. When this is done, it will be possible to prove the Weierstrass approximation theorem for functions defined on a closed and bounded subset of $\mathbb{R}^n$ rather than a box.

Lemma 5.7.4 *Let $S \subseteq \mathbb{R}^n$ be a nonempty subset. Define*

$$\operatorname{dist}(\mathbf{x}, S) \equiv \inf\{|\mathbf{x} - \mathbf{y}| : \mathbf{y} \in S\}.$$

Then $\mathbf{x} \to \operatorname{dist}(\mathbf{x}, S)$ is a continuous function satisfying the inequality,

$$|\operatorname{dist}(\mathbf{x}, S) - \operatorname{dist}(\mathbf{y}, S)| \leq |\mathbf{x} - \mathbf{y}|. \tag{5.11}$$

Proof: The continuity of $\mathbf{x} \to \operatorname{dist}(\mathbf{x}, S)$ is obvious if the inequality 5.11 is established. So let $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$. Without loss of generality, assume $\operatorname{dist}(\mathbf{x}, S) \geq \operatorname{dist}(\mathbf{y}, S)$ and pick $\mathbf{z} \in S$ such that $|\mathbf{y} - \mathbf{z}| - \varepsilon < \operatorname{dist}(\mathbf{y}, S)$. Then

$$\begin{aligned} |\operatorname{dist}(\mathbf{x}, S) - \operatorname{dist}(\mathbf{y}, S)| &= \operatorname{dist}(\mathbf{x}, S) - \operatorname{dist}(\mathbf{y}, S) \\ &\leq |\mathbf{x} - \mathbf{z}| - (|\mathbf{y} - \mathbf{z}| - \varepsilon) \\ &\leq |\mathbf{z} - \mathbf{y}| + |\mathbf{x} - \mathbf{y}| - |\mathbf{y} - \mathbf{z}| + \varepsilon = |\mathbf{x} - \mathbf{y}| + \varepsilon. \end{aligned}$$

Since ε is arbitrary, this proves 5.11.

Lemma 5.7.5 *Let H, K be two nonempty disjoint closed subsets of $\mathbb{R}^n$. Then there exists a continuous function, $g : \mathbb{R}^n \to [-1,1]$ such that $g(H) = -1/3$, $g(K) = 1/3, g(\mathbb{R}^n) \subseteq [-1/3, 1/3]$.*

Proof: Let

$$f(\mathbf{x}) \equiv \frac{\operatorname{dist}(\mathbf{x}, H)}{\operatorname{dist}(\mathbf{x}, H) + \operatorname{dist}(\mathbf{x}, K)}.$$

The denominator is never equal to zero because if $\operatorname{dist}(\mathbf{x}, H) = 0$, then $\mathbf{x} \in H$ because H is closed. (To see this, pick $\mathbf{h}_k \in B(\mathbf{x}, 1/k) \cap H$. Then $\mathbf{h}_k \to \mathbf{x}$ and since H is closed, $\mathbf{x} \in H$.) Similarly, if $\operatorname{dist}(\mathbf{x}, K) = 0$, then $\mathbf{x} \in K$ and so the denominator is never zero as claimed. Hence f is continuous and from its definition, $f = 0$ on H and $f = 1$ on K. Now let $g(\mathbf{x}) \equiv \frac{2}{3}\left(f(\mathbf{x}) - \frac{1}{2}\right)$. Then g has the desired properties.

Definition 5.7.6 *For f a real or complex valued bounded continuous function defined on $M \subseteq \mathbb{R}^n$.*

$$||f||_M \equiv \sup\{|f(\mathbf{x})| : \mathbf{x} \in M\}.$$

Lemma 5.7.7 *Suppose M is a closed set in $\mathbb{R}^n$ where $\mathbb{R}^n$ and suppose $f : M \to [-1,1]$ is continuous at every point of M. Then there exists a function, g which is defined and continuous on all of $\mathbb{R}^n$ such that $||f - g||_M < \frac{2}{3}$, $g(\mathbb{R}^n) \subseteq [-1/3, 1/3]$.*

Proof: Let $H = f^{-1}([-1, -1/3])$, $K = f^{-1}([1/3, 1])$. Thus H and K are disjoint closed subsets of M. Suppose first H, K are both nonempty. Then by Lemma 5.7.5 there exists g such that g is a continuous function defined on all of $\mathbb{R}^n$ and $g(H) = -1/3$, $g(K) = 1/3$, and $g(\mathbb{R}^n) \subseteq [-1/3, 1/3]$. It follows $||f - g||_M < 2/3$. If $H = \emptyset$, then f has all its values in $[-1/3, 1]$ and so letting $g \equiv 1/3$, the desired condition is obtained. If $K = \emptyset$, let $g \equiv -1/3$. This proves the lemma.

Lemma 5.7.8 *Suppose M is a closed set in $\mathbb{R}^n$ and suppose $f : M \to [-1,1]$ is continuous at every point of M. Then there exists a function, g which is defined and continuous on all of $\mathbb{R}^n$ such that $g = f$ on M and g has its values in $[-1,1]$.*

Proof: Using Lemma 5.7.7, let g_1 be such that $g_1(\mathbb{R}^n) \subseteq [-1/3, 1/3]$ and $||f - g_1||_M \leq \frac{2}{3}$. Suppose $g_1, \cdots, g_m$ have been chosen such that $g_j(\mathbb{R}^n) \subseteq [-1/3, 1/3]$ and

$$\left|\left| f - \sum_{i=1}^{m} \left(\frac{2}{3}\right)^{i-1} g_i \right|\right|_M < \left(\frac{2}{3}\right)^m. \tag{5.12}$$

This has been done for $m = 1$. Then

$$\left|\left| \left(\frac{3}{2}\right)^m \left(f - \sum_{i=1}^{m} \left(\frac{2}{3}\right)^{i-1} g_i \right) \right|\right|_M \leq 1$$

and so $\left(\frac{3}{2}\right)^m \left(f - \sum_{i=1}^{m} \left(\frac{2}{3}\right)^{i-1} g_i \right)$ can play the role of f in the first step of the proof. Therefore, there exists g_{m+1} defined and continuous on all of $\mathbb{R}^n$ such that its values are in $[-1/3, 1/3]$ and

$$\left|\left| \left(\frac{3}{2}\right)^m \left(f - \sum_{i=1}^{m} \left(\frac{2}{3}\right)^{i-1} g_i \right) - g_{m+1} \right|\right|_M \leq \frac{2}{3}.$$

Hence

$$\left|\left| \left(f - \sum_{i=1}^{m} \left(\frac{2}{3}\right)^{i-1} g_i \right) - \left(\frac{2}{3}\right)^m g_{m+1} \right|\right|_M \leq \left(\frac{2}{3}\right)^{m+1}.$$

It follows there exists a sequence, $\{g_i\}$ such that each has its values in $[-1/3, 1/3]$ and for every m 5.12 holds. Then let

$$g(\mathbf{x}) \equiv \sum_{i=1}^{\infty} \left(\frac{2}{3}\right)^{i-1} g_i(\mathbf{x}).$$

It follows

$$|g(\mathbf{x})| \leq \left| \sum_{i=1}^{\infty} \left(\frac{2}{3}\right)^{i-1} g_i(\mathbf{x}) \right| \leq \sum_{i=1}^{m} \left(\frac{2}{3}\right)^{i-1} \frac{1}{3} \leq 1$$

and

$$\left| \left(\frac{2}{3}\right)^{i-1} g_i(\mathbf{x}) \right| \leq \left(\frac{2}{3}\right)^{i-1} \frac{1}{3}$$

so the Weierstrass M test applies and shows convergence is uniform. Therefore g must be continuous. The estimate 5.12 implies $f = g$ on M.

The following is the Tietze extension theorem.

Theorem 5.7.9 *Let M be a closed nonempty subset of $\mathbb{R}^n$ and let $f : M \to [a, b]$ be continuous at every point of M. Then there exists a function, g continuous on all of $\mathbb{R}^n$ which coincides with f on M such that $g(\mathbb{R}^n) \subseteq [a, b]$.*

Proof: Let $f_1(\mathbf{x}) = 1 + \frac{2}{b-a}(f(\mathbf{x}) - b)$. Then f_1 satisfies the conditions of Lemma 5.7.8 and so there exists $g_1 : \mathbb{R}^n \to [-1, 1]$ such that g is continuous on $\mathbb{R}^n$ and equals f_1 on M. Let $g(\mathbf{x}) = (g_1(\mathbf{x}) - 1)\left(\frac{b-a}{2}\right) + b$. This works.

With the Tietze extension theorem, here is a better version of the Weierstrass approximation theorem.

Theorem 5.7.10 *Let K be a closed and bounded subset of $\mathbb{R}^n$ and let $f : K \to \mathbb{R}$ be continuous. Then there exists a sequence of polynomials $\{p_m\}$ such that*

$$\lim_{m \to \infty} (\sup\{|f(\mathbf{x}) - p_m(\mathbf{x})| : \mathbf{x} \in K\}) = 0.$$

In other words, the sequence of polynomials converges uniformly to f on K.

Proof: By the Tietze extension theorem, there exists an extension of f to a continuous function g defined on all $\mathbb{R}^n$ such that $g = f$ on K. Now since K is bounded, there exist intervals, $[a_k, b_k]$ such that

$$K \subseteq \prod_{k=1}^{n} [a_k, b_k] = R$$

Then by the Weierstrass approximation theorem, Theorem 5.7.3 there exists a sequence of polynomials $\{p_m\}$ converging uniformly to g on R. Therefore, this sequence of polynomials converges uniformly to $g = f$ on K as well. This proves the theorem.

By considering the real and imaginary parts of a function which has values in $\mathbb{C}$ one can generalize the above theorem.

Corollary 5.7.11 *Let K be a closed and bounded subset of $\mathbb{R}^n$ and let $f : K \to \mathbb{F}$ be continuous. Then there exists a sequence of polynomials $\{p_m\}$ such that*

$$\lim_{m \to \infty} (\sup\{|f(\mathbf{x}) - p_m(\mathbf{x})| : \mathbf{x} \in K\}) = 0.$$

In other words, the sequence of polynomials converges uniformly to f on K.

5.8 The Operator Norm

It is important to be able to measure the size of a linear operator. The most convenient way is described in the next definition.

Definition 5.8.1 *Let V, W be two finite dimensional normed vector spaces having norms $||\cdot||_V$ and $||\cdot||_W$ respectively. Let $L \in \mathcal{L}(V,W)$. Then the operator norm of L, denoted by $||L||$ is defined as*

$$||L|| \equiv \sup\{||L\mathbf{x}||_W : ||\mathbf{x}||_V \leq 1\}.$$

Then the following theorem discusses the main properties of this norm. In the future, I will dispense with the subscript on the symbols for the norm because it is clear from the context which norm is meant. Here is a useful lemma.

Lemma 5.8.2 *Let V be a normed vector space having a basis $\{\mathbf{v}_1, \cdots, \mathbf{v}_n\}$. Let*

$$A = \left\{ \mathbf{a} \in \mathbb{F}^n : \left|\left| \sum_{k=1}^n a_k \mathbf{v}_k \right|\right| \leq 1 \right\}$$

where $\mathbf{a} = (a_1, \cdots, a_n)$. Then A is a closed and bounded subset of $\mathbb{F}^n$.

Proof: First suppose $\mathbf{a} \notin A$. Then

$$\left|\left| \sum_{k=1}^n a_k \mathbf{v}_k \right|\right| > 1.$$

Then for $\mathbf{b} = (b_1, \cdots, b_n)$, and using the triangle inequality,

$$\left|\left| \sum_{k=1}^n b_k \mathbf{v}_k \right|\right| = \left|\left| \sum_{k=1}^n (a_k - (a_k - b_k)) \mathbf{v}_k \right|\right|$$

$$\geq \left|\left| \sum_{k=1}^n a_k \mathbf{v}_k \right|\right| - \sum_{k=1}^n |a_k - b_k| \, ||\mathbf{v}_k||$$

and now it is apparent that if $|\mathbf{a} - \mathbf{b}|$ is sufficiently small so that each $|a_k - b_k|$ is small enough, this expression is larger than 1. Thus there exists $\delta > 0$ such that $B(\mathbf{a}, \delta) \subseteq A^C$ showing that A^C is open. Therefore, A is closed.

Next consider the claim that A is bounded. Suppose this is not so. Then there exists a sequence $\{\mathbf{a}_k\}$ of points of A,

$$\mathbf{a}_k = \left(a_k^1, \cdots, a_k^n\right),$$

such that $\lim_{k\to\infty} |\mathbf{a}_k| = \infty$. Then from the definition of A,

$$\left|\left| \sum_{j=1}^n \frac{a_k^j}{|\mathbf{a}_k|} \mathbf{v}_j \right|\right| \leq \frac{1}{|\mathbf{a}_k|}. \tag{5.13}$$

Let

$$\mathbf{b}_k = \left(\frac{a_k^1}{|\mathbf{a}_k|}, \cdots, \frac{a_k^n}{|\mathbf{a}_k|} \right)$$

Then $|\mathbf{b}_k| = 1$ so $\mathbf{b}_k$ is contained in the closed and bounded set, $S(\mathbf{0},1)$ which is sequentially compact in $\mathbb{F}^n$. It follows there exists a subsequence, still denoted by $\{\mathbf{b}_k\}$ such that it converges to $\mathbf{b} \in S(\mathbf{0},1)$. Passing to the limit in 5.13 using the following inequality,

$$\left|\left|\sum_{j=1}^{n} \frac{a_k^j}{|\mathbf{a}_k|}\mathbf{v}_j - \sum_{j=1}^{n} b_j \mathbf{v}_j\right|\right| \leq \sum_{j=1}^{n} \left|\frac{a_k^j}{|\mathbf{a}_k|} - b_j\right| ||\mathbf{v}_j||$$

to see that the sum converges to $\sum_{j=1}^{n} b_j \mathbf{v}_j$, it follows

$$\sum_{j=1}^{n} b_j \mathbf{v}_j = \mathbf{0}$$

and this is a contradiction because $\{\mathbf{v}_1, \cdots, \mathbf{v}_n\}$ is a basis and not all the b_j can equal zero. Therefore, A must be bounded after all. This proves the lemma.

Theorem 5.8.3 *The operator norm has the following properties.*

1. $||L|| < \infty$
2. *For all* $\mathbf{x} \in X, ||L\mathbf{x}|| \leq ||L||\,||\mathbf{x}||$ *and if* $L \in \mathcal{L}(V,W)$ *while* $M \in \mathcal{L}(W,Z)$, *then* $||ML|| \leq ||M||\,||L||$.
3. $||\cdot||$ *is a norm. In particular,*
   (a) $||L|| \geq 0$ *and* $||L|| = 0$ *if and only if* $L = 0$, *the linear transformation which sends every vector to* $\mathbf{0}$.
   (b) $||aL|| = |a|\,||L||$ *whenever* $a \in \mathbb{F}$
   (c) $||L + M|| \leq ||L|| + ||M||$
4. *If* $L \in \mathcal{L}(V,W)$ *for* V, W *normed vector spaces,* L *is continuous, meaning that* $L^{-1}(U)$ *is open whenever* U *is an open set in* W.

Proof: First consider 1.). Let A be as in the above lemma. Then

$$\begin{aligned} ||L|| &\equiv \sup\left\{\left|\left|L\left(\sum_{j=1}^{n} a_j \mathbf{v}_j\right)\right|\right| : \mathbf{a} \in A\right\} \\ &= \sup\left\{\left|\left|\sum_{j=1}^{n} a_j L(\mathbf{v}_j)\right|\right| : \mathbf{a} \in A\right\} < \infty \end{aligned}$$

because $\mathbf{a} \to \left|\left|\sum_{j=1}^{n} a_j L(\mathbf{v}_j)\right|\right|$ is a real valued continuous function defined on a sequentially compact set and so it achieves its maximum.

Next consider 2.). If $\mathbf{x} = \mathbf{0}$ there is nothing to show. Assume $\mathbf{x} \neq \mathbf{0}$. Then from the definition of $||L||$,

$$\left|\left|L\left(\frac{\mathbf{x}}{||\mathbf{x}||}\right)\right|\right| \leq ||L||$$

and so, since L is linear, you can multiply on both sides by $||\mathbf{x}||$ and conclude

$$||L(\mathbf{x})|| \leq ||L||\,||\mathbf{x}||.$$

For the other claim,

$$\begin{aligned} ||ML|| &\equiv \sup\{||ML(\mathbf{x})|| : ||\mathbf{x}|| \leq 1\} \\ &\leq ||M|| \sup\{||L\mathbf{x}|| : ||\mathbf{x}|| \leq 1\} \equiv ||M||\,||L||. \end{aligned}$$

Finally consider 3.) If $||L|| = 0$ then from 2.), $||L\mathbf{x}|| \leq 0$ and so $L\mathbf{x} = \mathbf{0}$ for every $\mathbf{x}$ which is the same as saying $L = 0$. If $L\mathbf{x} = \mathbf{0}$ for every $\mathbf{x}$, then $L = 0$ by definition. Let $a \in \mathbb{F}$. Then from the properties of the norm, in the vector space,

$$\begin{aligned} ||aL|| &\equiv \sup\{||aL\mathbf{x}|| : ||\mathbf{x}|| \leq 1\} \\ &= \sup\{|a|\,||L\mathbf{x}|| : ||\mathbf{x}|| \leq 1\} \\ &= |a| \sup\{||L\mathbf{x}|| : ||\mathbf{x}|| \leq 1\} \equiv |a|\,||L|| \end{aligned}$$

Finally consider the triangle inequality.

$$||L + M|| \equiv \sup\{||L\mathbf{x} + M\mathbf{x}|| : ||\mathbf{x}|| \leq 1\}$$

$$\leq \sup\{||M\mathbf{x}|| + ||L\mathbf{x}|| : ||\mathbf{x}|| \leq 1\}$$

$$\leq \sup\{||L\mathbf{x}|| : ||\mathbf{x}|| \leq 1\} + \sup\{||M\mathbf{x}|| : ||\mathbf{x}|| \leq 1\}$$

because $||L\mathbf{x}|| \leq \sup\{||L\mathbf{x}|| : ||\mathbf{x}|| \leq 1\}$ with a similar inequality holding for M. Therefore, by definition,

$$||L + M|| \leq ||L|| + ||M||.$$

Finally consider 4.). Let $L \in \mathcal{L}(V, W)$ and let U be open in W and $\mathbf{v} \in L^{-1}(U)$. Thus since U is open, there exists $\delta > 0$ such that

$$L(\mathbf{v}) \in B(L(\mathbf{v}), \delta) \subseteq U.$$

Then if $\mathbf{w} \in V$,

$$||L(\mathbf{v} - \mathbf{w})|| = ||L(\mathbf{v}) - L(\mathbf{w})|| \leq ||L||\,||\mathbf{v} - \mathbf{w}||$$

and so if $||\mathbf{v} - \mathbf{w}||$ is sufficiently small, $||\mathbf{v} - \mathbf{w}|| < \delta/||L||$, then $L(\mathbf{w}) \in B(L(\mathbf{v}), \delta)$ which shows $B(\mathbf{v}, \delta/||L||) \subseteq L^{-1}(U)$ and since $\mathbf{v} \in L^{-1}(U)$ was arbitrary, this shows $L^{-1}(U)$ is open. This proves the theorem.

The operator norm will be very important in the chapter on the derivative.

Part 1.) of Theorem 5.8.3 says that if $L \in \mathcal{L}(V, W)$ where V and W are two normed vector spaces, then there exists K such that for all $\mathbf{v} \in V$,

$$||L\mathbf{v}||_W \leq K\,||\mathbf{v}||_V$$

An obvious case is to let $L = \mathrm{id}$, the identity map on V and let there be two different norms on $V, ||\cdot||_1$ and $||\cdot||_2$. Thus $(V, ||\cdot||_1)$ is a normed vector space and so is $(V, ||\cdot||_2)$. Then Theorem 5.8.3 implies that

$$||\mathbf{v}||_2 = ||\mathrm{id}(\mathbf{v})||_2 \leq K_2\,||\mathbf{v}||_1 \tag{5.14}$$

while the same reasoning implies there exists K_1 such that

$$||\mathbf{v}||_1 \leq K_1\,||\mathbf{v}||_2. \tag{5.15}$$

This leads to the following important theorem.

Theorem 5.8.4 *Let V be a finite dimensional vector space and let $||\cdot||_1$ and $||\cdot||_2$ be two norms for V. Then these norms are equivalent which means there exist constants, δ, Δ such that for all $\mathbf{v} \in V$*

$$\delta ||\mathbf{v}||_1 \leq ||\mathbf{v}||_2 \leq \Delta ||\mathbf{v}||_1$$

A set, K is sequentially compact if and only if it is closed and bounded. Also every finite dimensional normed vector space is complete.

Proof: From 5.14 and 5.15

$$||\mathbf{v}||_1 \leq K_1 ||\mathbf{v}||_2 \leq K_1 K_2 ||\mathbf{v}||_1$$

and so

$$\frac{1}{K_1} ||\mathbf{v}||_1 \leq ||\mathbf{v}||_2 \leq K_2 ||\mathbf{v}||_1 .$$

Next consider the claim that all closed and bounded sets in a normed vector space are sequentially compact. Let $L : \mathbb{F}^n \to V$ be defined by

$$L(\mathbf{a}) \equiv \sum_{k=1}^{n} a_k \mathbf{v}_k$$

where $\{\mathbf{v}_1, \cdots, \mathbf{v}_n\}$ is a basis for V. Thus $L \in \mathcal{L}(\mathbb{F}^n, V)$ and so by Theorem 5.8.3 this is a continuous function. Hence if K is a closed and bounded subset of V it follows

$$L^{-1}(K) = \mathbb{F}^n \setminus L^{-1}\left(K^C\right) = \mathbb{F}^n \setminus (\text{an open set}) = \text{a closed set.}$$

Also $L^{-1}(K)$ is bounded. To see this, note that L^{-1} is one to one onto V and so $L^{-1} \in \mathcal{L}(V, \mathbb{F}^n)$. Therefore,

$$\left|L^{-1}(\mathbf{v})\right| \leq \left|\left|L^{-1}\right|\right| ||\mathbf{v}|| \leq \left|\left|L^{-1}\right|\right| r$$

where $K \subseteq B(\mathbf{0}, r)$. Since K is bounded, such an r exists. Thus $L^{-1}(K)$ is a closed and bounded subset of $\mathbb{F}^n$ and is therefore sequentially compact. It follows that if $\{\mathbf{v}_k\}_{k=1}^{\infty} \subseteq K$, there is a subsequence $\{\mathbf{v}_{k_l}\}_{l=1}^{\infty}$ such that $\left\{L^{-1}\mathbf{v}_{k_l}\right\}$ converges to a point, $\mathbf{a} \in L^{-1}(K)$. Hence by continuity of L,

$$\mathbf{v}_{k_l} = L\left(L^{-1}(\mathbf{v}_{k_l})\right) \to L\mathbf{a} \in K.$$

Conversely, suppose K is sequentially compact. I need to verify it is closed and bounded. If it is not closed, then it is missing a limit point, $\mathbf{k}_0$. Since $\mathbf{k}_0$ is a limit point, there exists $\mathbf{k}_n \in B\left(\mathbf{k}_0, \frac{1}{n}\right)$ such that $\mathbf{k}_n \neq \mathbf{k}_0$. Therefore, $\{\mathbf{k}_n\}$ has no limit point in K because $\mathbf{k}_0 \notin K$. It follows K must be closed. If K is not bounded, then you could pick $\mathbf{k}_n \in K$ such that $\mathbf{k}_m \notin B(\mathbf{0}, m)$ and it follows $\{\mathbf{k}_k\}$ cannot have a subsequence which converges because if $\mathbf{k} \in K$, then for large enough m, $\mathbf{k} \in B(\mathbf{0}, m/2)$ and so if $\left\{\mathbf{k}_{k_j}\right\}$ is any subsequence, $\mathbf{k}_{k_j} \notin B(\mathbf{0}, m)$ for all but finitely many j. In other words, for any $\mathbf{k} \in K$, it is not the limit of any subsequence. Thus K must also be bounded.

Finally consider the claim about completeness. Let $\{\mathbf{v}_k\}_{k=1}^{\infty}$ be a Cauchy sequence in V. Since L^{-1}, defined above is in $\mathcal{L}(V, \mathbb{F}^n)$, it follows $\left\{L^{-1}\mathbf{v}_k\right\}_{k=1}^{\infty}$ is a Cauchy sequence in $\mathbb{F}^n$. This follows from the inequality,

$$\left|L^{-1}\mathbf{v}_k - L^{-1}\mathbf{v}_l\right| \leq \left|\left|L^{-1}\right|\right| ||\mathbf{v}_k - \mathbf{v}_l|| .$$

therefore, there exists $\mathbf{a} \in \mathbb{F}^n$ such that $L^{-1}\mathbf{v}_k \to \mathbf{a}$ and since L is continuous,

$$\mathbf{v}_k = L\left(L^{-1}(\mathbf{v}_k)\right) \to L(\mathbf{a}) .$$

This proves the theorem.

Example 5.8.5 *Let V be a vector space and let $\{\mathbf{v}_1, \cdots, \mathbf{v}_n\}$ be a basis. Define a norm on V as follows. For $\mathbf{v} = \sum_{k=1}^n a_k \mathbf{v}_k$,*

$$||\mathbf{v}|| \equiv \max \{|a_k| : k = 1, \cdots, n\}$$

In the above example, this is a norm on the vector space, V. It is clear $||a\mathbf{v}|| = |a|\, ||\mathbf{v}||$ and that $||\mathbf{v}|| \geq 0$ and equals 0 if and only if $\mathbf{v} = \mathbf{0}$. The hard part is the triangle inequality. Let $\mathbf{v} = \sum_{k=1}^n a_k \mathbf{v}_k$ and $\mathbf{w} = \mathbf{v} = \sum_{k=1}^n b_k \mathbf{v}_k$.

$$\begin{aligned} ||\mathbf{v} + \mathbf{w}|| &\equiv \max_k \{|a_k + b_k|\} \leq \max_k \{|a_k| + |b_k|\} \\ &\leq \max_k |a_k| + \max_k |b_k| \equiv ||\mathbf{v}|| + ||\mathbf{w}||. \end{aligned}$$

This shows this is indeed a norm.

5.9 Exercises

1. In Theorem 5.7.3 it is assumed f has values in $\mathbb{F}$. Show there is no change if f has values in V, a normed vector space provided you redefine the definition of a polynomial to be something of the form $\sum_{|\boldsymbol{\alpha}| \leq m} a_\alpha \mathbf{x}^{\boldsymbol{\alpha}}$ where $a_{\boldsymbol{\alpha}} \in V$.

2. How would you generalize the conclusion of Corollary 5.7.11 to include the situation where f has values in a finite dimensional normed vector space?

3. If $\{\mathbf{f}_n\}$ and $\{\mathbf{g}_n\}$ are sequences of $\mathbb{F}^n$ valued functions defined on D which converge uniformly, show that if a, b are constants, then $a\mathbf{f}_n + b\mathbf{g}_n$ also converges uniformly. If there exists a constant, M such that $|\mathbf{f}_n(\mathbf{x})|, |\mathbf{g}_n(\mathbf{x})| < M$ for all n and for all $\mathbf{x} \in D$, show $\{\mathbf{f}_n \cdot \mathbf{g}_n\}$ converges uniformly. Let $f_n(\mathbf{x}) \equiv 1/|\mathbf{x}|$ for $\mathbf{x} \in B(\mathbf{0},1)$ and let $g_n(\mathbf{x}) \equiv (n-1)/n$. Show $\{f_n\}$ converges uniformly on $B(\mathbf{0},1)$ and $\{g_n\}$ converges uniformly but $\{f_n g_n\}$ fails to converge uniformly.

4. Formulate a theorem for series of functions of n variables which will allow you to conclude the infinite series is uniformly continuous based on reasonable assumptions about the functions in the sum.

5. If f and g are real valued functions which are continuous on some set, D, show that

$$\min(f, g), \max(f, g)$$

are also continuous. Generalize this to any finite collection of continuous functions. **Hint:** Note $\max(f, g) = \frac{|f-g|+f+g}{2}$. Now recall the triangle inequality which can be used to show $|\cdot|$ is a continuous function.

6. Find an example of a sequence of continuous functions defined on $\mathbb{R}^n$ such that each function is nonnegative and each function has a maximum value equal to 1 but the sequence of functions converges to 0 pointwise on $\mathbb{R}^n \setminus \{\mathbf{0}\}$, that is, the set of vectors in $\mathbb{R}^n$ excluding $\mathbf{0}$.

7. Let $\mathbf{x} \to h(\mathbf{x})$ be a bounded continuous function. Show the function $f(\mathbf{x}) = \sum_{n=1}^\infty \frac{h(n\mathbf{x})}{n^2}$ is continuous.

8. Let S be a any countable subset of $\mathbb{R}^n$. Show there exists a function, $\mathbf{f}$ defined on $\mathbb{R}^n$ which is discontinuous at every point of S but continuous everywhere else. **Hint:** This is real easy if you do the right thing. It involves the Weierstrass M test.

9. By Theorem 5.7.10 there exists a sequence of polynomials converging uniformly to $f(\mathbf{x}) = |\mathbf{x}|$ on $R \equiv \prod_{k=1}^{n} [-M, M]$. Show there exists a sequence of polynomials, $\{p_n\}$ converging uniformly to f on R which has the additional property that for all $n, p_n(\mathbf{0}) = 0$.

10. If f is any continuous function defined on K a sequentially compact subset of $\mathbb{R}^n$, show there exists a series of the form $\sum_{k=1}^{\infty} p_k$, where each p_k is a polynomial, which converges uniformly to f on $[a, b]$. **Hint:** You should use the Weierstrass approximation theorem to obtain a sequence of polynomials. Then arrange it so the limit of this sequence is an infinite sum.

11. A function $\mathbf{f}$ is Holder continuous if there exists a constant, K such that

$$|\mathbf{f}(\mathbf{x}) - \mathbf{f}(\mathbf{y})| \leq K |\mathbf{x} - \mathbf{y}|^{\alpha}$$

for some $\alpha \leq 1$ for all $\mathbf{x}, \mathbf{y}$. Show every Holder continuous function is uniformly continuous.

12. Consider $f(\mathbf{x}) \equiv \text{dist}(\mathbf{x}, S)$ where S is a nonempty subset of $\mathbb{R}^n$. Show f is uniformly continuous.

13. Let K be a sequentially compact set in a normed vector space V and let $\mathbf{f} : V \to W$ be continuous where W is also a normed vector space. Show $\mathbf{f}(K)$ is also sequentially compact.

14. If $\mathbf{f}$ is uniformly continuous, does it follow that $|\mathbf{f}|$ is also uniformly continuous? If $|\mathbf{f}|$ is uniformly continuous does it follow that $\mathbf{f}$ is uniformly continuous? Answer the same questions with "uniformly continuous" replaced with "continuous". Explain why.

15. Let $\mathbf{f} : D \to \mathbb{R}$ be a function. This function is said to be lower semicontinuous[1] at $\mathbf{x} \in D$ if for any sequence $\{\mathbf{x}_n\} \subseteq D$ which converges to $\mathbf{x}$ it follows

$$\mathbf{f}(\mathbf{x}) \leq \lim_{n \to \infty} \inf \mathbf{f}(\mathbf{x}_n).$$

Suppose D is sequentially compact and $\mathbf{f}$ is lower semicontinuous at every point of D. Show that then $\mathbf{f}$ achieves its minimum on D.

16. Let $f : D \to \mathbb{R}$ be a function. This function is said to be upper semicontinuous at $\mathbf{x} \in D$ if for any sequence $\{\mathbf{x}_n\} \subseteq D$ which converges to $\mathbf{x}$ it follows

$$f(\mathbf{x}) \geq \lim_{n \to \infty} \sup f(\mathbf{x}_n).$$

Suppose D is sequentially compact and f is upper semicontinuous at every point of D. Show that then f achieves its maximum on D.

17. Show that a real valued function defined on $D \subseteq \mathbb{R}^n$ is continuous if and only if it is both upper and lower semicontinuous.

18. Show that a real valued lower semicontinuous function defined on a sequentially compact set achieves its minimum and that an upper semicontinuous function defined on a sequentially compact set achieves its maximum.

19. Give an example of a lower semicontinuous function defined on $\mathbb{R}^n$ which is not continuous and an example of an upper semicontinuous function which is not continuous.

[1] The notion of lower semicontinuity is very important for functions which are defined on infinite dimensional sets. In more general settings, one formulates the concept differently.

20. Suppose $\{f_\alpha : \alpha \in \Lambda\}$ is a collection of continuous functions. Let

$$F(\mathbf{x}) \equiv \inf \{f_\alpha(\mathbf{x}) : \alpha \in \Lambda\}$$

Show F is an upper semicontinuous function. Next let

$$G(\mathbf{x}) \equiv \sup \{f_\alpha(\mathbf{x}) : \alpha \in \Lambda\}$$

Show G is a lower semicontinuous function.

21. Let f be a function. $\text{epi}(f)$ is defined as

$$\{(\mathbf{x}, y) : y \geq \mathbf{f}(\mathbf{x})\}.$$

It is called the epigraph of f. We say $\text{epi}(f)$ is closed if whenever $(\mathbf{x}_n, y_n) \in \text{epi}(f)$ and $\mathbf{x}_n \to \mathbf{x}$ and $y_n \to y$, it follows $(\mathbf{x}, y) \in \text{epi}(f)$. Show f is lower semicontinuous if and only if $\text{epi}(f)$ is closed. What would be the corresponding result equivalent to upper semicontinuous?

22. The operator norm was defined for $\mathcal{L}(V, W)$ above. This is the usual norm used for this vector space of linear transformations. Show that any other norm used on $\mathcal{L}(V, W)$ is equivalent to the operator norm. That is, show that if $||\cdot||_1$ is another norm, there exist scalars δ, Δ such that

$$\delta ||L|| \leq ||L||_1 \leq \Delta ||L||$$

for all $L \in \mathcal{L}(V, W)$ where here $||\cdot||$ denotes the operator norm.

23. One alternative norm which is very popular is as follows. Let $L \in \mathcal{L}(V, W)$ and let (l_{ij}) denote the matrix of L with respect to some bases. Then the Frobenius norm is defined by

$$\left(\sum_{ij} |l_{ij}|^2 \right)^{1/2} \equiv ||L||_F.$$

Show this is a norm. Other norms are of the form

$$\left(\sum_{ij} |l_{ij}|^p \right)^{1/p}$$

where $p \geq 1$ or even

$$||L||_\infty = \max_{ij} |l_{ij}|.$$

Show these are also norms.

24. Explain why $\mathcal{L}(V, W)$ is always a complete normed vector space whenever V, W are finite dimensional normed vector spaces for any choice of norm for $\mathcal{L}(V, W)$. Also explain why every closed and bounded subset of $\mathcal{L}(V, W)$ is sequentially compact for any choice of norm on this space.

25. Let $L \in \mathcal{L}(V, V)$ where V is a finite dimensional normed vector space. Define

$$e^L \equiv \sum_{k=1}^{\infty} \frac{L^k}{k!}$$

Explain the meaning of this infinite sum and show it converges in $\mathcal{L}(V, V)$ for any choice of norm on this space. Now tell how to define $\sin(L)$.

The Derivative

6.1 Basic Definitions

The concept of derivative generalizes right away to functions of many variables. However, no attempt will be made to consider derivatives from one side or another. This is because when you consider functions of many variables, there isn't a well defined side. However, it is certainly the case that there are more general notions which include such things. I will present a fairly general notion of the derivative of a function which is defined on a normed vector space which has values in a normed vector space. The case of most interest is that of a function which maps $\mathbb{F}^n$ to $\mathbb{F}^m$ but it is no more trouble to consider the extra generality and it is sometimes useful to have this extra generality because sometimes you want to consider functions defined, for example on subspaces of $\mathbb{F}^n$and it is nice to not have to trouble with ad hoc considerations. Also, you might want to consider $\mathbb{F}^n$ with some norm other than the usual one.

In what follows, X, Y will denote normed vector spaces. Thanks to Theorem 5.8.4 all the definitions and theorems given below work the same for any norm given on the vector spaces.

Let U be an open set in X, and let $\mathbf{f} : U \to Y$ be a function.

Definition 6.1.1 *A function* $\mathbf{g}$ *is* $\mathbf{o}(\mathbf{v})$ *if*

$$\lim_{||\mathbf{v}||\to 0} \frac{\mathbf{g}(\mathbf{v})}{||\mathbf{v}||} = \mathbf{0} \tag{6.1}$$

A function $\mathbf{f} : U \to Y$ *is differentiable at* $\mathbf{x} \in U$ *if there exists a linear transformation* $L \in \mathcal{L}(X, Y)$ *such that*

$$\mathbf{f}(\mathbf{x}+\mathbf{v}) = \mathbf{f}(\mathbf{x}) + L\mathbf{v} + \mathbf{o}(\mathbf{v})$$

This linear transformation L *is the definition of* $D\mathbf{f}(\mathbf{x})$*. This derivative is often called the Frechet derivative.*

Note that from Theorem 5.8.4 the question whether a given function is differentiable is independent of the norm used on the finite dimensional vector space. That is, a function is differentiable with one norm if and only if it is differentiable with another norm.

The definition 6.1 means the error,

$$\mathbf{f}(\mathbf{x}+\mathbf{v}) - \mathbf{f}(\mathbf{x}) - L\mathbf{v}$$

converges to $\mathbf{0}$ faster than $||\mathbf{v}||$. Thus the above definition is equivalent to saying

$$\lim_{||\mathbf{v}||\to 0} \frac{||\mathbf{f}(\mathbf{x}+\mathbf{v}) - \mathbf{f}(\mathbf{x}) - L\mathbf{v}||}{||\mathbf{v}||} = 0 \tag{6.2}$$

or equivalently,

$$\lim_{\mathbf{y}\to\mathbf{x}} \frac{||\mathbf{f}(\mathbf{y}) - \mathbf{f}(\mathbf{x}) - D\mathbf{f}(\mathbf{x})(\mathbf{y}-\mathbf{x})||}{||\mathbf{y}-\mathbf{x}||} = 0. \tag{6.3}$$

The symbol, $\mathbf{o}(\mathbf{v})$ should be thought of as an adjective. Thus, if t and k are constants,

$$\mathbf{o}(\mathbf{v}) = \mathbf{o}(\mathbf{v}) + \mathbf{o}(\mathbf{v}),\ \mathbf{o}(t\mathbf{v}) = \mathbf{o}(\mathbf{v}),\ k\mathbf{o}(\mathbf{v}) = \mathbf{o}(\mathbf{v})$$

and other similar observations hold.

Theorem 6.1.2 *The derivative is well defined.*

Proof: First note that for a fixed vector, $\mathbf{v}$, $\mathbf{o}(t\mathbf{v}) = \mathbf{o}(t)$. This is because

$$\lim_{t\to 0} \frac{\mathbf{o}(t\mathbf{v})}{|t|} = \lim_{t\to 0} ||\mathbf{v}|| \frac{\mathbf{o}(t\mathbf{v})}{||t\mathbf{v}||} = \mathbf{0}$$

Now suppose both L_1 and L_2 work in the above definition. Then let $\mathbf{v}$ be any vector and let t be a real scalar which is chosen small enough that $t\mathbf{v} + \mathbf{x} \in U$. Then

$$\mathbf{f}(\mathbf{x}+t\mathbf{v}) = \mathbf{f}(\mathbf{x}) + L_1 t\mathbf{v} + \mathbf{o}(t\mathbf{v}),\ \mathbf{f}(\mathbf{x}+t\mathbf{v}) = \mathbf{f}(\mathbf{x}) + L_2 t\mathbf{v} + \mathbf{o}(t\mathbf{v}).$$

Therefore, subtracting these two yields $(L_2 - L_1)(t\mathbf{v}) = \mathbf{o}(t\mathbf{v}) = \mathbf{o}(t)$. Therefore, dividing by t yields $(L_2 - L_1)(\mathbf{v}) = \frac{\mathbf{o}(t)}{t}$. Now let $t \to 0$ to conclude that $(L_2 - L_1)(\mathbf{v}) = 0$. Since this is true for all $\mathbf{v}$, it follows $L_2 = L_1$. This proves the theorem.

Lemma 6.1.3 *Let $\mathbf{f}$ be differentiable at $\mathbf{x}$. Then $\mathbf{f}$ is continuous at $\mathbf{x}$ and in fact, there exists $K > 0$ such that whenever $||\mathbf{v}||$ is small enough,*

$$||\mathbf{f}(\mathbf{x}+\mathbf{v}) - \mathbf{f}(\mathbf{x})|| \le K||\mathbf{v}||$$

Proof: From the definition of the derivative,

$$\mathbf{f}(\mathbf{x}+\mathbf{v}) - \mathbf{f}(\mathbf{x}) = D\mathbf{f}(\mathbf{x})\mathbf{v} + \mathbf{o}(\mathbf{v}).$$

Let $||\mathbf{v}||$ be small enough that $\frac{\mathbf{o}(||\mathbf{v}||)}{||\mathbf{v}||} < 1$ so that $||\mathbf{o}(\mathbf{v})|| \le ||\mathbf{v}||$. Then for such $\mathbf{v}$,

$$\begin{aligned} ||\mathbf{f}(\mathbf{x}+\mathbf{v}) - \mathbf{f}(\mathbf{x})|| &\le ||D\mathbf{f}(\mathbf{x})\mathbf{v}|| + ||\mathbf{v}|| \\ &\le (||D\mathbf{f}(\mathbf{x})|| + 1)||\mathbf{v}|| \end{aligned}$$

This proves the lemma with $K = ||D\mathbf{f}(\mathbf{x})|| + 1$.

Here $||D\mathbf{f}(\mathbf{x})||$ is the operator norm of the linear transformation, $D\mathbf{f}(\mathbf{x})$.

6.2 The Chain Rule

With the above lemma, it is easy to prove the chain rule.

Theorem 6.2.1 *(The chain rule) Let U and V be open sets, $U \subseteq X$ and $V \subseteq Y$. Suppose $\mathbf{f} : U \to V$ is differentiable at $\mathbf{x} \in U$ and suppose $\mathbf{g} : V \to \mathbb{F}^q$ is differentiable at $\mathbf{f}(\mathbf{x}) \in V$. Then $\mathbf{g} \circ \mathbf{f}$ is differentiable at $\mathbf{x}$ and*

$$D(\mathbf{g}\circ\mathbf{f})(\mathbf{x}) = D(\mathbf{g}(\mathbf{f}(\mathbf{x})))\,D(\mathbf{f}(\mathbf{x})).$$

Proof: This follows from a computation. Let $B(\mathbf{x},r) \subseteq U$ and let r also be small enough that for $||\mathbf{v}|| \leq r$, it follows that $\mathbf{f}(\mathbf{x}+\mathbf{v}) \in V$. Such an r exists because $\mathbf{f}$ is continuous at $\mathbf{x}$. For $||\mathbf{v}|| < r$, the definition of differentiability of $\mathbf{g}$ and $\mathbf{f}$ implies

$$\mathbf{g}(\mathbf{f}(\mathbf{x}+\mathbf{v})) - \mathbf{g}(\mathbf{f}(\mathbf{x})) =$$

$$\begin{aligned} & D\mathbf{g}(\mathbf{f}(\mathbf{x}))(\mathbf{f}(\mathbf{x}+\mathbf{v}) - \mathbf{f}(\mathbf{x})) + \mathbf{o}(\mathbf{f}(\mathbf{x}+\mathbf{v}) - \mathbf{f}(\mathbf{x})) \\ = \; & D\mathbf{g}(\mathbf{f}(\mathbf{x}))[D\mathbf{f}(\mathbf{x})\mathbf{v} + \mathbf{o}(\mathbf{v})] + \mathbf{o}(\mathbf{f}(\mathbf{x}+\mathbf{v}) - \mathbf{f}(\mathbf{x})) \\ = \; & D(\mathbf{g}(\mathbf{f}(\mathbf{x})))D(\mathbf{f}(\mathbf{x}))\mathbf{v} + \mathbf{o}(\mathbf{v}) + \mathbf{o}(\mathbf{f}(\mathbf{x}+\mathbf{v}) - \mathbf{f}(\mathbf{x})). \end{aligned} \tag{6.4}$$

It remains to show $\mathbf{o}(\mathbf{f}(\mathbf{x}+\mathbf{v}) - \mathbf{f}(\mathbf{x})) = \mathbf{o}(\mathbf{v})$.

By Lemma 6.1.3, with K given there, letting $\varepsilon > 0$, it follows that for $||\mathbf{v}||$ small enough,

$$||\mathbf{o}(\mathbf{f}(\mathbf{x}+\mathbf{v}) - \mathbf{f}(\mathbf{x}))|| \leq (\varepsilon/K)||\mathbf{f}(\mathbf{x}+\mathbf{v}) - \mathbf{f}(\mathbf{x})|| \leq (\varepsilon/K)K||\mathbf{v}|| = \varepsilon||\mathbf{v}||.$$

Since $\varepsilon > 0$ is arbitrary, this shows $\mathbf{o}(\mathbf{f}(\mathbf{x}+\mathbf{v}) - \mathbf{f}(\mathbf{x})) = \mathbf{o}(\mathbf{v})$ because whenever $||\mathbf{v}||$ is small enough,

$$\frac{||\mathbf{o}(\mathbf{f}(\mathbf{x}+\mathbf{v}) - \mathbf{f}(\mathbf{x}))||}{||\mathbf{v}||} \leq \varepsilon.$$

By 6.4, this shows

$$\mathbf{g}(\mathbf{f}(\mathbf{x}+\mathbf{v})) - \mathbf{g}(\mathbf{f}(\mathbf{x})) = D(\mathbf{g}(\mathbf{f}(\mathbf{x})))D(\mathbf{f}(\mathbf{x}))\mathbf{v} + \mathbf{o}(\mathbf{v})$$

which proves the theorem.

6.3 The Matrix Of The Derivative

Let X, Y be normed vector spaces, a basis for X being $\{\mathbf{v}_1, \cdots, \mathbf{v}_n\}$ and a basis for Y being $\{\mathbf{w}_1, \cdots, \mathbf{w}_m\}$. First note that if $\pi_i : X \to \mathbb{F}$ is defined by

$$\pi_i \mathbf{v} \equiv x_i \text{ where } \mathbf{v} = \sum_k x_k \mathbf{v}_k,$$

then $\pi_i \in \mathcal{L}(X, \mathbb{F})$ and so by Theorem 5.8.3, it follows that π_i is continuous and if $\lim_{s\to t} \mathbf{g}(s) = \mathbf{L}$, then $|\pi_i \mathbf{g}(s) - \pi_i \mathbf{L}| \leq ||\pi_i||\,||\mathbf{g}(s) - \mathbf{L}||$ and so the i^{th} components converge also.

Suppose that $\mathbf{f} : U \to Y$ is differentiable. What is the matrix of $D\mathbf{f}(\mathbf{x})$ with respect to the given bases? That is, if

$$D\mathbf{f}(\mathbf{x}) = \sum_{ij} J_{ij}(\mathbf{x})\mathbf{w}_i \mathbf{v}_j,$$

what is $J_{ij}(\mathbf{x})$?

$$\begin{aligned} D_{\mathbf{v}_k}\mathbf{f}(\mathbf{x}) &\equiv \lim_{t\to 0} \frac{\mathbf{f}(\mathbf{x}+t\mathbf{v}_k) - \mathbf{f}(\mathbf{x})}{t} = \lim_{t\to 0} \frac{D\mathbf{f}(\mathbf{x})(t\mathbf{v}_k) + \mathbf{o}(t\mathbf{v}_k)}{t} \\ &= D\mathbf{f}(\mathbf{x})(\mathbf{v}_k) = \sum_{ij} J_{ij}(\mathbf{x})\mathbf{w}_i \mathbf{v}_j(\mathbf{v}_k) = \sum_{ij} J_{ij}(\mathbf{x})\mathbf{w}_i \delta_{jk} \\ &= \sum_i J_{ik}(\mathbf{x})\mathbf{w}_i \end{aligned}$$

It follows

$$\begin{aligned} & \lim_{t\to 0} \pi_j \left(\frac{\mathbf{f}(\mathbf{x}+t\mathbf{v}_k) - \mathbf{f}(\mathbf{x})}{t} \right) \\ \equiv & \lim_{t\to 0} \frac{f_j(\mathbf{x}+t\mathbf{v}_k) - f_j(\mathbf{x})}{t} \equiv D_{\mathbf{v}_k} f_j(\mathbf{x}) \\ = & \pi_j \left(\sum_i J_{ik}(\mathbf{x}) \mathbf{w}_i \right) = J_{jk}(\mathbf{x}) \end{aligned}$$

Thus $J_{ik}(\mathbf{x}) = D_{\mathbf{v}_k} f_i(\mathbf{x})$.

In the case where $X = \mathbb{R}^n$ and $Y = \mathbb{R}^m$ and $\mathbf{v}$ is a unit vector, $D_{\mathbf{v}} f_i(\mathbf{x})$ is the familiar directional derivative in the direction $\mathbf{v}$ of the function, f_i.

Of course the case where $X = \mathbb{F}^n$ and $\mathbf{f} : U \subseteq \mathbb{F}^n \to \mathbb{F}^m$, is differentiable and the basis vectors are the usual basis vectors is the case most commonly encountered. What is the matrix of $D\mathbf{f}(\mathbf{x})$ taken with respect to the usual basis vectors? Let $\mathbf{e}_i$ denote the vector of $\mathbb{F}^n$ which has a one in the i^{th} entry and zeroes elsewhere. This is the standard basis for $\mathbb{F}^n$. Denote by $J_{ij}(\mathbf{x})$ the matrix with respect to these basis vectors. Thus

$$D\mathbf{f}(\mathbf{x}) = \sum_{ij} J_{ij}(\mathbf{x}) \mathbf{e}_i \mathbf{e}_j.$$

Then from what was just shown,

$$\begin{aligned} J_{ik}(\mathbf{x}) &= D_{\mathbf{e}_k} f_i(\mathbf{x}) \equiv \lim_{t\to 0} \frac{f_i(\mathbf{x}+t\mathbf{e}_k) - f_i(\mathbf{x})}{t} \\ &\equiv \frac{\partial f_i}{\partial x_k}(\mathbf{x}) \equiv f_{i,x_k}(\mathbf{x}) \equiv f_{i,k}(\mathbf{x}) \end{aligned}$$

where the last several symbols are just the usual notations for the partial derivative of the function, f_i with respect to the k^{th} variable where

$$\mathbf{f}(\mathbf{x}) \equiv \sum_{i=1}^{m} f_i(\mathbf{x}) \mathbf{e}_i.$$

In other words, the matrix of $D\mathbf{f}(\mathbf{x})$ is nothing more than the matrix of partial derivatives. The k^{th} column of the matrix (J_{ij}) is

$$\frac{\partial \mathbf{f}}{\partial x_k}(\mathbf{x}) = \lim_{t\to 0} \frac{\mathbf{f}(\mathbf{x}+t\mathbf{e}_k) - \mathbf{f}(\mathbf{x})}{t} \equiv D_{\mathbf{e}_k} \mathbf{f}(\mathbf{x}).$$

Thus the matrix of $D\mathbf{f}(\mathbf{x})$ with respect to the usual basis vectors is the matrix of the form

$$\begin{pmatrix} f_{1,x_1}(\mathbf{x}) & f_{1,x_2}(\mathbf{x}) & \cdots & f_{1,x_n}(\mathbf{x}) \\ \vdots & \vdots & & \vdots \\ f_{m,x_1}(\mathbf{x}) & f_{m,x_2}(\mathbf{x}) & \cdots & f_{m,x_n}(\mathbf{x}) \end{pmatrix}$$

where the notation $g_{,x_k}$ denotes the k^{th} partial derivative given by the limit,

$$\lim_{t\to 0} \frac{g(\mathbf{x}+t\mathbf{e}_k) - g(\mathbf{x})}{t} \equiv \frac{\partial g}{\partial x_k}.$$

The above discussion is summarized in the following theorem.

Theorem 6.3.1 *Let $\mathbf{f} : \mathbb{F}^n \to \mathbb{F}^m$ and suppose $\mathbf{f}$ is differentiable at $\mathbf{x}$. Then all the partial derivatives $\frac{\partial f_i(\mathbf{x})}{\partial x_j}$ exist and if $J\mathbf{f}(\mathbf{x})$ is the matrix of the linear transformation, $D\mathbf{f}(\mathbf{x})$ with respect to the standard basis vectors, then the ij^{th} entry is given by $\frac{\partial f_i}{\partial x_j}(\mathbf{x})$ also denoted as $f_{i,j}$ or f_{i,x_j}.*

Definition 6.3.2 *In general, the symbol*

$$D_{\mathbf{v}}\mathbf{f}(\mathbf{x})$$

is defined by

$$\lim_{t\to 0}\frac{\mathbf{f}(\mathbf{x}+t\mathbf{v})-\mathbf{f}(\mathbf{x})}{t}$$

where $t \in \mathbb{F}$. This is often called the Gateaux derivative.

What if all the partial derivatives of $\mathbf{f}$ exist? Does it follow that $\mathbf{f}$ is differentiable? Consider the following function, $f : \mathbb{R}^2 \to \mathbb{R}$,

$$f(x,y) = \begin{cases} \frac{xy}{x^2+y^2} \text{ if } (x,y) \neq (0,0) \\ 0 \text{ if } (x,y) = (0,0) \end{cases}.$$

Then from the definition of partial derivatives,

$$\lim_{h\to 0}\frac{f(h,0)-f(0,0)}{h} = \lim_{h\to 0}\frac{0-0}{h} = 0$$

and

$$\lim_{h\to 0}\frac{f(0,h)-f(0,0)}{h} = \lim_{h\to 0}\frac{0-0}{h} = 0$$

However f is not even continuous at $(0,0)$ which may be seen by considering the behavior of the function along the line $y = x$ and along the line $x = 0$. By Lemma 6.1.3 this implies f is not differentiable. Therefore, it is necessary to consider the correct definition of the derivative given above if you want to get a notion which generalizes the concept of the derivative of a function of one variable in such a way as to preserve continuity whenever the function is differentiable.

6.4 A Mean Value Inequality

The following theorem will be very useful in much of what follows. It is a version of the mean value theorem as is the next lemma.

Lemma 6.4.1 *Let Y be a normed vector space and suppose $\mathbf{h} : [0,1] \to Y$ is differentiable and satisfies*

$$||\mathbf{h}'(t)|| \leq M.$$

Then

$$||\mathbf{h}(1)-\mathbf{h}(0)|| \leq M.$$

Proof: Let $\varepsilon > 0$ be given and let

$$S \equiv \{t \in [0,1] : \text{ for all } s \in [0,t], ||\mathbf{h}(s)-\mathbf{h}(0)|| \leq (M+\varepsilon)s\}$$

Then $0 \in S$. Let $t = \sup S$. Then by continuity of $\mathbf{h}$ it follows

$$||\mathbf{h}(t)-\mathbf{h}(0)|| = (M+\varepsilon)t \tag{6.5}$$

Suppose $t < 1$. Then there exist positive numbers, h_k decreasing to 0 such that

$$||\mathbf{h}(t+h_k) - \mathbf{h}(0)|| > (M+\varepsilon)(t+h_k)$$

and now it follows from 6.5 and the triangle inequality that

$$\begin{aligned} & ||\mathbf{h}(t+h_k) - \mathbf{h}(t)|| + ||\mathbf{h}(t) - \mathbf{h}(0)|| \\ = \; & ||\mathbf{h}(t+h_k) - \mathbf{h}(t)|| + (M+\varepsilon)\, t > (M+\varepsilon)(t+h_k) \end{aligned}$$

and so

$$||\mathbf{h}(t+h_k) - \mathbf{h}(t)|| > (M+\varepsilon)\, h_k$$

Now dividing by h_k and letting $k \to \infty$

$$||\mathbf{h}'(t)|| \geq M + \varepsilon,$$

a contradiction. This proves the lemma.

Theorem 6.4.2 *Suppose U is an open subset of X and $\mathbf{f} : U \to Y$ has the property that $D\mathbf{f}(\mathbf{x})$ exists for all $\mathbf{x}$ in U and that, $\mathbf{x} + t(\mathbf{y} - \mathbf{x}) \in U$ for all $t \in [0,1]$. (The line segment joining the two points lies in U.) Suppose also that for all points on this line segment,*

$$||D\mathbf{f}(\mathbf{x}+t(\mathbf{y}-\mathbf{x}))|| \leq M.$$

Then

$$||\mathbf{f}(\mathbf{y}) - \mathbf{f}(\mathbf{x})|| \leq M\,|\mathbf{y} - \mathbf{x}|.$$

Proof: Let

$$\mathbf{h}(t) \equiv \mathbf{f}(\mathbf{x} + t(\mathbf{y} - \mathbf{x})).$$

Then by the chain rule,

$$\mathbf{h}'(t) = D\mathbf{f}(\mathbf{x} + t(\mathbf{y} - \mathbf{x}))(\mathbf{y} - \mathbf{x})$$

and so

$$\begin{aligned} ||\mathbf{h}'(t)|| &= ||D\mathbf{f}(\mathbf{x} + t(\mathbf{y} - \mathbf{x}))(\mathbf{y} - \mathbf{x})|| \\ &\leq M\,||\mathbf{y} - \mathbf{x}|| \end{aligned}$$

by Lemma 6.4.1

$$||\mathbf{h}(1) - \mathbf{h}(0)|| = ||\mathbf{f}(\mathbf{y}) - \mathbf{f}(\mathbf{x})|| \leq M\,||\mathbf{y} - \mathbf{x}||.$$

This proves the theorem.

6.5 Existence Of The Derivative, C^1 Functions

There is a way to get the differentiability of a function from the existence and continuity of the Gateaux derivatives. This is very convenient because these Gateaux derivatives are taken with respect to a one dimensional variable. The following theorem is the main result.

Theorem 6.5.1 *Let X be a normed vector space having basis $\{\mathbf{v}_1, \cdots, \mathbf{v}_n\}$ and let Y be another normed vector space having basis $\{\mathbf{w}_1, \cdots, \mathbf{w}_m\}$. Let U be an open set in X and let $\mathbf{f} : U \to Y$ have the property that the Gateaux derivatives,*

$$D_{\mathbf{v}_k}\mathbf{f}(\mathbf{x}) \equiv \lim_{t \to 0} \frac{\mathbf{f}(\mathbf{x} + t\mathbf{v}_k) - \mathbf{f}(\mathbf{x})}{t}$$

exist and are continuous functions of $\mathbf{x}$. *Then* $D\mathbf{f}(\mathbf{x})$ *exists and*

$$D\mathbf{f}(\mathbf{x})\,\mathbf{v} = \sum_{k=1}^{n} D_{\mathbf{v}_k}\mathbf{f}(\mathbf{x})\,a_k$$

where

$$\mathbf{v} = \sum_{k=1}^{n} a_k \mathbf{v}_k.$$

Furthermore, $\mathbf{x} \to D\mathbf{f}(\mathbf{x})$ *is continuous; that is*

$$\lim_{\mathbf{y}\to\mathbf{x}} ||D\mathbf{f}(\mathbf{y}) - D\mathbf{f}(\mathbf{x})|| = 0.$$

Proof: Let $\mathbf{v} = \sum_{k=1}^{n} a_k \mathbf{v}_k$. Then

$$\mathbf{f}(\mathbf{x}+\mathbf{v}) - \mathbf{f}(\mathbf{x}) = \mathbf{f}\left(\mathbf{x} + \sum_{k=1}^{n} a_k\mathbf{v}_k\right) - \mathbf{f}(\mathbf{x}).$$

Then letting $\sum_{k=1}^{0} \equiv 0$, $\mathbf{f}(\mathbf{x}+\mathbf{v}) - \mathbf{f}(\mathbf{x})$ is given by

$$\sum_{k=1}^{n}\left[\mathbf{f}\left(\mathbf{x} + \sum_{j=1}^{k} a_j\mathbf{v}_j\right) - \mathbf{f}\left(\mathbf{x} + \sum_{j=1}^{k-1} a_j\mathbf{v}_j\right)\right]$$

$$= \sum_{k=1}^{n}\left[\mathbf{f}(\mathbf{x} + a_k\mathbf{v}_k) - \mathbf{f}(\mathbf{x})\right] +$$

$$\sum_{k=1}^{n}\left[\left(\mathbf{f}\left(\mathbf{x} + \sum_{j=1}^{k} a_j\mathbf{v}_j\right) - \mathbf{f}(\mathbf{x} + a_k\mathbf{v}_k)\right) - \left(\mathbf{f}\left(\mathbf{x} + \sum_{j=1}^{k-1} a_j\mathbf{v}_j\right) - \mathbf{f}(\mathbf{x})\right)\right] \tag{6.6}$$

Consider the k^{th} term in 6.6. Let

$$\mathbf{h}(t) \equiv \mathbf{f}\left(\mathbf{x} + \sum_{j=1}^{k-1} a_j\mathbf{v}_j + ta_k\mathbf{v}_k\right) - \mathbf{f}(\mathbf{x} + ta_k\mathbf{v}_k)$$

for $t \in [0,1]$. Then

$$\begin{aligned}\mathbf{h}'(t) &= a_k \lim_{h\to 0} \frac{1}{a_k h}\left(\mathbf{f}\left(\mathbf{x} + \sum_{j=1}^{k-1} a_j\mathbf{v}_j + (t+h)\,a_k\mathbf{v}_k\right) - \mathbf{f}(\mathbf{x} + (t+h)\,a_k\mathbf{v}_k)\right.\\ &\quad \left. - \left(\mathbf{f}\left(\mathbf{x} + \sum_{j=1}^{k-1} a_j\mathbf{v}_j + ta_k\mathbf{v}_k\right) - \mathbf{f}(\mathbf{x} + ta_k\mathbf{v}_k)\right)\right)\end{aligned}$$

and this equals

$$\left(D_{\mathbf{v}_k}\mathbf{f}\left(\mathbf{x} + \sum_{j=1}^{k-1} a_j\mathbf{v}_j + ta_k\mathbf{v}_k\right) - D_{\mathbf{v}_k}\mathbf{f}(\mathbf{x} + ta_k\mathbf{v}_k)\right)a_k \tag{6.7}$$

Now without loss of generality, it can be assumed the norm on X is given by that of Example 5.8.5,

$$||\mathbf{v}|| \equiv \max\left\{|a_k| : \mathbf{v} = \sum_{j=1}^{n} a_k \mathbf{v}_k\right\}$$

because by Theorem 5.8.4 all norms on X are equivalent. Therefore, from 6.7 and the assumption that the Gateaux derivatives are continuous,

$$\begin{aligned} ||\mathbf{h}'(t)|| &= \left|\left|\left(D_{\mathbf{v}_k}\mathbf{f}\left(\mathbf{x}+\sum_{j=1}^{k-1} a_j\mathbf{v}_j + ta_k\mathbf{v}_k\right) - D_{\mathbf{v}_k}\mathbf{f}\left(\mathbf{x}+ta_k\mathbf{v}_k\right)\right)a_k\right|\right| \\ &\leq \varepsilon\,|a_k| \leq \varepsilon\,||\mathbf{v}|| \end{aligned}$$

provided $||\mathbf{v}||$ is sufficiently small. Since ε is arbitrary, it follows from Lemma 6.4.1 the expression in 6.6 is $\mathbf{o}(\mathbf{v})$ because this expression equals a finite sum of terms of the form $\mathbf{h}(1) - \mathbf{h}(0)$ where $||\mathbf{h}'(t)|| \leq \varepsilon\,||\mathbf{v}||$. Thus

$$\mathbf{f}(\mathbf{x}+\mathbf{v}) - \mathbf{f}(\mathbf{x}) = \sum_{k=1}^{n}\left[\mathbf{f}(\mathbf{x}+a_k\mathbf{v}_k) - \mathbf{f}(\mathbf{x})\right] + \mathbf{o}(\mathbf{v})$$

$$= \sum_{k=1}^{n} D_{\mathbf{v}_k}\mathbf{f}(\mathbf{x})\,a_k + \sum_{k=1}^{n}\left[\mathbf{f}(\mathbf{x}+a_k\mathbf{v}_k) - \mathbf{f}(\mathbf{x}) - D_{\mathbf{v}_k}\mathbf{f}(\mathbf{x})\,a_k\right] + \mathbf{o}(\mathbf{v}).$$

Consider the k^{th} term in the second sum.

$$\mathbf{f}(\mathbf{x}+a_k\mathbf{v}_k) - \mathbf{f}(\mathbf{x}) - D_{\mathbf{v}_k}\mathbf{f}(\mathbf{x})\,a_k = a_k\left(\frac{\mathbf{f}(\mathbf{x}+a_k\mathbf{v}_k) - \mathbf{f}(\mathbf{x})}{a_k} - D_{\mathbf{v}_k}\mathbf{f}(\mathbf{x})\right)$$

where the expression in the parentheses converges to 0 as $a_k \to 0$. Thus whenever $||\mathbf{v}||$ is sufficiently small,

$$||\mathbf{f}(\mathbf{x}+a_k\mathbf{v}_k) - \mathbf{f}(\mathbf{x}) - D_{\mathbf{v}_k}\mathbf{f}(\mathbf{x})\,a_k|| \leq \varepsilon\,|a_k| \leq \varepsilon\,||\mathbf{v}||$$

which shows the second sum is also $\mathbf{o}(\mathbf{v})$. Therefore,

$$\mathbf{f}(\mathbf{x}+\mathbf{v}) - \mathbf{f}(\mathbf{x}) = \sum_{k=1}^{n} D_{\mathbf{v}_k}\mathbf{f}(\mathbf{x})\,a_k + \mathbf{o}(\mathbf{v}).$$

Defining

$$D\mathbf{f}(\mathbf{x})\,\mathbf{v} \equiv \sum_{k=1}^{n} D_{\mathbf{v}_k}\mathbf{f}(\mathbf{x})\,a_k$$

where $\mathbf{v} = \sum_k a_k\mathbf{v}_k$, it follows $D\mathbf{f}(\mathbf{x}) \in \mathcal{L}(X,Y)$ and is given by the above formula.

It remains to verify $\mathbf{x} \to D\mathbf{f}(\mathbf{x})$ is continuous.

$$\begin{aligned} &||(D\mathbf{f}(\mathbf{x}) - D\mathbf{f}(\mathbf{y}))\,\mathbf{v}|| \\ \leq\ &\sum_{k=1}^{n} ||(D_{\mathbf{v}_k}\mathbf{f}(\mathbf{x}) - D_{\mathbf{v}_k}\mathbf{f}(\mathbf{y}))\,a_k|| \\ \leq\ &\max\{|a_k|, k = 1,\cdots,n\}\sum_{k=1}^{n} ||D_{\mathbf{v}_k}\mathbf{f}(\mathbf{x}) - D_{\mathbf{v}_k}\mathbf{f}(\mathbf{y})|| \\ =\ &||\mathbf{v}||\sum_{k=1}^{n} ||D_{\mathbf{v}_k}\mathbf{f}(\mathbf{x}) - D_{\mathbf{v}_k}\mathbf{f}(\mathbf{y})|| \end{aligned}$$

and so

$$||D\mathbf{f}(\mathbf{x}) - D\mathbf{f}(\mathbf{y})|| \leq \sum_{k=1}^{n} ||D_{\mathbf{v}_k}\mathbf{f}(\mathbf{x}) - D_{\mathbf{v}_k}\mathbf{f}(\mathbf{y})||$$

which proves the continuity of $D\mathbf{f}$ because of the assumption the Gateaux derivatives are continuous. This proves the theorem.

This motivates the following definition of what it means for a function to be C^1.

Definition 6.5.2 *Let U be an open subset of a normed finite dimensional vector space, X and let $\mathbf{f} : U \to Y$ another finite dimensional normed vector space. Then $\mathbf{f}$ is said to be C^1 if there exists a basis for $X, \{\mathbf{v}_1, \cdots, \mathbf{v}_n\}$ such that the Gateaux derivatives,*

$$D_{\mathbf{v}_k}\mathbf{f}(\mathbf{x})$$

exist on U and are continuous.

Here is another definition of what it means for a function to be C^1.

Definition 6.5.3 *Let U be an open subset of a normed finite dimensional vector space, X and let $\mathbf{f} : U \to Y$ another finite dimensional normed vector space. Then $\mathbf{f}$ is said to be C^1 if $\mathbf{f}$ is differentiable and $\mathbf{x} \to D\mathbf{f}(\mathbf{x})$ is continuous as a map from U to $\mathcal{L}(X, Y)$.*

Now the following major theorem states these two definitions are equivalent.

Theorem 6.5.4 *Let U be an open subset of a normed finite dimensional vector space, X and let $\mathbf{f} : U \to Y$ another finite dimensional normed vector space. Then the two definitions above are equivalent.*

Proof: It was shown in Theorem 6.5.1 that Definition 6.5.2 implies 6.5.3. Suppose then that Definition 6.5.3 holds. Then if $\mathbf{v}$ is any vector,

$$\begin{aligned} \lim_{t\to 0} \frac{\mathbf{f}(\mathbf{x}+t\mathbf{v}) - \mathbf{f}(\mathbf{x})}{t} &= \lim_{t\to 0} \frac{D\mathbf{f}(\mathbf{x})\, t\mathbf{v} + \mathbf{o}(t\mathbf{v})}{t} \\ &= D\mathbf{f}(\mathbf{x})\,\mathbf{v} + \lim_{t\to 0} \frac{\mathbf{o}(t\mathbf{v})}{t} = D\mathbf{f}(\mathbf{x})\,\mathbf{v} \end{aligned}$$

Thus $D_{\mathbf{v}}\mathbf{f}(\mathbf{x})$ exists and equals $D\mathbf{f}(\mathbf{x})\,\mathbf{v}$. By continuity of $\mathbf{x} \to D\mathbf{f}(\mathbf{x})$, this establishes continuity of $\mathbf{x} \to D_{\mathbf{v}}\mathbf{f}(\mathbf{x})$ and proves the theorem.

Note that the proof of the theorem also implies the following corollary.

Corollary 6.5.5 *Let U be an open subset of a normed finite dimensional vector space, X and let $\mathbf{f} : U \to Y$ another finite dimensional normed vector space. Then if there is a basis of $X, \{\mathbf{v}_1, \cdots, \mathbf{v}_n\}$ such that the Gateaux derivatives, $D_{\mathbf{v}_k}\mathbf{f}(\mathbf{x})$ exist and are continuous. Then all Gateaux derivatives, $D_{\mathbf{v}}\mathbf{f}(\mathbf{x})$ exist and are continuous for all $\mathbf{v} \in X$.*

From now on, whichever definition is more convenient will be used.

6.6 Higher Order Derivatives

If $f : U \subseteq X \to Y$ for U an open set, then

$$\mathbf{x} \to D\mathbf{f}(\mathbf{x})$$

is a mapping from U to $\mathcal{L}(X, Y)$, a normed vector space. Therefore, it makes perfect sense to ask whether this function is also differentiable.

Definition 6.6.1 *The following is the definition of the second derivative.*

$$D^2\mathbf{f}(\mathbf{x}) \equiv D(D\mathbf{f}(\mathbf{x})).$$

Thus,

$$D\mathbf{f}(\mathbf{x}+\mathbf{v}) - D\mathbf{f}(\mathbf{x}) = D^2\mathbf{f}(\mathbf{x})\,\mathbf{v} + \mathbf{o}(\mathbf{v}).$$

This implies

$$D^2\mathbf{f}(\mathbf{x}) \in \mathcal{L}(X, \mathcal{L}(X,Y)),\ D^2\mathbf{f}(\mathbf{x})(\mathbf{u})(\mathbf{v}) \in Y,$$

and the map

$$(\mathbf{u},\mathbf{v}) \to D^2\mathbf{f}(\mathbf{x})(\mathbf{u})(\mathbf{v})$$

is a bilinear map having values in Y. In other words, the two functions,

$$\mathbf{u} \to D^2\mathbf{f}(\mathbf{x})(\mathbf{u})(\mathbf{v}),\ \mathbf{v} \to D^2\mathbf{f}(\mathbf{x})(\mathbf{u})(\mathbf{v})$$

are both linear.

The same pattern applies to taking higher order derivatives. Thus,

$$D^3\mathbf{f}(\mathbf{x}) \equiv D\left(D^2\mathbf{f}(\mathbf{x})\right)$$

and $D^3\mathbf{f}(\mathbf{x})$ may be considered as a trilinear map having values in Y. In general $D^k\mathbf{f}(\mathbf{x})$ may be considered a k linear map. This means the function

$$(\mathbf{u}_1,\cdots,\mathbf{u}_k) \to D^k\mathbf{f}(\mathbf{x})(\mathbf{u}_1)\cdots(\mathbf{u}_k)$$

has the property

$$\mathbf{u}_j \to D^k\mathbf{f}(\mathbf{x})(\mathbf{u}_1)\cdots(\mathbf{u}_j)\cdots(\mathbf{u}_k)$$

is linear.

Also, instead of writing

$$D^2\mathbf{f}(\mathbf{x})(\mathbf{u})(\mathbf{v}),\ \text{or } D^3\mathbf{f}(\mathbf{x})(\mathbf{u})(\mathbf{v})(\mathbf{w})$$

the following notation is often used.

$$D^2\mathbf{f}(\mathbf{x})(\mathbf{u},\mathbf{v})\ \text{or } D^3\mathbf{f}(\mathbf{x})(\mathbf{u},\mathbf{v},\mathbf{w})$$

with similar conventions for higher derivatives than 3. Another convention which is often used is the notation

$$D^k\mathbf{f}(\mathbf{x})\,\mathbf{v}^k$$

instead of

$$D^k\mathbf{f}(\mathbf{x})(\mathbf{v},\cdots,\mathbf{v}).$$

Note that for every k, $D^k\mathbf{f}$ maps U to a normed vector space. As mentioned above, $D\mathbf{f}(\mathbf{x})$ has values in $\mathcal{L}(X,Y)$, $D^2\mathbf{f}(\mathbf{x})$ has values in $\mathcal{L}(X,\mathcal{L}(X,Y))$, etc. Thus it makes sense to consider whether $D^k\mathbf{f}$ is continuous. This is described in the following definition.

Definition 6.6.2 *Let U be an open subset of X, a normed vector space and let $\mathbf{f}: U \to Y$. Then $\mathbf{f}$ is $C^k(U)$ if $\mathbf{f}$ and its first k derivatives are all continuous. Also, $D^k\mathbf{f}(\mathbf{x})$ when it exists can be considered a Y valued multilinear function.*

6.7 C^k Functions

Recall that for a C^1 function, $\mathbf{f}$

$$\begin{aligned} D\mathbf{f}(\mathbf{x})\,\mathbf{v} &= \sum_j D_{\mathbf{v}_j}\mathbf{f}(\mathbf{x})\,a_j = \sum_{ij} D_{\mathbf{v}_j} f_i(\mathbf{x})\,\mathbf{w}_i a_j \\ &= \sum_{ij} D_{\mathbf{v}_j} f_i(\mathbf{x})\,\mathbf{w}_i\mathbf{v}_j\left(\sum_k a_k\mathbf{v}_k\right) = \sum_{ij} D_{\mathbf{v}_j} f_i(\mathbf{x})\,\mathbf{w}_i\mathbf{v}_j(\mathbf{v}) \end{aligned}$$

where $\sum_k a_k\mathbf{v}_k = \mathbf{v}$ and

$$\mathbf{f}(\mathbf{x}) = \sum_i f_i(\mathbf{x})\,\mathbf{w}_i. \tag{6.8}$$

This is because

$$\mathbf{w}_i\mathbf{v}_j\left(\sum_k a_k\mathbf{v}_k\right) \equiv \sum_k a_k\mathbf{w}_i\delta_{jk} = \mathbf{w}_i a_j.$$

Thus

$$D\mathbf{f}(\mathbf{x}) = \sum_{ij} D_{\mathbf{v}_j} f_i(\mathbf{x})\,\mathbf{w}_i\mathbf{v}_j$$

I propose to iterate this observation, starting with $\mathbf{f}$ and then going to $D\mathbf{f}$ and then $D^2\mathbf{f}$ and so forth. Hopefully it will yield a rational way to understand higher order derivatives in the same way that matrices can be used to understand linear transformations. Thus beginning with the derivative,

$$D\mathbf{f}(\mathbf{x}) = \sum_{ij_1} D_{\mathbf{v}_{j_1}} f_i(\mathbf{x})\,\mathbf{w}_i\mathbf{v}_{j_1}.$$

Then letting $\mathbf{w}_i\mathbf{v}_{j_1}$ play the role of $\mathbf{w}_i$ in 6.8,

$$\begin{aligned} D^2\mathbf{f}(\mathbf{x}) &= \sum_{ij_1j_2} D_{\mathbf{v}_{j_2}}\left(D_{\mathbf{v}_{j_1}} f_i\right)(\mathbf{x})\,\mathbf{w}_i\mathbf{v}_{j_1}\mathbf{v}_{j_2} \\ &\equiv \sum_{ij_1j_2} D_{\mathbf{v}_{j_1}\mathbf{v}_{j_2}} f_i(\mathbf{x})\,\mathbf{w}_i\mathbf{v}_{j_1}\mathbf{v}_{j_2} \end{aligned}$$

Then letting $\mathbf{w}_i\mathbf{v}_{j_1}\mathbf{v}_{j_2}$ play the role of $\mathbf{w}_i$ in 6.8,

$$\begin{aligned} D^3\mathbf{f}(\mathbf{x}) &= \sum_{ij_1j_2j_3} D_{\mathbf{v}_{j_3}}\left(D_{\mathbf{v}_{j_1}\mathbf{v}_{j_2}} f_i\right)(\mathbf{x})\,\mathbf{w}_i\mathbf{v}_{j_1}\mathbf{v}_{j_2}\mathbf{v}_{j_3} \\ &\equiv \sum_{ij_1j_2j_3} D_{\mathbf{v}_{j_1}\mathbf{v}_{j_2}\mathbf{v}_{j_3}} f_i(\mathbf{x})\,\mathbf{w}_i\mathbf{v}_{j_1}\mathbf{v}_{j_2}\mathbf{v}_{j_3} \end{aligned}$$

etc. In general, the notation,

$$\mathbf{w}_i\mathbf{v}_{j_1}\mathbf{v}_{j_2}\cdots\mathbf{v}_{j_n}$$

defines an appropriate linear transformation given by

$$\mathbf{w}_i\mathbf{v}_{j_1}\mathbf{v}_{j_2}\cdots\mathbf{v}_{j_n}(\mathbf{v}_k) = \mathbf{w}_i\mathbf{v}_{j_1}\mathbf{v}_{j_2}\cdots\mathbf{v}_{j_{n-1}}\delta_{kj_n}.$$

The following theorem is important.

Theorem 6.7.1 *The function $\mathbf{x} \to D^k \mathbf{f}(\mathbf{x})$ exists and is continuous for $k \leq p$ if and only if there exists a basis for $X, \{\mathbf{v}_1, \cdots, \mathbf{v}_n\}$ and a basis for $Y, \{\mathbf{w}_1, \cdots, \mathbf{w}_m\}$ such that for*

$$\mathbf{f}(\mathbf{x}) \equiv \sum_i f_i(\mathbf{x}) \mathbf{w}_i,$$

it follows that for each $i = 1, 2, \cdots, m$ all Gateaux derivatives,

$$D_{\mathbf{v}_{j_1}\mathbf{v}_{j_2}\cdots\mathbf{v}_{j_k}} f_i(\mathbf{x})$$

for any choice of $\mathbf{v}_{j_1}\mathbf{v}_{j_2}\cdots\mathbf{v}_{j_k}$ and for any $k \leq p$ exist and are continuous.

Proof: This follows from a repeated application of Theorems 6.5.1 and 6.5.4 at each new differentiation.

Definition 6.7.2 *Let X, Y be finite dimensional normed vector spaces and let U be an open set in X and $\mathbf{f} : U \to Y$ be a function,*

$$\mathbf{f}(\mathbf{x}) = \sum_i f_i(\mathbf{x}) \mathbf{w}_i$$

where $\{\mathbf{w}_1, \cdots, \mathbf{w}_m\}$ is a basis for Y. Then $\mathbf{f}$ is said to be a $C^n(U)$ function if for every $k \leq n, D^k \mathbf{f}(\mathbf{x})$ exists for all $\mathbf{x} \in U$ and is continuous. This is equivalent to the other condition which states that for each $i = 1, 2, \cdots, m$ all Gateaux derivatives,

$$D_{\mathbf{v}_{j_1}\mathbf{v}_{j_2}\cdots\mathbf{v}_{j_k}} f_i(\mathbf{x})$$

for any choice of $\mathbf{v}_{j_1}\mathbf{v}_{j_2}\cdots\mathbf{v}_{j_k}$ where $\{\mathbf{v}_1, \cdots, \mathbf{v}_n\}$ is a basis for X and for any $k \leq n$ exist and are continuous.

6.7.1 Some Standard Notation

In the case where $X = \mathbb{R}^n$ and the basis chosen is the standard basis, these Gateaux derivatives are just the partial derivatives. Recall the notation for partial derivatives in the following definition.

Definition 6.7.3 *Let $\mathbf{g} : U \to X$. Then*

$$\mathbf{g}_{x_k}(\mathbf{x}) \equiv \frac{\partial \mathbf{g}}{\partial x_k}(\mathbf{x}) \equiv \lim_{h \to 0} \frac{\mathbf{g}(\mathbf{x} + h\mathbf{e}_k) - \mathbf{g}(\mathbf{x})}{h}$$

Higher order partial derivatives are defined in the usual way.

$$\mathbf{g}_{x_k x_l}(\mathbf{x}) \equiv \frac{\partial^2 \mathbf{g}}{\partial x_l \partial x_k}(\mathbf{x})$$

and so forth.

A convenient notation which is often used which helps to make sense of higher order partial derivatives is presented in the following definition.

Definition 6.7.4 *$\boldsymbol{\alpha} = (\alpha_1, \cdots, \alpha_n)$ for $\alpha_1 \cdots \alpha_n$ positive integers is called a multi-index. For $\boldsymbol{\alpha}$ a multi-index, $|\boldsymbol{\alpha}| \equiv \alpha_1 + \cdots + \alpha_n$ and if $\mathbf{x} \in X$,*

$$\mathbf{x} = (x_1, \cdots, x_n),$$

and $\mathbf{f}$ a function, define

$$\mathbf{x}^{\boldsymbol{\alpha}} \equiv x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n}, \; D^{\boldsymbol{\alpha}} \mathbf{f}(\mathbf{x}) \equiv \frac{\partial^{|\boldsymbol{\alpha}|} \mathbf{f}(\mathbf{x})}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \cdots \partial x_n^{\alpha_n}}.$$

Then in this special case, the following definition is equivalent to the above as a definition of what is meant by a C^k function.

Definition 6.7.5 *Let U be an open subset of $\mathbb{R}^n$ and let $\mathbf{f} : U \to Y$. Then for k a nonnegative integer, $\mathbf{f}$ is C^k if for every $|\boldsymbol{\alpha}| \leq k$, $D^{\boldsymbol{\alpha}}\mathbf{f}$ exists and is continuous.*

6.8 The Derivative Of A Function Defined On A Cartesian Product

There are theorems which can be used to get differentiability of a function based on existence and continuity of the partial derivatives. A generalization of this was given above. Here a function defined on a product space is considered. It is very much like what was presented above and could be obtained as a special case but to reinforce the ideas, I will do it from scratch because certain aspects of it are important in the statement of the implicit function theorem.

The following is an important abstract generalization of the concept of partial derivative presented above. Insead of taking the derivative with respect to one variable, it is taken with respect to several but not with respect to others. This vague notion is made precise in the following definition. First here is a lemma.

Lemma 6.8.1 *Suppose U is an open set in $X \times Y$. Then the set, $U_{\mathbf{y}}$ defined by*

$$U_{\mathbf{y}} \equiv \{\mathbf{x} \in X : (\mathbf{x}, \mathbf{y}) \in U\}$$

is an open set in X. Here $X \times Y$ is a finite dimensional vector space in which the vector space operations are defined componentwise. Thus for $a, b \in \mathbb{F}$,

$$a(\mathbf{x}_1, \mathbf{y}_1) + b(\mathbf{x}_2, \mathbf{y}_2) = (a\mathbf{x}_1 + b\mathbf{x}_2, a\mathbf{y}_1 + b\mathbf{y}_2)$$

and the norm can be taken to be

$$||(\mathbf{x}, \mathbf{y})|| \equiv \max(||\mathbf{x}||, ||\mathbf{y}||)$$

Proof: Recall by Theorem 5.8.4 it does not matter how this norm is defined and the definition above is convenient. It obviously satisfies most axioms of a norm. The only one which is not obvious is the triangle inequality. I will show this now.

$$\begin{aligned} ||(\mathbf{x}, \mathbf{y}) + (\mathbf{x}_1, \mathbf{y}_1)|| &= ||(\mathbf{x} + \mathbf{x}_1, \mathbf{y} + \mathbf{y}_1)|| \equiv \max(||\mathbf{x} + \mathbf{x}_1||, ||\mathbf{y} + \mathbf{y}_1||) \\ &\leq \max(||\mathbf{x}|| + ||\mathbf{x}_1||, ||\mathbf{y}|| + ||\mathbf{y}_1||) \end{aligned}$$

suppose then that $||\mathbf{x}|| + ||\mathbf{x}_1|| \geq ||\mathbf{y}|| + ||\mathbf{y}_1||$. Then the above equals

$$||\mathbf{x}|| + ||\mathbf{x}_1|| \leq \max(||\mathbf{x}||, ||\mathbf{y}||) + \max(||\mathbf{x}_1||, ||\mathbf{y}_1||) \equiv ||(\mathbf{x}, \mathbf{y})|| + ||(\mathbf{x}_1, \mathbf{y}_1)||$$

In case $||\mathbf{x}|| + ||\mathbf{x}_1|| < ||\mathbf{y}|| + ||\mathbf{y}_1||$, the argument is similar.

Let $\mathbf{x} \in U_{\mathbf{y}}$. Then $(\mathbf{x}, \mathbf{y}) \in U$ and so there exists $r > 0$ such that

$$B((\mathbf{x}, \mathbf{y}), r) \in U.$$

This says that if $(\mathbf{u}, \mathbf{v}) \in X \times Y$ such that $||(\mathbf{u}, \mathbf{v}) - (\mathbf{x}, \mathbf{y})|| < r$, then $(\mathbf{u}, \mathbf{v}) \in U$. Thus if

$$||(\mathbf{u}, \mathbf{y}) - (\mathbf{x}, \mathbf{y})|| = ||\mathbf{u} - \mathbf{x}|| < r,$$

then $(\mathbf{u},\mathbf{y}) \in U$. This has just said that $B(\mathbf{x},r)$, the ball taken in X is contained in $U_{\mathbf{y}}$. This proves the lemma.

Or course one could also consider

$$U_{\mathbf{x}} \equiv \{\mathbf{y} : (\mathbf{x},\mathbf{y}) \in U\}$$

in the same way and conclude this set is open in Y. Also, the generalization to many factors yields the same conclusion. In this case, for $\mathbf{x} \in \prod_{i=1}^n X_i$, let

$$||\mathbf{x}|| \equiv \max\left(||\mathbf{x}_i||_{X_i} : \mathbf{x} = (\mathbf{x}_1,\cdots,\mathbf{x}_n)\right)$$

Then a similar argument to the above shows this is a norm on $\prod_{i=1}^n X_i$.

Corollary 6.8.2 *Let $U \subseteq \prod_{i=1}^n X_i$ and let*

$$U_{(\mathbf{x}_1,\cdots,\mathbf{x}_{i-1},\mathbf{x}_{i+1},\cdots,\mathbf{x}_n)} \equiv \left\{\mathbf{x} \in \mathbb{F}^{r_i} : \left(\mathbf{x}_1,\cdots,\mathbf{x}_{i-1},\mathbf{x},\mathbf{x}_{i+1},\cdots,\mathbf{x}_n\right) \in U\right\}.$$

Then $U_{(\mathbf{x}_1,\cdots,\mathbf{x}_{i-1},\mathbf{x}_{i+1},\cdots,\mathbf{x}_n)}$ is an open set in $\mathbb{F}^{r_i}$.

The proof is similar to the above.

Definition 6.8.3 *Let $\mathbf{g} : U \subseteq \prod_{i=1}^n X_i \to Y$, where U is an open set. Then the map*

$$\mathbf{z} \to \mathbf{g}\left(\mathbf{x}_1,\cdots,\mathbf{x}_{i-1},\mathbf{z},\mathbf{x}_{i+1},\cdots,\mathbf{x}_n\right)$$

is a function from the open set in X_i,

$$\left\{\mathbf{z} : \mathbf{x} = \left(\mathbf{x}_1,\cdots,\mathbf{x}_{i-1},\mathbf{z},\mathbf{x}_{i+1},\cdots,\mathbf{x}_n\right) \in U\right\}$$

to Y. When this map is differentiable, its derivative is denoted by $D_i\mathbf{g}(\mathbf{x})$. To aid in the notation, for $\mathbf{v} \in X_i$, let $\theta_i\mathbf{v} \in \prod_{i=1}^n X_i$ be the vector $(\mathbf{0},\cdots,\mathbf{v},\cdots,\mathbf{0})$ where the $\mathbf{v}$ is in the i^{th} slot and for $\mathbf{v} \in \prod_{i=1}^n X_i$, let $\mathbf{v}_i$ denote the entry in the i^{th} slot of $\mathbf{v}$. Thus, by saying

$$\mathbf{z} \to \mathbf{g}\left(\mathbf{x}_1,\cdots,\mathbf{x}_{i-1},\mathbf{z},\mathbf{x}_{i+1},\cdots,\mathbf{x}_n\right)$$

is differentiable is meant that for $\mathbf{v} \in X_i$ sufficiently small,

$$\mathbf{g}(\mathbf{x}+\theta_i\mathbf{v}) - \mathbf{g}(\mathbf{x}) = D_i\mathbf{g}(\mathbf{x})\,\mathbf{v} + \mathbf{o}(\mathbf{v}).$$

Note $D_i\mathbf{g}(\mathbf{x}) \in \mathcal{L}(X_i,Y)$.

Definition 6.8.4 *Let $U \subseteq X$ be an open set. Then $\mathbf{f} : U \to Y$ is $C^1(U)$ if $\mathbf{f}$ is differentiable and the mapping*

$$\mathbf{x} \to D\mathbf{f}(\mathbf{x}),$$

is continuous as a function from U to $\mathcal{L}(X,Y)$.

With this definition of partial derivatives, here is the major theorem.

Theorem 6.8.5 *Let $\mathbf{g}, U, \prod_{i=1}^n X_i$, be given as in Definition 6.8.3. Then $\mathbf{g}$ is $C^1(U)$ if and only if $D_i\mathbf{g}$ exists and is continuous on U for each i. In this case, $\mathbf{g}$ is differentiable and*

$$D\mathbf{g}(\mathbf{x})(\mathbf{v}) = \sum_k D_k\mathbf{g}(\mathbf{x})\,\mathbf{v}_k \tag{6.9}$$

where $\mathbf{v} = (\mathbf{v}_1,\cdots,\mathbf{v}_n)$.

Proof: Suppose then that $D_i\mathbf{g}$ exists and is continuous for each i. Note that

$$\sum_{j=1}^{k}\theta_j\mathbf{v}_j = (\mathbf{v}_1,\cdots,\mathbf{v}_k,\mathbf{0},\cdots,\mathbf{0}).$$

Thus $\sum_{j=1}^{n}\theta_j\mathbf{v}_j = \mathbf{v}$ and define $\sum_{j=1}^{0}\theta_j\mathbf{v}_j \equiv \mathbf{0}$. Therefore,

$$\mathbf{g}(\mathbf{x}+\mathbf{v}) - \mathbf{g}(\mathbf{x}) = \sum_{k=1}^{n}\left[\mathbf{g}\left(\mathbf{x}+\sum_{j=1}^{k}\theta_j\mathbf{v}_j\right) - \mathbf{g}\left(\mathbf{x}+\sum_{j=1}^{k-1}\theta_j\mathbf{v}_j\right)\right] \tag{6.10}$$

Consider the terms in this sum.

$$\mathbf{g}\left(\mathbf{x}+\sum_{j=1}^{k}\theta_j\mathbf{v}_j\right) - \mathbf{g}\left(\mathbf{x}+\sum_{j=1}^{k-1}\theta_j\mathbf{v}_j\right) = \mathbf{g}(\mathbf{x}+\theta_k\mathbf{v}_k) - \mathbf{g}(\mathbf{x}) + \tag{6.11}$$

$$\left(\mathbf{g}\left(\mathbf{x}+\sum_{j=1}^{k}\theta_j\mathbf{v}_j\right) - \mathbf{g}(\mathbf{x}+\theta_k\mathbf{v}_k)\right) - \left(\mathbf{g}\left(\mathbf{x}+\sum_{j=1}^{k-1}\theta_j\mathbf{v}_j\right) - \mathbf{g}(\mathbf{x})\right) \tag{6.12}$$

and the expression in 6.12 is of the form $\mathbf{h}(\mathbf{v}_k) - \mathbf{h}(\mathbf{0})$ where for small $\mathbf{w} \in X_k$,

$$\mathbf{h}(\mathbf{w}) \equiv \mathbf{g}\left(\mathbf{x}+\sum_{j=1}^{k-1}\theta_j\mathbf{v}_j + \theta_k\mathbf{w}\right) - \mathbf{g}(\mathbf{x}+\theta_k\mathbf{w}).$$

Therefore,

$$D\mathbf{h}(\mathbf{w}) = D_k\mathbf{g}\left(\mathbf{x}+\sum_{j=1}^{k-1}\theta_j\mathbf{v}_j + \theta_k\mathbf{w}\right) - D_k\mathbf{g}(\mathbf{x}+\theta_k\mathbf{w})$$

and by continuity, $||D\mathbf{h}(\mathbf{w})|| < \varepsilon$ provided $||\mathbf{v}||$ is small enough. Therefore, by Theorem 6.4.2, whenever $||\mathbf{v}||$ is small enough,

$$||\mathbf{h}(\mathbf{v}_k) - \mathbf{h}(\mathbf{0})|| \le \varepsilon\,||\mathbf{v}_k|| \le \varepsilon\,||\mathbf{v}||$$

which shows that since ε is arbitrary, the expression in 6.12 is $\mathbf{o}(\mathbf{v})$. Now in 6.11

$$\mathbf{g}(\mathbf{x}+\theta_k\mathbf{v}_k) - \mathbf{g}(\mathbf{x}) = D_k\mathbf{g}(\mathbf{x})\,\mathbf{v}_k + \mathbf{o}(\mathbf{v}_k) = D_k\mathbf{g}(\mathbf{x})\,\mathbf{v}_k + \mathbf{o}(\mathbf{v}).$$

Therefore, referring to 6.10,

$$\mathbf{g}(\mathbf{x}+\mathbf{v}) - \mathbf{g}(\mathbf{x}) = \sum_{k=1}^{n} D_k\mathbf{g}(\mathbf{x})\,\mathbf{v}_k + \mathbf{o}(\mathbf{v})$$

which shows $D\mathbf{g}(\mathbf{x})$ exists and equals the formula given in 6.9.

Next suppose $\mathbf{g}$ is C^1. I need to verify that $D_k\mathbf{g}(\mathbf{x})$ exists and is continuous. Let $\mathbf{v} \in X_k$ sufficiently small. Then

$$\begin{aligned}\mathbf{g}(\mathbf{x}+\theta_k\mathbf{v}) - \mathbf{g}(\mathbf{x}) &= D\mathbf{g}(\mathbf{x})\,\theta_k\mathbf{v} + \mathbf{o}(\theta_k\mathbf{v}) \\ &= D\mathbf{g}(\mathbf{x})\,\theta_k\mathbf{v} + \mathbf{o}(\mathbf{v})\end{aligned}$$

since $||\theta_k\mathbf{v}|| = ||\mathbf{v}||$. Then $D_k\mathbf{g}(\mathbf{x})$ exists and equals

$$D\mathbf{g}(\mathbf{x}) \circ \theta_k$$

Now $\mathbf{x} \to D\mathbf{g}(\mathbf{x})$ is continuous. Since θ_k is linear, it follows from Theorem 5.8.3 that $\theta_k : X_k \to \prod_{i=1}^n X_i$ is also continuous, this proves the theorem.

The way this is usually used is in the following corollary, a case of Theorem 6.8.5 obtained by letting $X_i = \mathbb{F}$ in the above theorem.

Corollary 6.8.6 *Let U be an open subset of $\mathbb{F}^n$ and let $\mathbf{f} : U \to \mathbb{F}^m$ be C^1 in the sense that all the partial derivatives of $\mathbf{f}$ exist and are continuous. Then $\mathbf{f}$ is differentiable and*

$$\mathbf{f}(\mathbf{x}+\mathbf{v}) = \mathbf{f}(\mathbf{x}) + \sum_{k=1}^{n} \frac{\partial \mathbf{f}}{\partial x_k}(\mathbf{x})\,\mathbf{v}_k + \mathbf{o}(\mathbf{v}).$$

6.9 Mixed Partial Derivatives

Continuing with the special case where f is defined on an open set in $\mathbb{F}^n$, I will next consider an interesting result due to Euler in 1734 about mixed partial derivatives. It turns out that the mixed partial derivatives, if continuous will end up being equal. Recall the notation

$$f_x = \frac{\partial f}{\partial x} = D_{\mathbf{e}_1} f$$

and

$$f_{xy} = \frac{\partial^2 f}{\partial y \partial x} = D_{\mathbf{e}_1\mathbf{e}_2} f.$$

Theorem 6.9.1 *Suppose $f : U \subseteq \mathbb{F}^2 \to \mathbb{R}$ where U is an open set on which f_x, f_y, f_{xy} and f_{yx} exist. Then if f_{xy} and f_{yx} are continuous at the point $(x,y) \in U$, it follows*

$$f_{xy}(x,y) = f_{yx}(x,y).$$

Proof: Since U is open, there exists $r > 0$ such that $B((x,y),r) \subseteq U$. Now let $|t|, |s| < r/2, t, s$ real numbers and consider

$$\Delta(s,t) \equiv \frac{1}{st}\{\overbrace{f(x+t,y+s) - f(x+t,y)}^{h(t)} - \overbrace{(f(x,y+s) - f(x,y))}^{h(0)}\}. \tag{6.13}$$

Note that $(x+t, y+s) \in U$ because

$$\begin{aligned} |(x+t,y+s)-(x,y)| &= |(t,s)| = \left(t^2+s^2\right)^{1/2} \\ &\leq \left(\frac{r^2}{4}+\frac{r^2}{4}\right)^{1/2} = \frac{r}{\sqrt{2}} < r. \end{aligned}$$

As implied above, $h(t) \equiv f(x+t,y+s) - f(x+t,y)$. Therefore, by the mean value theorem and the (one variable) chain rule,

$$\begin{aligned} \Delta(s,t) &= \frac{1}{st}(h(t)-h(0)) = \frac{1}{st}h'(\alpha t)\,t \\ &= \frac{1}{s}(f_x(x+\alpha t, y+s) - f_x(x+\alpha t, y)) \end{aligned}$$

for some $\alpha \in (0,1)$. Applying the mean value theorem again,

$$\Delta(s,t) = f_{xy}(x+\alpha t, y+\beta s)$$

where $\alpha, \beta \in (0,1)$.

If the terms $f(x+t,y)$ and $f(x,y+s)$ are interchanged in 6.13, $\Delta(s,t)$ is unchanged and the above argument shows there exist $\gamma, \delta \in (0,1)$ such that

$$\Delta(s,t) = f_{yx}(x+\gamma t, y+\delta s).$$

Letting $(s,t) \to (0,0)$ and using the continuity of f_{xy} and f_{yx} at (x,y),

$$\lim_{(s,t)\to(0,0)} \Delta(s,t) = f_{xy}(x,y) = f_{yx}(x,y).$$

This proves the theorem.

The following is obtained from the above by simply fixing all the variables except for the two of interest.

Corollary 6.9.2 *Suppose U is an open subset of X and $f : U \to \mathbb{R}$ has the property that for two indices, k, l, f_{x_k}, $f_{x_l}, f_{x_l x_k}$, and $f_{x_k x_l}$ exist on U and $f_{x_k x_l}$ and $f_{x_l x_k}$ are both continuous at $\mathbf{x} \in U$. Then $f_{x_k x_l}(\mathbf{x}) = f_{x_l x_k}(\mathbf{x})$.*

By considering the real and imaginary parts of f in the case where f has values in $\mathbb{C}$ you obtain the following corollary.

Corollary 6.9.3 *Suppose U is an open subset of $\mathbb{F}^n$ and $f : U \to \mathbb{F}$ has the property that for two indices, k, l, f_{x_k}, $f_{x_l}, f_{x_l x_k}$, and $f_{x_k x_l}$ exist on U and $f_{x_k x_l}$ and $f_{x_l x_k}$ are both continuous at $\mathbf{x} \in U$. Then $f_{x_k x_l}(\mathbf{x}) = f_{x_l x_k}(\mathbf{x})$.*

Finally, by considering the components of $\mathbf{f}$ you get the following generalization.

Corollary 6.9.4 *Suppose U is an open subset of $\mathbb{F}^n$ and $\mathbf{f} : U \to \mathbb{F}^m$ has the property that for two indices, k, l, $\mathbf{f}_{x_k}$, $\mathbf{f}_{x_l}, \mathbf{f}_{x_l x_k}$, and $\mathbf{f}_{x_k x_l}$ exist on U and $\mathbf{f}_{x_k x_l}$ and $\mathbf{f}_{x_l x_k}$ are both continuous at $\mathbf{x} \in U$. Then $\mathbf{f}_{x_k x_l}(\mathbf{x}) = \mathbf{f}_{x_l x_k}(\mathbf{x})$.*

It is necessary to assume the mixed partial derivatives are continuous in order to assert they are equal. The following is a well known example [3].

Example 6.9.5 *Let*

$$f(x,y) = \begin{cases} \frac{xy(x^2-y^2)}{x^2+y^2} \text{ if } (x,y) \neq (0,0) \\ 0 \text{ if } (x,y) = (0,0) \end{cases}$$

From the definition of partial derivatives it follows immediately that $f_x(0,0) = f_y(0,0) = 0$. Using the standard rules of differentiation, for $(x,y) \neq (0,0)$,

$$f_x = y\frac{x^4 - y^4 + 4x^2y^2}{(x^2+y^2)^2}, \; f_y = x\frac{x^4 - y^4 - 4x^2y^2}{(x^2+y^2)^2}$$

Now

$$\begin{aligned} f_{xy}(0,0) &\equiv \lim_{y\to 0} \frac{f_x(0,y) - f_x(0,0)}{y} \\ &= \lim_{y\to 0} \frac{-y^4}{(y^2)^2} = -1 \end{aligned}$$

while

$$\begin{aligned} f_{yx}(0,0) &\equiv \lim_{x\to 0} \frac{f_y(x,0) - f_y(0,0)}{x} \\ &= \lim_{x\to 0} \frac{x^4}{(x^2)^2} = 1 \end{aligned}$$

showing that although the mixed partial derivatives do exist at $(0,0)$, they are not equal there.

6.10 Implicit Function Theorem

The following lemma is very useful.

Lemma 6.10.1 *Let $A \in \mathcal{L}(X,X)$ where X is a finite dimensional normed vector space and suppose $||A|| \leq r < 1$. Then*

$$(I-A)^{-1} \text{ exists} \tag{6.14}$$

and

$$\left|\left|(I-A)^{-1}\right|\right| \leq (1-r)^{-1}. \tag{6.15}$$

Furthermore, if

$$\mathcal{I} \equiv \left\{A \in \mathcal{L}(X,X) : A^{-1} \text{ exists}\right\}$$

the map $A \to A^{-1}$ is continuous on $\mathcal{I}$ and $\mathcal{I}$ is an open subset of $\mathcal{L}(X,X)$.

Proof: Let $||A|| \leq r < 1$. If $(I-A)\mathbf{x} = \mathbf{0}$, then $\mathbf{x} = A\mathbf{x}$ and so if $\mathbf{x} \neq \mathbf{0}$,

$$||\mathbf{x}|| = ||A\mathbf{x}|| \leq ||A||\,||\mathbf{x}|| < r\,||\mathbf{x}||$$

which is a contradiction. Therefore, $(I-A)$ is one to one. Hence it maps a basis of X to a basis of X and is therefore, onto. Here is why. Let $\{\mathbf{v}_1, \cdots, \mathbf{v}_n\}$ be a basis for X and suppose

$$\sum_{k=1}^{n} c_k (I-A)\mathbf{v}_k = \mathbf{0}.$$

Then

$$(I-A)\left(\sum_{k=1}^{n} c_k \mathbf{v}_k\right) = \mathbf{0}$$

and since $(I-A)$ is one to one, it follows

$$\sum_{k=1}^{n} c_k \mathbf{v}_k = \mathbf{0}$$

which requires each $c_k = 0$ because the $\{\mathbf{v}_k\}$ are independent. Hence $\{(I-A)\mathbf{v}_k\}_{k=1}^{n}$ is a basis for X because there are n of these vectors and every basis has the same size. Therefore, if $\mathbf{y} \in X$, there exist scalars, c_k such that

$$\mathbf{y} = \sum_{k=1}^{n} c_k (I-A)\mathbf{v}_k = (I-A)\left(\sum_{k=1}^{n} c_k \mathbf{v}_k\right)$$

so $(I-A)$ is onto as claimed. Thus $(I-A)^{-1} \in \mathcal{L}(X,X)$ and it remains to estimate its norm.

$$||\mathbf{x} - A\mathbf{x}|| \geq ||\mathbf{x}|| - ||A\mathbf{x}|| \geq ||\mathbf{x}|| - ||A||\,||\mathbf{x}|| \geq ||\mathbf{x}||\,(1-r)$$

Letting $\mathbf{y} = \mathbf{x} - A\mathbf{x}$ so $\mathbf{x} = (I-A)^{-1}\mathbf{y}$, this shows, since $(I-A)$ is onto that for all $\mathbf{y} \in X$,

$$||\mathbf{y}|| \geq \left|\left|(I-A)^{-1}\mathbf{y}\right|\right|(1-r)$$

and so $\left|\left|(I-A)^{-1}\right|\right| \leq (1-r)^{-1}$. This proves the first part.

To verify the continuity of the inverse map, let $A \in \mathcal{I}$. Then

$$B = A\left(I - A^{-1}(A - B)\right)$$

and so if $\left|\left|A^{-1}(A - B)\right|\right| < 1$ which, by Theorem 5.8.3, happens if

$$||A - B|| < 1/\left|\left|A^{-1}\right|\right|,$$

it follows from the first part of this proof that $\left(I - A^{-1}(A - B)\right)^{-1}$ exists and so

$$B^{-1} = \left(I - A^{-1}(A - B)\right)^{-1} A^{-1}$$

which shows $\mathcal{I}$ is open. Also, if

$$\left|\left|A^{-1}(A - B)\right|\right| \leq r < 1, \tag{6.16}$$

$$\left|\left|B^{-1}\right|\right| \leq \left|\left|A^{-1}\right|\right|(1 - r)^{-1}$$

Now for such B this close to A such that 6.16 holds,

$$\begin{aligned}\left|\left|B^{-1} - A^{-1}\right|\right| &= \left|\left|B^{-1}(A - B)A^{-1}\right|\right| \leq ||A - B||\left|\left|B^{-1}\right|\right|\left|\left|A^{-1}\right|\right| \\ &\leq ||A - B||\left|\left|A^{-1}\right|\right|^2(1 - r)^{-1}\end{aligned}$$

which shows the map which takes a linear transformation in $\mathcal{I}$ to its inverse is continuous. This proves the lemma.

The next theorem is a very useful result in many areas. It will be used in this section to give a short proof of the implicit function theorem but it is also useful in studying differential equations and integral equations. It is sometimes called the uniform contraction principle.

Theorem 6.10.2 *Let X, Y be finite dimensional normed vector spaces. Also let E be a closed subset of X and F a closed subset of Y. Suppose for each $(\mathbf{x}, \mathbf{y}) \in E \times F$, $\mathbf{T}(\mathbf{x}, \mathbf{y}) \in E$ and satisfies*

$$||\mathbf{T}(\mathbf{x}, \mathbf{y}) - \mathbf{T}(\mathbf{x}', \mathbf{y})|| \leq r\,||\mathbf{x} - \mathbf{x}'|| \tag{6.17}$$

where $0 < r < 1$ and also

$$||\mathbf{T}(\mathbf{x}, \mathbf{y}) - \mathbf{T}(\mathbf{x}, \mathbf{y}')|| \leq M\,||\mathbf{y} - \mathbf{y}'||. \tag{6.18}$$

Then for each $\mathbf{y} \in F$ there exists a unique "fixed point" for $\mathbf{T}(\cdot, \mathbf{y}), \mathbf{x} \in E$, satisfying

$$\mathbf{T}(\mathbf{x}, \mathbf{y}) = \mathbf{x} \tag{6.19}$$

and also if $\mathbf{x}(\mathbf{y})$ is this fixed point,

$$||\mathbf{x}(\mathbf{y}) - \mathbf{x}(\mathbf{y}')|| \leq \frac{M}{1 - r}\,||\mathbf{y} - \mathbf{y}'||. \tag{6.20}$$

Proof: First consider the claim there exists a fixed point for the mapping, $\mathbf{T}(\cdot, y)$. For a fixed $\mathbf{y}$, let $\mathbf{g}(\mathbf{x}) \equiv \mathbf{T}(\mathbf{x}, \mathbf{y})$. Now pick any $\mathbf{x}_0 \in E$ and consider the sequence,

$$\mathbf{x}_1 = \mathbf{g}(\mathbf{x}_0),\ \mathbf{x}_{k+1} = \mathbf{g}(\mathbf{x}_k).$$

Then by 6.17,

$$||\mathbf{x}_{k+1} - \mathbf{x}_k|| = ||\mathbf{g}(\mathbf{x}_k) - \mathbf{g}(\mathbf{x}_{k-1})|| \leq r\,||\mathbf{x}_k - \mathbf{x}_{k-1}|| \leq$$

$$r^2 ||\mathbf{x}_{k-1} - \mathbf{x}_{k-2}|| \leq \cdots \leq r^k ||\mathbf{g}(\mathbf{x}_0) - \mathbf{x}_0||.$$

Now by the triangle inequality,

$$||\mathbf{x}_{k+p} - \mathbf{x}_k|| \leq \sum_{i=1}^{p} ||\mathbf{x}_{k+i} - \mathbf{x}_{k+i-1}||$$

$$\leq \sum_{i=1}^{p} r^{k+i-1} ||\mathbf{g}(\mathbf{x}_0) - \mathbf{x}_0|| \leq \frac{r^k ||\mathbf{g}(\mathbf{x}_0) - \mathbf{x}_0||}{1-r}.$$

Since $0 < r < 1$, this shows that $\{\mathbf{x}_k\}_{k=1}^{\infty}$ is a Cauchy sequence. Therefore, by completeness of E it converges to a point $\mathbf{x} \in E$. To see $\mathbf{x}$ is a fixed point, use the continuify of $\mathbf{g}$ to obtain

$$\mathbf{x} \equiv \lim_{k\to\infty} \mathbf{x}_k = \lim_{k\to\infty} \mathbf{x}_{k+1} = \lim_{k\to\infty} \mathbf{g}(\mathbf{x}_k) = \mathbf{g}(\mathbf{x}).$$

This proves 6.19. To verify 6.20,

$$||\mathbf{x}(\mathbf{y}) - \mathbf{x}(\mathbf{y}')|| = ||\mathbf{T}(\mathbf{x}(\mathbf{y}), \mathbf{y}) - \mathbf{T}(\mathbf{x}(\mathbf{y}'), \mathbf{y}')|| \leq$$

$$||\mathbf{T}(\mathbf{x}(\mathbf{y}), \mathbf{y}) - \mathbf{T}(\mathbf{x}(\mathbf{y}), \mathbf{y}')|| + ||\mathbf{T}(\mathbf{x}(\mathbf{y}), \mathbf{y}') - \mathbf{T}(\mathbf{x}(\mathbf{y}'), \mathbf{y}')||$$

$$\leq M ||\mathbf{y} - \mathbf{y}'|| + r ||\mathbf{x}(\mathbf{y}) - \mathbf{x}(\mathbf{y}')||.$$

Thus

$$(1-r) ||\mathbf{x}(\mathbf{y}) - \mathbf{x}(\mathbf{y}')|| \leq M ||\mathbf{y} - \mathbf{y}'||.$$

This also shows the fixed point for a given y is unique. This proves the theorem.

The implicit function theorem deals with the question of solving, $\mathbf{f}(\mathbf{x}, \mathbf{y}) = \mathbf{0}$ for $\mathbf{x}$ in terms of $\mathbf{y}$ and how smooth the solution is. It is one of the most important theorems in mathematics. The proof I will give holds with no change in the context of infinite dimensional complete normed vector spaces when suitable modifications are made on what is meant by $\mathcal{L}(X, Y)$. There are also even more general versions of this theorem than to normed vector spaces.

Recall that for X, Y normed vector spaces, the norm on $X \times Y$ is of the form

$$||(\mathbf{x}, \mathbf{y})|| = \max(||\mathbf{x}||, ||\mathbf{y}||).$$

Theorem 6.10.3 *(implicit function theorem) Let X, Y, Z be finite dimensional normed vector spaces and suppose U is an open set in $X \times Y$. Let $\mathbf{f} : U \to Z$ be in $C^1(U)$ and suppose*

$$\mathbf{f}(\mathbf{x}_0, \mathbf{y}_0) = \mathbf{0},\ D_1\mathbf{f}(\mathbf{x}_0, \mathbf{y}_0)^{-1} \in \mathcal{L}(Z, X). \tag{6.21}$$

Then there exist positive constants, δ, η, such that for every $\mathbf{y} \in B(\mathbf{y}_0, \eta)$ there exists a unique $\mathbf{x}(\mathbf{y}) \in B(\mathbf{x}_0, \delta)$ such that

$$\mathbf{f}(\mathbf{x}(\mathbf{y}), \mathbf{y}) = \mathbf{0}. \tag{6.22}$$

Furthermore, the mapping, $\mathbf{y} \to \mathbf{x}(\mathbf{y})$ is in $C^1(B(\mathbf{y}_0, \eta))$.

Proof: Let $\mathbf{T}(\mathbf{x}, \mathbf{y}) \equiv \mathbf{x} - D_1\mathbf{f}(\mathbf{x}_0, \mathbf{y}_0)^{-1} \mathbf{f}(\mathbf{x}, \mathbf{y})$. Therefore,

$$D_1\mathbf{T}(\mathbf{x}, \mathbf{y}) = I - D_1\mathbf{f}(\mathbf{x}_0, \mathbf{y}_0)^{-1} D_1\mathbf{f}(\mathbf{x}, \mathbf{y}). \tag{6.23}$$

by continuity of the derivative which implies continuity of $D_1\mathbf{T}$, it follows there exists $\delta > 0$ such that if $||(\mathbf{x} - \mathbf{x}_0, \mathbf{y} - \mathbf{y}_0)|| < \delta$, then

$$||D_1\mathbf{T}(\mathbf{x}, \mathbf{y})|| < \frac{1}{2}. \tag{6.24}$$

Also, it can be assumed δ is small enough that

$$\left|\left|D_1\mathbf{f}\left(\mathbf{x}_0,\mathbf{y}_0\right)^{-1}\right|\right| ||D_2\mathbf{f}\left(\mathbf{x},\mathbf{y}\right)|| < M \tag{6.25}$$

where $M > \left|\left|D_1\mathbf{f}\left(\mathbf{x}_0,\mathbf{y}_0\right)^{-1}\right|\right| ||D_2\mathbf{f}\left(\mathbf{x}_0,\mathbf{y}_0\right)||$. By Theorem 6.4.2, whenever $\mathbf{x},\mathbf{x}' \in B\left(\mathbf{x}_0,\delta\right)$ and $\mathbf{y} \in B\left(\mathbf{y}_0,\delta\right)$,

$$||\mathbf{T}\left(\mathbf{x},\mathbf{y}\right) - \mathbf{T}\left(\mathbf{x}',\mathbf{y}\right)|| \leq \frac{1}{2}||\mathbf{x}-\mathbf{x}'||. \tag{6.26}$$

Solving 6.23 for $D_1\mathbf{f}\left(\mathbf{x},\mathbf{y}\right)$,

$$D_1\mathbf{f}\left(\mathbf{x},\mathbf{y}\right) = D_1\mathbf{f}\left(\mathbf{x}_0,\mathbf{y}_0\right)\left(I - D_1\mathbf{T}\left(\mathbf{x},\mathbf{y}\right)\right).$$

By Lemma 6.10.1 and the assumption that $D_1\mathbf{f}\left(\mathbf{x}_0,\mathbf{y}_0\right)^{-1}$ exists, it follows, $D_1\mathbf{f}\left(\mathbf{x},\mathbf{y}\right)^{-1}$ exists and equals

$$\left(I - D_1\mathbf{T}\left(\mathbf{x},\mathbf{y}\right)\right)^{-1} D_1\mathbf{f}\left(\mathbf{x}_0,\mathbf{y}_0\right)^{-1}$$

By the estimate of Lemma 6.10.1 and 6.24,

$$\left|\left|D_1\mathbf{f}\left(\mathbf{x},\mathbf{y}\right)^{-1}\right|\right| \leq 2\left|\left|D_1\mathbf{f}\left(\mathbf{x}_0,\mathbf{y}_0\right)^{-1}\right|\right|. \tag{6.27}$$

Next more restrictions are placed on $\mathbf{y}$ to make it even closer to $\mathbf{y}_0$. Let

$$0 < \eta < \min\left(\delta, \frac{\delta}{3M}\right).$$

Then suppose $\mathbf{x} \in \overline{B\left(\mathbf{x}_0,\delta\right)}$ and $\mathbf{y} \in \overline{B\left(\mathbf{y}_0,\eta\right)}$. Consider

$$\mathbf{x} - D_1\mathbf{f}\left(\mathbf{x}_0,\mathbf{y}_0\right)^{-1}\mathbf{f}\left(\mathbf{x},\mathbf{y}\right) - \mathbf{x}_0 = \mathbf{T}\left(\mathbf{x},\mathbf{y}\right) - \mathbf{x}_0 \equiv \mathbf{g}\left(\mathbf{x},\mathbf{y}\right).$$

$$D_1\mathbf{g}\left(\mathbf{x},\mathbf{y}\right) = I - D_1\mathbf{f}\left(\mathbf{x}_0,\mathbf{y}_0\right)^{-1} D_1\mathbf{f}\left(\mathbf{x},\mathbf{y}\right) = D_1\mathbf{T}\left(\mathbf{x},\mathbf{y}\right),$$

and

$$D_2\mathbf{g}\left(\mathbf{x},\mathbf{y}\right) = -D_1\mathbf{f}\left(\mathbf{x}_0,\mathbf{y}_0\right)^{-1} D_2\mathbf{f}\left(\mathbf{x},\mathbf{y}\right).$$

Also note that $\mathbf{T}\left(\mathbf{x},\mathbf{y}\right) = \mathbf{x}$ is the same as saying $\mathbf{f}\left(\mathbf{x}_0,\mathbf{y}_0\right) = \mathbf{0}$ and also $\mathbf{g}\left(\mathbf{x}_0,\mathbf{y}_0\right) = \mathbf{0}$. Thus by 6.25 and Theorem 6.4.2, it follows that for such $\left(\mathbf{x},\mathbf{y}\right) \in \overline{B\left(\mathbf{x}_0,\delta\right)} \times \overline{B\left(\mathbf{y}_0,\eta\right)}$,

$$\begin{aligned} ||\mathbf{T}\left(\mathbf{x},\mathbf{y}\right) - \mathbf{x}_0|| &= ||\mathbf{g}\left(\mathbf{x},\mathbf{y}\right)|| = ||\mathbf{g}\left(\mathbf{x},\mathbf{y}\right) - \mathbf{g}\left(\mathbf{x}_0,\mathbf{y}_0\right)|| \\ &\leq ||\mathbf{g}\left(\mathbf{x},\mathbf{y}\right) - \mathbf{g}\left(\mathbf{x},\mathbf{y}_0\right)|| + ||\mathbf{g}\left(\mathbf{x},\mathbf{y}_0\right) - \mathbf{g}\left(\mathbf{x}_0,\mathbf{y}_0\right)|| \\ &\leq M||\mathbf{y}-\mathbf{y}_0|| + \frac{1}{2}||\mathbf{x}-\mathbf{x}_0|| < \frac{\delta}{2} + \frac{\delta}{3} = \frac{5\delta}{6} < \delta. \end{aligned} \tag{6.28}$$

Also for such $\left(\mathbf{x},\mathbf{y}_i\right), i = 1,2$, Theorem 6.4.2 and 6.25 implies

$$\begin{aligned} ||\mathbf{T}\left(\mathbf{x},\mathbf{y}_1\right) - \mathbf{T}\left(\mathbf{x},\mathbf{y}_2\right)|| &= \left|\left|D_1\mathbf{f}\left(\mathbf{x}_0,\mathbf{y}_0\right)^{-1}\left(\mathbf{f}\left(\mathbf{x},\mathbf{y}_2\right) - \mathbf{f}\left(\mathbf{x},\mathbf{y}_1\right)\right)\right|\right| \\ &\leq M||\mathbf{y}_2 - \mathbf{y}_1||. \end{aligned} \tag{6.29}$$

From now on assume $||\mathbf{x}-\mathbf{x}_0|| < \delta$ and $||\mathbf{y}-\mathbf{y}_0|| < \eta$ so that 6.29, 6.27, 6.28, 6.26, and 6.25 all hold. By 6.29, 6.26, 6.28, and the uniform contraction principle, Theorem 6.10.2 applied to $E \equiv \overline{B\left(\mathbf{x}_0,\frac{5\delta}{6}\right)}$ and $F \equiv \overline{B\left(\mathbf{y}_0,\eta\right)}$ implies that for each $\mathbf{y} \in B\left(\mathbf{y}_0,\eta\right)$, there exists

a unique $\mathbf{x}(\mathbf{y}) \in B(\mathbf{x}_0, \delta)$ (actually in $\overline{B\left(\mathbf{x}_0, \frac{5\delta}{6}\right)}$) such that $\mathbf{T}(\mathbf{x}(\mathbf{y}), \mathbf{y}) = \mathbf{x}(\mathbf{y})$ which is equivalent to

$$\mathbf{f}(\mathbf{x}(\mathbf{y}), \mathbf{y}) = \mathbf{0}.$$

Furthermore,

$$||\mathbf{x}(\mathbf{y}) - \mathbf{x}(\mathbf{y}')|| \leq 2M ||\mathbf{y} - \mathbf{y}'||. \tag{6.30}$$

This proves the implicit function theorem except for the verification that $\mathbf{y} \to \mathbf{x}(\mathbf{y})$ is C^1. This is shown next. Letting $\mathbf{v}$ be sufficiently small, Theorem 6.8.5 and Theorem 6.4.2 imply

$$\mathbf{0} = \mathbf{f}(\mathbf{x}(\mathbf{y}+\mathbf{v}), \mathbf{y}+\mathbf{v}) - \mathbf{f}(\mathbf{x}(\mathbf{y}), \mathbf{y}) =$$

$$D_1\mathbf{f}(\mathbf{x}(\mathbf{y}), \mathbf{y})(\mathbf{x}(\mathbf{y}+\mathbf{v}) - \mathbf{x}(\mathbf{y})) +$$

$$+D_2\mathbf{f}(\mathbf{x}(\mathbf{y}), \mathbf{y})\mathbf{v} + o((\mathbf{x}(\mathbf{y}+\mathbf{v}) - \mathbf{x}(\mathbf{y}), \mathbf{v})).$$

The last term in the above is $o(\mathbf{v})$ because of 6.30. Therefore, using 6.27, solve the above equation for $\mathbf{x}(\mathbf{y}+\mathbf{v}) - \mathbf{x}(\mathbf{y})$ and obtain

$$\mathbf{x}(\mathbf{y}+\mathbf{v}) - \mathbf{x}(\mathbf{y}) = -D_1(\mathbf{x}(\mathbf{y}), \mathbf{y})^{-1} D_2\mathbf{f}(\mathbf{x}(\mathbf{y}), \mathbf{y})\mathbf{v} + o(\mathbf{v})$$

Which shows that $\mathbf{y} \to \mathbf{x}(\mathbf{y})$ is differentiable on $B(\mathbf{y}_0, \eta)$ and

$$D\mathbf{x}(\mathbf{y}) = -D_1\mathbf{f}(\mathbf{x}(\mathbf{y}), \mathbf{y})^{-1} D_2\mathbf{f}(\mathbf{x}(\mathbf{y}), \mathbf{y}). \tag{6.31}$$

Now it follows from the continuity of $D_2\mathbf{f}$, $D_1\mathbf{f}$, the inverse map, 6.30, and this formula for $D\mathbf{x}(\mathbf{y})$that $\mathbf{x}(\cdot)$ is $C^1(B(\mathbf{y}_0, \eta))$. This proves the theorem.

The next theorem is a very important special case of the implicit function theorem known as the inverse function theorem. Actually one can also obtain the implicit function theorem from the inverse function theorem. It is done this way in [27] and in [2].

Theorem 6.10.4 *(inverse function theorem) Let $\mathbf{x}_0 \in U$, an open set in X , and let $\mathbf{f} : U \to Y$ where X, Y are finite dimensional normed vector spaces. Suppose*

$$\mathbf{f} \text{ is } C^1(U) \ , \quad \text{and} \quad D\mathbf{f}(\mathbf{x}_0)^{-1} \in \mathcal{L}(Y, X). \tag{6.32}$$

Then there exist open sets, W, and V such that

$$\mathbf{x}_0 \in W \subseteq U, \tag{6.33}$$

$$\mathbf{f} : W \to V \text{ is one to one and onto}, \tag{6.34}$$

$$\mathbf{f}^{-1} \text{ is } C^1, \tag{6.35}$$

Proof: Apply the implicit function theorem to the function

$$\mathbf{F}(\mathbf{x}, \mathbf{y}) \equiv \mathbf{f}(\mathbf{x}) - \mathbf{y}$$

where $\mathbf{y}_0 \equiv \mathbf{f}(\mathbf{x}_0)$. Thus the function $\mathbf{y} \to \mathbf{x}(\mathbf{y})$ defined in that theorem is $\mathbf{f}^{-1}$. Now let

$$W \equiv B(\mathbf{x}_0, \delta) \cap \mathbf{f}^{-1}(B(\mathbf{y}_0, \eta))$$

and

$$V \equiv B(\mathbf{y}_0, \eta).$$

This proves the theorem.

6.10.1 More Derivatives

In the implicit function theorem, suppose $\mathbf{f}$ is C^k. Will the implicitly defined function also be C^k? It was shown above that this is the case if $k = 1$. In fact it holds for any positive integer k.

First of all, consider $D_2\mathbf{f}(\mathbf{x}(\mathbf{y}),\mathbf{y}) \in \mathcal{L}(Y,Z)$. Let $\{\mathbf{w}_1,\cdots,\mathbf{w}_m\}$ be a basis for Y and let $\{\mathbf{z}_1,\cdots,\mathbf{z}_n\}$ be a basis for Z. Then $D_2\mathbf{f}(\mathbf{x}(\mathbf{y}),\mathbf{y})$ has a matrix with respect to these bases. Thus conserving on notation, denote this matrix by $\left(D_2\mathbf{f}(\mathbf{x}(\mathbf{y}),\mathbf{y})_{ij}\right)$. Thus

$$D_2\mathbf{f}(\mathbf{x}(\mathbf{y}),\mathbf{y}) = \sum_{ij} D_2\mathbf{f}(\mathbf{x}(\mathbf{y}),\mathbf{y})_{ij}\,\mathbf{z}_i\mathbf{w}_j$$

The scalar valued entries of the matrix of $D_2\mathbf{f}(\mathbf{x}(\mathbf{y}),\mathbf{y})$ have the same differentiability as the function $\mathbf{y} \to D_2\mathbf{f}(\mathbf{x}(\mathbf{y}),\mathbf{y})$. This is because the linear projection map, π_{ij} mapping $\mathcal{L}(Y,Z)$ to $\mathbb{F}$ given by $\pi_{ij}L \equiv L_{ij}$, the ij^{th} entry of the matrix of L with respect to the given bases is continuous thanks to Theorem 5.8.3. Similar considerations apply to $D_1\mathbf{f}(\mathbf{x}(\mathbf{y}),\mathbf{y})$ and the entries of its matrix, $D_1\mathbf{f}(\mathbf{x}(\mathbf{y}),\mathbf{y})_{ij}$ taken with respect to suitable bases. From the formula for the inverse of a matrix, Theorem 3.5.14, the ij^{th} entries of the matrix of $D_1\mathbf{f}(\mathbf{x}(\mathbf{y}),\mathbf{y})^{-1}$, $D_1\mathbf{f}(\mathbf{x}(\mathbf{y}),\mathbf{y})^{-1}_{ij}$ also have the same differentiability as $\mathbf{y} \to D_1\mathbf{f}(\mathbf{x}(\mathbf{y}),\mathbf{y})$.

Now consider the formula for the derivative of the implicitly defined function in 6.31,

$$D\mathbf{x}(\mathbf{y}) = -D_1\mathbf{f}(\mathbf{x}(\mathbf{y}),\mathbf{y})^{-1} D_2\mathbf{f}(\mathbf{x}(\mathbf{y}),\mathbf{y}). \tag{6.36}$$

The above derivative is in $\mathcal{L}(Y,X)$. Let $\{\mathbf{w}_1,\cdots,\mathbf{w}_m\}$ be a basis for Y and let $\{\mathbf{v}_1,\cdots,\mathbf{v}_n\}$ be a basis for X. Letting x_i be the i^{th} component of $\mathbf{x}$ with respect to the basis for X, it follows from Theorem 6.7.1, $\mathbf{y} \to \mathbf{x}(\mathbf{y})$ will be C^k if all such Gateaux derivatives, $D_{\mathbf{w}_{j_1}\mathbf{w}_{j_2}\cdots\mathbf{w}_{j_r}}x_i(\mathbf{y})$ exist and are continuous for $r \le k$ and for any i. Consider what is required for this to happen. By 6.36,

$$\begin{aligned} D_{\mathbf{w}_j}x_i(\mathbf{y}) &= \sum_k \left(-D_1\mathbf{f}(\mathbf{x}(\mathbf{y}),\mathbf{y})^{-1}\right)_{ik} (D_2\mathbf{f}(\mathbf{x}(\mathbf{y}),\mathbf{y}))_{kj} \\ &\equiv G_1(\mathbf{x}(\mathbf{y}),\mathbf{y}) \end{aligned} \tag{6.37}$$

where $(\mathbf{x},\mathbf{y}) \to G_1(\mathbf{x},\mathbf{y})$ is C^{k-1} because it is assumed $\mathbf{f}$ is C^k and one derivative has been taken to write the above. If $k \ge 2$, then another Gateaux derivative can be taken.

$$\begin{aligned} D_{\mathbf{w}_j\mathbf{w}_k}x_i(\mathbf{y}) &\equiv \lim_{t\to 0} \frac{G_1(\mathbf{x}(\mathbf{y}+t\mathbf{w}_k),\mathbf{y}+t\mathbf{w}_k) - G_1(\mathbf{x}(\mathbf{y}),\mathbf{y})}{t} \\ &= D_1G_1(\mathbf{x}(\mathbf{y}),\mathbf{y})\,D\mathbf{x}(\mathbf{y})\,\mathbf{w}_k + D_2G_1(\mathbf{x}(\mathbf{y}),\mathbf{y}) \\ &\equiv G_2(\mathbf{x}(\mathbf{y}),\mathbf{y},D\mathbf{x}(\mathbf{y})) \end{aligned}$$

Since a similar result holds for all i and any choice of $\mathbf{w}_j,\mathbf{w}_k$, this shows $\mathbf{x}$ is at least C^2. If $k \ge 3$, then another Gateaux derivative can be taken because then $(\mathbf{x},\mathbf{y},\mathbf{z}) \to G_2(\mathbf{x},\mathbf{y},\mathbf{z})$ is C^1 and it has been established $D\mathbf{x}$ is C^1. Continuing this way shows $D_{\mathbf{w}_{j_1}\mathbf{w}_{j_2}\cdots\mathbf{w}_{j_r}}x_i(\mathbf{y})$ exists and is continuous for $r \le k$. This proves the following corollary to the implicit and inverse function theorems.

Corollary 6.10.5 *In the implicit and inverse function theorems, you can replace C^1 with C^k in the statements of the theorems for any $k \in \mathbb{N}$.*

6.10.2 The Case Of $\mathbb{R}^n$

In many applications of the implicit function theorem,

$$\mathbf{f} : U \subseteq \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$$

and $\mathbf{f}(\mathbf{x}_0, \mathbf{y}_0) = \mathbf{0}$ while $\mathbf{f}$ is C^1. How can you recognize the condition of the implicit function theorem which says $D_1\mathbf{f}(\mathbf{x}_0, \mathbf{y}_0)^{-1}$ exists? This is really not hard. You recall the matrix of the transformation $D_1\mathbf{f}(\mathbf{x}_0, \mathbf{y}_0)$ with respect to the usual basis vectors is

$$\begin{pmatrix} f_{1,x_1}(\mathbf{x}_0, \mathbf{y}_0) & \cdots & f_{1,x_n}(\mathbf{x}_0, \mathbf{y}_0) \\ \vdots & & \vdots \\ f_{n,x_1}(\mathbf{x}_0, \mathbf{y}_0) & \cdots & f_{n,x_n}(\mathbf{x}_0, \mathbf{y}_0) \end{pmatrix}$$

and so $D_1\mathbf{f}(\mathbf{x}_0, \mathbf{y}_0)^{-1}$ exists exactly when the determinant of the above matrix is nonzero. This is the condition to check. In the general case, you just need to verify $D_1\mathbf{f}(\mathbf{x}_0, \mathbf{y}_0)$ is one to one and this can also be accomplished by looking at the matrix of the transformation with respect to some bases on X and Z.

6.11 Taylor's Formula

First recall the Taylor formula with the Lagrange form of the remainder. It will only be needed on $[0,1]$ so that is what I will show.

Theorem 6.11.1 *Let $h : [0,1] \to \mathbb{R}$ have $m+1$ derivatives. Then there exists $t \in (0,1)$ such that*

$$h(1) = h(0) + \sum_{k=1}^{m} \frac{h^{(k)}(0)}{k!} + \frac{h^{(m+1)}(t)}{(m+1)!}.$$

Proof: Let K be a number chosen such that

$$h(1) - \left(h(0) + \sum_{k=1}^{m} \frac{h^{(k)}(0)}{k!} + K \right) = 0$$

Now the idea is to find K. To do this, let

$$F(t) = h(1) - \left(h(t) + \sum_{k=1}^{m} \frac{h^{(k)}(t)}{k!} (1-t)^k + K(1-t)^{m+1} \right)$$

Then $F(1) = F(0) = 0$. Therefore, by Rolle's theorem there exists t between 0 and 1 such that $F'(t) = 0$. Thus,

$$\begin{aligned} 0 &= -F'(t) = h'(t) + \sum_{k=1}^{m} \frac{h^{(k+1)}(t)}{k!} (1-t)^k \\ &\quad - \sum_{k=1}^{m} \frac{h^{(k)}(t)}{k!} k(1-t)^{k-1} - K(m+1)(1-t)^m \end{aligned}$$

And so

$$\begin{aligned} &= h'(t) + \sum_{k=1}^{m} \frac{h^{(k+1)}(t)}{k!} (1-t)^k - \sum_{k=0}^{m-1} \frac{h^{(k+1)}(t)}{k!} (1-t)^k \\ &\quad -K(m+1)(1-t)^m \end{aligned}$$

$$= h'(t) + \frac{h^{(m+1)}(t)}{m!}(1-t)^m - h'(t) - K(m+1)(1-t)^m$$

and so

$$K = \frac{h^{(m+1)}(t)}{(m+1)!}.$$

This proves the theorem.

Now let $f : U \to \mathbb{R}$ where $U \subseteq X$ a normed vector space and suppose $f \in C^m(U)$. Let $\mathbf{x} \in U$ and let $r > 0$ be such that

$$B(\mathbf{x},r) \subseteq U.$$

Then for $||\mathbf{v}|| < r$ consider

$$f(\mathbf{x}+t\mathbf{v}) - f(\mathbf{x}) \equiv h(t)$$

for $t \in [0,1]$. Then by the chain rule,

$$h'(t) = Df(\mathbf{x}+t\mathbf{v})(\mathbf{v}),\ h''(t) = D^2 f(\mathbf{x}+t\mathbf{v})(\mathbf{v})(\mathbf{v})$$

and continuing in this way,

$$h^{(k)}(t) = D^{(k)} f(\mathbf{x}+t\mathbf{v})(\mathbf{v})(\mathbf{v})\cdots(\mathbf{v}) \equiv D^{(k)} f(\mathbf{x}+t\mathbf{v})\,\mathbf{v}^k.$$

It follows from Taylor's formula for a function of one variable given above that

$$f(\mathbf{x}+\mathbf{v}) = f(\mathbf{x}) + \sum_{k=1}^{m} \frac{D^{(k)} f(\mathbf{x})\,\mathbf{v}^k}{k!} + \frac{D^{(m+1)} f(\mathbf{x}+t\mathbf{v})\,\mathbf{v}^{m+1}}{(m+1)!}. \tag{6.38}$$

This proves the following theorem.

Theorem 6.11.2 *Let $f : U \to \mathbb{R}$ and let $f \in C^{m+1}(U)$. Then if*

$$B(\mathbf{x},r) \subseteq U,$$

and $||\mathbf{v}|| < r$, there exists $t \in (0,1)$ such that 6.38 holds.

6.11.1 Second Derivative Test

Now consider the case where $U \subseteq \mathbb{R}^n$ and $f : U \to \mathbb{R}$ is $C^2(U)$. Then from Taylor's theorem, if $\mathbf{v}$ is small enough, there exists $t \in (0,1)$ such that

$$f(\mathbf{x}+\mathbf{v}) = f(\mathbf{x}) + Df(\mathbf{x})\,\mathbf{v} + \frac{D^2 f(\mathbf{x}+t\mathbf{v})\,\mathbf{v}^2}{2}. \tag{6.39}$$

Consider

$$\begin{aligned} D^2 f(\mathbf{x}+t\mathbf{v})(\mathbf{e}_i)(\mathbf{e}_j) &\equiv D(D(f(\mathbf{x}+t\mathbf{v}))\,\mathbf{e}_i)\,\mathbf{e}_j \\ &= D\left(\frac{\partial f(\mathbf{x}+t\mathbf{v})}{\partial x_i}\right)\mathbf{e}_j \\ &= \frac{\partial^2 f(\mathbf{x}+t\mathbf{v})}{\partial x_j \partial x_i} \end{aligned}$$

where $\mathbf{e}_i$ are the usual basis vectors. Letting

$$\mathbf{v} = \sum_{i=1}^{n} v_i \mathbf{e}_i,$$

the second derivative term in 6.39 reduces to

$$\frac{1}{2}\sum_{i,j} D^2 f(\mathbf{x}+t\mathbf{v})(\mathbf{e}_i)(\mathbf{e}_j) v_i v_j = \frac{1}{2}\sum_{i,j} H_{ij}(\mathbf{x}+t\mathbf{v}) v_i v_j$$

where

$$H_{ij}(\mathbf{x}+t\mathbf{v}) = D^2 f(\mathbf{x}+t\mathbf{v})(\mathbf{e}_i)(\mathbf{e}_j) = \frac{\partial^2 f(\mathbf{x}+t\mathbf{v})}{\partial x_j \partial x_i}.$$

Definition 6.11.3 *The matrix whose ij^{th} entry is $\frac{\partial^2 f(\mathbf{x})}{\partial x_j \partial x_i}$ is called the Hessian matrix, denoted as $\mathbf{H}(\mathbf{x})$.*

From Theorem 6.9.1, this is a symmetric real matrix, thus self adjoint. By the continuity of the second partial derivative,

$$f(\mathbf{x}+\mathbf{v}) = f(\mathbf{x}) + Df(\mathbf{x})\mathbf{v} + \frac{1}{2}\mathbf{v}^T H(\mathbf{x})\mathbf{v} +$$

$$\frac{1}{2}\left(\mathbf{v}^T\left(H(\mathbf{x}+t\mathbf{v}) - H(\mathbf{x})\right)\mathbf{v}\right). \tag{6.40}$$

where the last two terms involve ordinary matrix multiplication and

$$\mathbf{v}^T = (v_1 \cdots v_n)$$

for v_i the components of $\mathbf{v}$ relative to the standard basis.

Definition 6.11.4 *Let $f : D \to \mathbb{R}$ where D is a subset of some normed vector space. Then f has a local minimum at $\mathbf{x} \in D$ if there exists $\delta > 0$ such that for all $\mathbf{y} \in B(\mathbf{x}, \delta)$*

$$f(\mathbf{y}) \geq f(\mathbf{x}).$$

f has a local maximum at $\mathbf{x} \in D$ if there exists $\delta > 0$ such that for all $\mathbf{y} \in B(\mathbf{x}, \delta)$

$$f(\mathbf{y}) \leq f(\mathbf{x}).$$

Theorem 6.11.5 *If $f : U \to \mathbb{R}$ where U is an open subset of $\mathbb{R}^n$ and f is C^2, suppose $Df(\mathbf{x}) = 0$. Then if $H(\mathbf{x})$ has all positive eigenvalues, $\mathbf{x}$ is a local minimum. If the Hessian matrix $H(\mathbf{x})$ has all negative eigenvalues, then $\mathbf{x}$ is a local maximum. If $H(\mathbf{x})$ has a positive eigenvalue, then there exists a direction in which f has a local minimum at $\mathbf{x}$, while if $H(\mathbf{x})$ has a negative eigenvalue, there exists a direction in which $H(\mathbf{x})$ has a local maximum at $\mathbf{x}$.*

Proof: Since $Df(\mathbf{x}) = 0$, formula 6.40 holds and by continuity of the second derivative, $H(\mathbf{x})$ is a symmetric matrix. Thus $H(\mathbf{x})$ has all real eigenvalues. Suppose first that $H(\mathbf{x})$ has all positive eigenvalues and that all are larger than $\delta^2 > 0$. Then by Theorem 3.8.19, $H(\mathbf{x})$ has an orthonormal basis of eigenvectors, $\{\mathbf{v}_i\}_{i=1}^n$ and if $\mathbf{u}$ is an arbitrary vector, such that $\mathbf{u} = \sum_{j=1}^n u_j \mathbf{v}_j$ where $u_j = \mathbf{u} \cdot \mathbf{v}_j$, then

$$\mathbf{u}^T H(\mathbf{x})\mathbf{u} = \sum_{j=1}^n u_j \mathbf{v}_j^T H(\mathbf{x}) \sum_{j=1}^n u_j \mathbf{v}_j$$

$$= \sum_{j=1}^n u_j^2 \lambda_j \geq \delta^2 \sum_{j=1}^n u_j^2 = \delta^2 |\mathbf{u}|^2.$$

From 6.40 and the continuity of H, if $\mathbf{v}$ is small enough,

$$f(\mathbf{x}+\mathbf{v}) \geq f(\mathbf{x}) + \frac{1}{2}\delta^2 |\mathbf{v}|^2 - \frac{1}{4}\delta^2 |\mathbf{v}|^2 = f(\mathbf{x}) + \frac{\delta^2}{4} |\mathbf{v}|^2 .$$

This shows the first claim of the theorem. The second claim follows from similar reasoning. Suppose $H(\mathbf{x})$ has a positive eigenvalue λ^2. Then let $\mathbf{v}$ be an eigenvector for this eigenvalue. Then from 6.40,

$$f(\mathbf{x}+t\mathbf{v}) = f(\mathbf{x}) + \frac{1}{2}t^2 \mathbf{v}^T H(\mathbf{x})\,\mathbf{v} + \frac{1}{2}t^2 \left(\mathbf{v}^T \left(H(\mathbf{x}+t\mathbf{v}) - H(\mathbf{x})\right)\mathbf{v}\right)$$

which implies

$$\begin{aligned} f(\mathbf{x}+t\mathbf{v}) &= f(\mathbf{x}) + \frac{1}{2}t^2\lambda^2 |\mathbf{v}|^2 + \frac{1}{2}t^2 \left(\mathbf{v}^T \left(H(\mathbf{x}+t\mathbf{v}) - H(\mathbf{x})\right)\mathbf{v}\right) \\ &\geq f(\mathbf{x}) + \frac{1}{4}t^2\lambda^2 |\mathbf{v}|^2 \end{aligned}$$

whenever t is small enough. Thus in the direction $\mathbf{v}$ the function has a local minimum at $\mathbf{x}$. The assertion about the local maximum in some direction follows similarly. This proves the theorem.

This theorem is an analogue of the second derivative test for higher dimensions. As in one dimension, when there is a zero eigenvalue, it may be impossible to determine from the Hessian matrix what the local qualitative behavior of the function is. For example, consider

$$f_1(x,y) = x^4 + y^2,\ f_2(x,y) = -x^4 + y^2.$$

Then $Df_i(0,0) = \mathbf{0}$ and for both functions, the Hessian matrix evaluated at $(0,0)$ equals

$$\begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix}$$

but the behavior of the two functions is very different near the origin. The second has a saddle point while the first has a minimum there.

6.12 The Method Of Lagrange Multipliers

As an application of the implicit function theorem, consider the method of Lagrange multipliers from calculus. Recall the problem is to maximize or minimize a function subject to equality constraints. Let $f : U \to \mathbb{R}$ be a C^1 function where $U \subseteq \mathbb{R}^n$ and let

$$g_i(\mathbf{x}) = 0,\ i = 1, \cdots, m \tag{6.41}$$

be a collection of equality constraints with $m < n$. Now consider the system of nonlinear equations

$$\begin{aligned} f(\mathbf{x}) &= a \\ g_i(\mathbf{x}) &= 0,\ i = 1, \cdots, m. \end{aligned}$$

$\mathbf{x}_0$ is a local maximum if $f(\mathbf{x}_0) \geq f(\mathbf{x})$ for all $\mathbf{x}$ near $\mathbf{x}_0$ which also satisfies the constraints 6.41. A local minimum is defined similarly. Let $\mathbf{F} : U \times \mathbb{R} \to \mathbb{R}^{m+1}$ be defined by

$$\mathbf{F}(\mathbf{x},a) \equiv \begin{pmatrix} f(\mathbf{x}) - a \\ g_1(\mathbf{x}) \\ \vdots \\ g_m(\mathbf{x}) \end{pmatrix}. \tag{6.42}$$

Now consider the $m+1 \times n$ Jacobian matrix, the matrix of the linear transformation, $D_1\mathbf{F}(\mathbf{x},a)$ with respect to the usual basis for $\mathbb{R}^n$ and $\mathbb{R}^{m+1}$.

$$\begin{pmatrix} f_{x_1}(\mathbf{x}_0) & \cdots & f_{x_n}(\mathbf{x}_0) \\ g_{1x_1}(\mathbf{x}_0) & \cdots & g_{1x_n}(\mathbf{x}_0) \\ \vdots & & \vdots \\ g_{mx_1}(\mathbf{x}_0) & \cdots & g_{mx_n}(\mathbf{x}_0) \end{pmatrix}.$$

If this matrix has rank $m+1$ then some $m+1 \times m+1$ submatrix has nonzero determinant. It follows from the implicit function theorem that there exist $m+1$ variables, $x_{i_1}, \cdots, x_{i_{m+1}}$ such that the system

$$\mathbf{F}(\mathbf{x},a) = \mathbf{0} \tag{6.43}$$

specifies these $m+1$ variables as a function of the remaining $n-(m+1)$ variables and a in an open set of $\mathbb{R}^{n-m}$. Thus there is a solution $(\mathbf{x},a)$ to 6.43 for some $\mathbf{x}$ close to $\mathbf{x}_0$ whenever a is in some open interval. Therefore, $\mathbf{x}_0$ cannot be either a local minimum or a local maximum. It follows that if $\mathbf{x}_0$ is either a local maximum or a local minimum, then the above matrix must have rank less than $m+1$ which, by Corollary 3.5.20, requires the rows to be linearly dependent. Thus, there exist m scalars,

$$\lambda_1, \cdots, \lambda_m,$$

and a scalar μ, not all zero such that

$$\mu \begin{pmatrix} f_{x_1}(\mathbf{x}_0) \\ \vdots \\ f_{x_n}(\mathbf{x}_0) \end{pmatrix} = \lambda_1 \begin{pmatrix} g_{1x_1}(\mathbf{x}_0) \\ \vdots \\ g_{1x_n}(\mathbf{x}_0) \end{pmatrix} + \cdots + \lambda_m \begin{pmatrix} g_{mx_1}(\mathbf{x}_0) \\ \vdots \\ g_{mx_n}(\mathbf{x}_0) \end{pmatrix}. \tag{6.44}$$

If the column vectors

$$\begin{pmatrix} g_{1x_1}(\mathbf{x}_0) \\ \vdots \\ g_{1x_n}(\mathbf{x}_0) \end{pmatrix}, \cdots \begin{pmatrix} g_{mx_1}(\mathbf{x}_0) \\ \vdots \\ g_{mx_n}(\mathbf{x}_0) \end{pmatrix} \tag{6.45}$$

are linearly independent, then, $\mu \neq 0$ and dividing by μ yields an expression of the form

$$\begin{pmatrix} f_{x_1}(\mathbf{x}_0) \\ \vdots \\ f_{x_n}(\mathbf{x}_0) \end{pmatrix} = \lambda_1 \begin{pmatrix} g_{1x_1}(\mathbf{x}_0) \\ \vdots \\ g_{1x_n}(\mathbf{x}_0) \end{pmatrix} + \cdots + \lambda_m \begin{pmatrix} g_{mx_1}(\mathbf{x}_0) \\ \vdots \\ g_{mx_n}(\mathbf{x}_0) \end{pmatrix} \tag{6.46}$$

at every point $\mathbf{x}_0$ which is either a local maximum or a local minimum. This proves the following theorem.

Theorem 6.12.1 *Let U be an open subset of $\mathbb{R}^n$ and let $f : U \to \mathbb{R}$ be a C^1 function. Then if $\mathbf{x}_0 \in U$ is either a local maximum or local minimum of f subject to the constraints 6.41, then 6.44 must hold for some scalars $\mu, \lambda_1, \cdots, \lambda_m$ not all equal to zero. If the vectors in 6.45 are linearly independent, it follows that an equation of the form 6.46 holds.*

6.13 Exercises

1. Suppose $L \in \mathcal{L}(X,Y)$ and suppose L is one to one. Show there exists $r > 0$ such that for all $\mathbf{x} \in X$,
$$||L\mathbf{x}|| \geq r\,||\mathbf{x}||.$$
Hint: You might argue that $|||\mathbf{x}||| \equiv ||L\mathbf{x}||$ is a norm.

2. Show every polynomial, $\sum_{|\boldsymbol{\alpha}|\leq k} d_{\boldsymbol{\alpha}}\mathbf{x}^{\boldsymbol{\alpha}}$ is C^k for every k.

3. If $f : U \to \mathbb{R}$ where U is an open set in X and f is C^2, show the mixed Gateaux derivatives, $D_{\mathbf{v}_1\mathbf{v}_2}f(\mathbf{x})$ and $D_{\mathbf{v}_2\mathbf{v}_1}f(\mathbf{x})$ are equal.

4. Give an example of a function which is differentiable everywhere but at some point it fails to have continuous partial derivatives. Thus this function will be an example of a differentiable function which is not C^1.

5. The existence of partial derivatives does not imply continuity as was shown in an example. However, much more can be said than this. Consider

$$f(x,y) = \begin{cases} \frac{\left(x^2-y^4\right)^2}{\left(x^2+y^4\right)^2} \text{ if } (x,y) \neq (0,0), \\ 1 \text{ if } (x,y) = (0,0). \end{cases}$$

   Show each Gateaux derivative, $D_{\mathbf{v}}f(\mathbf{0})$ exists and equals 0 for every $\mathbf{v}$. Also show each Gateaux derivative exists at every other point in $\mathbb{R}^2$. Now consider the curve $x^2 = y^4$ and the curve $y = 0$ to verify the function fails to be continuous at $(0,0)$. This is an example of an everywhere Gateaux differentiable function which is not differentiable and not continuous.

6. Let f be a real valued function defined on $\mathbb{R}^2$ by

$$f(x,y) \equiv \begin{cases} \frac{x^3-y^3}{x^2+y^2} \text{ if } (x,y) \neq (0,0) \\ 0 \text{ if } (x,y) = (0,0) \end{cases}$$

   Determine whether f is continuous at $(0,0)$. Find $f_x(0,0)$ and $f_y(0,0)$. Are the partial derivatives of f continuous at $(0,0)$? Find $D_{(u,v)}f((0,0)), \lim_{t\to 0} \frac{f(t(u,v))}{t}$. Is the mapping $(u,v) \to D_{(u,v)}f((0,0))$ linear? Is f differentiable at $(0,0)$?

7. Let $f : V \to \mathbb{R}$ where V is a finite dimensional normed vector space. Suppose f is convex which means

$$f(t\mathbf{x} + (1-t)\mathbf{y}) \leq tf(\mathbf{x}) + (1-t)f(\mathbf{y})$$

   whenever $t \in [0,1]$. Suppose also that f is differentiable. Show then that for every $\mathbf{x}, \mathbf{y} \in V$,

$$(Df(\mathbf{x}) - Df(\mathbf{y}))(\mathbf{x}-\mathbf{y}) \geq 0.$$

8. Suppose $f : U \subseteq V \to \mathbb{F}$ where U is an open subset of V, a finite dimensional inner product space with the inner product denoted by $(\cdot,\cdot)$. Suppose f is differentiable. Show there exists a unique vector $\mathbf{v}(\mathbf{x}) \in V$ such that

$$(\mathbf{u} \cdot \mathbf{v}(\mathbf{x})) = Df(\mathbf{x})\mathbf{u}.$$

   This special vector is called the gradient and is usually denoted by $\nabla f(\mathbf{x})$. **Hint:** You might review the Riesz representation theorem presented earlier.

9. Suppose $\mathbf{f} : U \to Y$ where U is an open subset of X, a finite dimensional normed vector space. Suppose that for all $\mathbf{v} \in X, D_{\mathbf{v}}\mathbf{f}(\mathbf{x})$ exists. Show that whenever $a \in \mathbb{F}$ $D_{a\mathbf{v}}\mathbf{f}(\mathbf{x}) = aD_{\mathbf{v}}\mathbf{f}(\mathbf{x})$. Explain why if $\mathbf{x} \to D_{\mathbf{v}}\mathbf{f}(\mathbf{x})$ is continuous then $\mathbf{v} \to D_{\mathbf{v}}\mathbf{f}(\mathbf{x})$ is linear. Show that if $\mathbf{f}$ is differentiable at $\mathbf{x}$, then $D_{\mathbf{v}}\mathbf{f}(\mathbf{x}) = D\mathbf{f}(\mathbf{x})\mathbf{v}$.

10. Suppose B is an open ball in X and $\mathbf{f} : B \to Y$ is differentiable. Suppose also there exists $L \in \mathcal{L}(X,Y)$ such that

$$||D\mathbf{f}(\mathbf{x}) - L|| < k$$

for all $\mathbf{x} \in B$. Show that if $\mathbf{x}_1, \mathbf{x}_2 \in B$,

$$|\mathbf{f}(\mathbf{x}_1) - \mathbf{f}(\mathbf{x}_2) - L(\mathbf{x}_1 - \mathbf{x}_2)| \leq k|\mathbf{x}_1 - \mathbf{x}_2|.$$

Hint: Consider $T\mathbf{x} = \mathbf{f}(\mathbf{x}) - L\mathbf{x}$ and argue $||DT(\mathbf{x})|| < k$. Then consider Theorem 6.4.2.

11. Let U be an open subset of X, $\mathbf{f} : U \to Y$ where X, Y are finite dimensional normed vector spaces and suppose $\mathbf{f} \in C^1(U)$ and $D\mathbf{f}(\mathbf{x}_0)$ is one to one. Then show $\mathbf{f}$ is one to one near $\mathbf{x}_0$. **Hint:** Show using the assumption that $\mathbf{f}$ is C^1 that there exists $\delta > 0$ such that if

$$\mathbf{x}_1, \mathbf{x}_2 \in B(\mathbf{x}_0, \delta),$$

then

$$|\mathbf{f}(\mathbf{x}_1) - \mathbf{f}(\mathbf{x}_2) - D\mathbf{f}(\mathbf{x}_0)(\mathbf{x}_1 - \mathbf{x}_2)| \leq \frac{r}{2}|\mathbf{x}_1 - \mathbf{x}_2| \tag{6.47}$$

then use Problem 1.

12. Suppose $M \in \mathcal{L}(X,Y)$ and suppose M is onto. Show there exists $L \in \mathcal{L}(Y,X)$ such that

$$LM\mathbf{x} = P\mathbf{x}$$

where $P \in \mathcal{L}(X,X)$, and $P^2 = P$. Also show L is one to one and onto. **Hint:** Let $\{\mathbf{y}_1, \cdots, \mathbf{y}_m\}$ be a basis of Y and let $M\mathbf{x}_i = \mathbf{y}_i$. Then define

$$L\mathbf{y} = \sum_{i=1}^{m} \alpha_i \mathbf{x}_i \text{ where } \mathbf{y} = \sum_{i=1}^{m} \alpha_i \mathbf{y}_i.$$

Show $\{\mathbf{x}_1, \cdots, \mathbf{x}_m\}$ is a linearly independent set and show you can obtain $\{\mathbf{x}_1, \cdots, \mathbf{x}_m, \cdots, \mathbf{x}_n\}$, a basis for X in which $M\mathbf{x}_j = \mathbf{0}$ for $j > m$. Then let

$$P\mathbf{x} \equiv \sum_{i=1}^{m} \alpha_i \mathbf{x}_i$$

where

$$\mathbf{x} = \sum_{i=1}^{m} \alpha_i \mathbf{x}_i.$$

13. This problem depends on the result of Problem 12. Let $\mathbf{f} : U \subseteq X \to Y$, $\mathbf{f}$ is C^1, and $D\mathbf{f}(\mathbf{x})$ is onto for each $\mathbf{x} \in U$. Then show $\mathbf{f}$ maps open subsets of U onto open sets in Y. **Hint:** Let $P = LD\mathbf{f}(\mathbf{x})$ as in Problem 12. Argue L maps open sets from Y to open sets of the vector space $X_1 \equiv PX$ and L^{-1} maps open sets from X_1 to open sets of Y. Then $L\mathbf{f}(\mathbf{x} + \mathbf{v}) = L\mathbf{f}(\mathbf{x}) + LD\mathbf{f}(\mathbf{x})\mathbf{v} + \mathbf{o}(\mathbf{v})$. Now for $\mathbf{z} \in X_1$, let $\mathbf{h}(\mathbf{z}) = L\mathbf{f}(\mathbf{x} + \mathbf{z}) - L\mathbf{f}(\mathbf{x})$. Then $\mathbf{h}$ is C^1 on some small open subset of X_1 containing $\mathbf{0}$ and $D\mathbf{h}(\mathbf{0}) = LD\mathbf{f}(\mathbf{x})$ which is seen to be one to one and onto and in $\mathcal{L}(X_1, X_1)$. Therefore, if r is small enough, $\mathbf{h}(B(\mathbf{0},r))$ equals an open set in X_1, V. This is by the inverse function theorem. Hence $L(\mathbf{f}(\mathbf{x} + B(\mathbf{0},r)) - \mathbf{f}(\mathbf{x})) = V$ and so $\mathbf{f}(\mathbf{x} + B(\mathbf{0},r)) - \mathbf{f}(\mathbf{x}) = L^{-1}(V)$, an open set in Y.

14. Suppose $U \subseteq \mathbb{R}^2$ is an open set and $\mathbf{f} : U \to \mathbb{R}^3$ is C^1. Suppose $D\mathbf{f}(s_0, t_0)$ has rank two and
$$\mathbf{f}(s_0, t_0) = \begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix}.$$
Show that for (s,t) near (s_0, t_0), the points $\mathbf{f}(s,t)$ may be realized in one of the following forms.
$$\{(x, y, \phi(x,y)) : (x,y) \text{ near } (x_0, y_0)\},$$
$$\{(\phi(y,z), y, z) : (y,z) \text{ near } (y_0, z_0)\},$$
or
$$\{(x, \phi(x,z), z,) : (x,z) \text{ near } (x_0, z_0)\}.$$
This shows that parametrically defined surfaces can be obtained locally in a particularly simple form.

15. Let $\mathbf{f} : U \to Y$, $D\mathbf{f}(\mathbf{x})$ exists for all $\mathbf{x} \in U$, $B(\mathbf{x}_0, \delta) \subseteq U$, and there exists $L \in \mathcal{L}(X,Y)$, such that $L^{-1} \in \mathcal{L}(Y,X)$, and for all $\mathbf{x} \in B(\mathbf{x}_0, \delta)$
$$||D\mathbf{f}(\mathbf{x}) - L|| < \frac{r}{||L^{-1}||},\ r < 1.$$
Show that there exists $\varepsilon > 0$ and an open subset of $B(\mathbf{x}_0, \delta), V$, such that $\mathbf{f} : V \to B(\mathbf{f}(\mathbf{x}_0), \varepsilon)$ is one to one and onto. Also $D\mathbf{f}^{-1}(\mathbf{y})$ exists for each $\mathbf{y} \in B(f(\mathbf{x}_0), \varepsilon)$ and is given by the formula
$$D\mathbf{f}^{-1}(\mathbf{y}) = \left[D\mathbf{f}\left(\mathbf{f}^{-1}(\mathbf{y})\right)\right]^{-1}.$$
Hint: Let
$$T_{\mathbf{y}}(\mathbf{x}) \equiv T(\mathbf{x}, \mathbf{y}) \equiv \mathbf{x} - L^{-1}(\mathbf{f}(\mathbf{x}) - \mathbf{y})$$
for $|\mathbf{y} - \mathbf{f}(\mathbf{x}_0)| < \frac{(1-r)\delta}{2||L^{-1}||}$, consider $\{T_{\mathbf{y}}^n(\mathbf{x}_0)\}$. This is a version of the inverse function theorem for $\mathbf{f}$ only differentiable, not C^1.

16. Recall the n^{th} derivative can be considered a multilinear function defined on X^n with values in some normed vector space. Now define a function denoted as $\mathbf{w}_i \mathbf{v}_{j_1} \cdots \mathbf{v}_{j_n}$ which maps $X^n \to Y$ in the following way
$$\mathbf{w}_i \mathbf{v}_{j_1} \cdots \mathbf{v}_{j_n}(\mathbf{v}_{k_1}, \cdots, \mathbf{v}_{k_n}) \equiv \mathbf{w}_i \delta_{j_1 k_1} \delta_{j_2 k_2} \cdots \delta_{j_n k_n} \tag{6.48}$$
and $\mathbf{w}_i \mathbf{v}_{j_1} \cdots \mathbf{v}_{j_n}$ is to be linear in each variable. Thus, for
$$\left(\sum_{k_1=1}^n a_{k_1}\mathbf{v}_{k_1}, \cdots, \sum_{k_n=1}^n a_{k_n}\mathbf{v}_{k_n}\right) \in X^n,$$
$$\begin{aligned} &\mathbf{w}_i \mathbf{v}_{j_1} \cdots \mathbf{v}_{j_n}\left(\sum_{k_1=1}^n a_{k_1}\mathbf{v}_{k_1}, \cdots, \sum_{k_n=1}^n a_{k_n}\mathbf{v}_{k_n}\right) \\ \equiv &\sum_{k_1 k_2 \cdots k_n} \mathbf{w}_i (a_{k_1} a_{k_2} \cdots a_{k_n}) \delta_{j_1 k_1} \delta_{j_2 k_2} \cdots \delta_{j_n k_n} \\ = &\ \mathbf{w}_i a_{j_1} a_{j_2} \cdots a_{j_n} \end{aligned} \tag{6.49}$$
Show each $\mathbf{w}_i \mathbf{v}_{j_1} \cdots \mathbf{v}_{j_n}$ is an n linear Y valued function. Next show the set of n linear Y valued functions is a vector space and these special functions, $\mathbf{w}_i \mathbf{v}_{j_1} \cdots \mathbf{v}_{j_n}$ for all choices of i and the j_k is a basis of this vector space. Find the dimension of the vector space.

17. Minimize $\sum_{j=1}^{n} x_j$ subject to the constraint $\sum_{j=1}^{n} x_j^2 = a^2$. Your answer should be some function of a which you may assume is a positive number.

18. Find the point, (x, y, z) on the level surface, $4x^2 + y^2 - z^2 = 1$which is closest to $(0, 0, 0)$.

19. A curve is formed from the intersection of the plane, $2x + 3y + z = 3$ and the cylinder $x^2 + y^2 = 4$. Find the point on this curve which is closest to $(0, 0, 0)$.

20. A curve is formed from the intersection of the plane, $2x + 3y + z = 3$ and the sphere $x^2 + y^2 + z^2 = 16$. Find the point on this curve which is closest to $(0, 0, 0)$.

21. Find the point on the plane, $2x + 3y + z = 4$ which is closest to the point $(1, 2, 3)$.

22. Let $A = (A_{ij})$ be an $n \times n$ matrix which is symmetric. Thus $A_{ij} = A_{ji}$ and recall $(A\mathbf{x})_i = A_{ij}x_j$ where as usual sum over the repeated index. Show $\frac{\partial}{\partial x_i}(A_{ij}x_j x_i) = 2A_{ij}x_j$. Show that when you use the method of Lagrange multipliers to maximize the function, $A_{ij}x_j x_i$ subject to the constraint, $\sum_{j=1}^{n} x_j^2 = 1$, the value of λ which corresponds to the maximum value of this functions is such that $A_{ij}x_j = \lambda x_i$. Thus $A\mathbf{x} = \lambda\mathbf{x}$. Thus λ is an eigenvalue of the matrix, A.

23. Let $x_1, \cdots, x_5$ be 5 positive numbers. Maximize their product subject to the constraint that
$$x_1 + 2x_2 + 3x_3 + 4x_4 + 5x_5 = 300.$$

24. Let $f(x_1, \cdots, x_n) = x_1^n x_2^{n-1} \cdots x_n^1$. Then f achieves a maximum on the set,
$$S \equiv \left\{ \mathbf{x} \in \mathbb{R}^n : \sum_{i=1}^{n} i x_i = 1 \text{ and each } x_i \geq 0 \right\}.$$
If $\mathbf{x} \in S$ is the point where this maximum is achieved, find x_1/x_n.

25. Let (x, y) be a point on the ellipse, $x^2/a^2 + y^2/b^2 = 1$ which is in the first quadrant. Extend the tangent line through (x, y) till it intersects the x and y axes and let $A(x, y)$ denote the area of the triangle formed by this line and the two coordinate axes. Find the minimum value of the area of this triangle as a function of a and b.

26. Maximize $\prod_{i=1}^{n} x_i^2$ $(\equiv x_1^2 \times x_2^2 \times x_3^2 \times \cdots \times x_n^2)$ subject to the constraint, $\sum_{i=1}^{n} x_i^2 = r^2$. Show the maximum is $\left(r^2/n\right)^n$. Now show from this that
$$\left(\prod_{i=1}^{n} x_i^2 \right)^{1/n} \leq \frac{1}{n} \sum_{i=1}^{n} x_i^2$$
and finally, conclude that if each number $x_i \geq 0$, then
$$\left(\prod_{i=1}^{n} x_i \right)^{1/n} \leq \frac{1}{n} \sum_{i=1}^{n} x_i$$
and there exist values of the x_i for which equality holds. This says the "geometric mean" is always smaller than the arithmetic mean.

27. Maximize x^2y^2 subject to the constraint

$$\frac{x^{2p}}{p} + \frac{y^{2q}}{q} = r^2$$

where p, q are real numbers larger than 1 which have the property that

$$\frac{1}{p} + \frac{1}{q} = 1.$$

show the maximum is achieved when $x^{2p} = y^{2q}$ and equals r^2. Now conclude that if $x, y > 0$, then

$$xy \leq \frac{x^p}{p} + \frac{y^q}{q}$$

and there are values of x and y where this inequality is an equation.

Measures And Measurable Functions

The integral to be discussed next is the Lebesgue integral. This integral is more general than the Riemann integral of beginning calculus. It is not as easy to define as this integral but is vastly superior in every application. In fact, the Riemann integral has been obsolete for over 100 years. There exist convergence theorems for this integral which are not available for the Riemann integral and unlike the Riemann integral, the Lebesgue integral generalizes readily to abstract settings used in probability theory. Much of the analysis done in the last 100 years applies to the Lebesgue integral. For these reasons, and because it is very easy to generalize the Lebesgue integral to functions of many variables I will present the Lebesgue integral here. First it is convenient to discuss outer measures, measures, and measurable function in a general setting.

7.1 Compact Sets

This is a good place to put an important theorem about compact sets. The definition of what is meant by a compact set follows.

Definition 7.1.1 *Let $\mathcal{U}$ denote a collection of open sets in a normed vector space. Then $\mathcal{U}$ is said to be an open cover of a set K if $K \subseteq \cup\mathcal{U}$. Let K be a subset of a normed vector space. Then K is compact if whenever $\mathcal{U}$ is an open cover of K there exist finitely many sets of $\mathcal{U}$, $\{U_1, \cdots, U_m\}$ such that*

$$K \subseteq \cup_{k=1}^{m} U_k.$$

In words, every open cover admits a finite subcover.

It was shown earlier that in any finite dimensional normed vector space the closed and bounded sets are those which are sequentially compact. The next theorem says that in any normed vector space, sequentially compact and compact are the same.[1] First here is a very interesting lemma about the existence of something called a Lebesgue number, the number r in the next lemma.

Lemma 7.1.2 *Let K be a sequentially compact set in a normed vector space and let $\mathcal{U}$ be an open cover of K. Then there exists $r > 0$ such that if $\mathbf{x} \in K$, then $B(\mathbf{x}, r)$ is a subset of some set of $\mathcal{U}$.*

[1] Actually, this is true more generally than for normed vector spaces. It is also true for metric spaces, those on which there is a distance defined.

Proof: Suppose no such r exists. Then in particular, $1/n$ does not work for each $n \in \mathbb{N}$. Therefore, there exists $\mathbf{x}_n \in K$ such that $B(\mathbf{x}_n, r)$ is not a subset of any of the sets of $\mathcal{U}$. Since K is sequentially compact, there exists a subsequence, $\{\mathbf{x}_{n_k}\}$ converging to a point $\mathbf{x}$ of K. Then there exists $r > 0$ such that $B(\mathbf{x}, r) \subseteq U \in \mathcal{U}$ because $\mathcal{U}$ is an open cover. Also $\mathbf{x}_{n_k} \in B(\mathbf{x}, r/2)$ for all k large enough and also for all k large enough, $1/n_k < r/2$. Therefore, there exists $\mathbf{x}_{n_k} \in B(\mathbf{x}, r/2)$ and $1/n_k < r/2$. But this is a contradiction because

$$B(\mathbf{x}_{n_k}, 1/n_k) \subseteq B(\mathbf{x}, r) \subseteq U$$

contrary to the choice of $\mathbf{x}_{n_k}$ which required $B(\mathbf{x}_{n_k}, 1/n_k)$ is not contained in any set of $\mathcal{U}$. This proves the lemma.

Theorem 7.1.3 *Let K be a set in a normed vector space. Then K is compact if and only if K is sequentially compact. In particular if K is a closed and bounded subset of a finite dimensional normed vector space, then K is compact.*

Proof: Suppose first K is sequentially compact and let $\mathcal{U}$ be an open cover. Let r be a Lebesgue number as described in Lemma 7.1.2. Pick $\mathbf{x}_1 \in K$. Then $B(\mathbf{x}_1, r) \subseteq U_1$ for some $U_1 \in \mathcal{U}$. Suppose $\{B(\mathbf{x}_i, r)\}_{i=1}^m$ have been chosen such that

$$B(\mathbf{x}_i, r) \subseteq U_i \in \mathcal{U}.$$

If their union contains K then $\{U_i\}_{i=1}^m$ is a finite subcover of $\mathcal{U}$. If $\{B(\mathbf{x}_i, r)\}_{i=1}^m$ does not cover K, then there exists $\mathbf{x}_{m+1} \notin \cup_{i=1}^m B(\mathbf{x}_i, r)$ and so $B(\mathbf{x}_{m+1}, r) \subseteq U_{m+1} \in \mathcal{U}$. This process must stop after finitely many choices of $B(\mathbf{x}_i, r)$ because if not, $\{\mathbf{x}_k\}_{k=1}^\infty$ would have a subsequence which converges to a point of K which cannot occur because whenever $i \neq j$,

$$||\mathbf{x}_i - \mathbf{x}_j|| > r$$

Therefore, eventually

$$K \subseteq \cup_{k=1}^m B(\mathbf{x}_k, r) \subseteq \cup_{k=1}^m U_k.$$

this proves one half of the theorem.

Now suppose K is compact. I need to show it is sequentially compact. Suppose it is not. Then there exists a sequence, $\{\mathbf{x}_k\}$ which has no convergent subsequence. This requires that $\{\mathbf{x}_k\}$ have no limit point for if it did have a limit point, $\mathbf{x}$, then $B(\mathbf{x}, 1/n)$ would contain infinitely many distinct points of $\{\mathbf{x}_k\}$ and so a subsequence of $\{\mathbf{x}_k\}$ converging to $\mathbf{x}$ could be obtained. Also no $\mathbf{x}_k$ is repeated infinitely often because if there were such, a convergent subsequence could be obtained. Hence $\cup_{k=m}^\infty \{\mathbf{x}_k\} \equiv C_m$ is a closed set, closed because it contains all its limit points. (It has no limit points so it contains them all.) Then letting $U_m = C_m^C$, it follows $\{U_m\}$ is an open cover of K which has no finite subcover. Thus K must be sequentially compact after all.

If K is a closed and bounded set in a finite dimensional normed vector space, then K is sequentially compact by Theorem 5.8.4. Therefore, by the first part of this theorem, it is sequentially compact. This proves the theorem.

Summarizing the above theorem along with Theorem 5.8.4 yields the following corollary which is often called the Heine Borel theorem.

Corollary 7.1.4 *Let X be a finite dimensional normed vector space and let $K \subseteq X$. Then the following are equivalent.*

1. *K is closed and bounded.*
2. *K is sequentially compact.*
3. *K is compact.*

7.2 An Outer Measure On $\mathcal{P}(\mathbb{R})$

A measure on $\mathbb{R}$ is like length. I will present something more general because it is no trouble to do so and the generalization is useful in many areas of mathematics such as probability. Recall that $\mathcal{P}(S)$ denotes the set of all subsets of S.

Theorem 7.2.1 *Let F be an increasing function defined on $\mathbb{R}$, an integrator function. There exists a function $\mu : \mathcal{P}(\mathbb{R}) \to [0,\infty]$ which satisfies the following properties.*

1. *If $A \subseteq B$, then $0 \leq \mu(A) \leq \mu(B), \mu(\emptyset) = 0$.*
2. *$\mu(\cup_{k=1}^{\infty} A_i) \leq \sum_{i=1}^{\infty} \mu(A_i)$*
3. *$\mu([a,b]) = F(b+) - F(a-)$,*
4. *$\mu((a,b)) = F(b-) - F(a+)$*
5. *$\mu((a,b]) = F(b+) - F(a+)$*
6. *$\mu([a,b)) = F(b-) - F(a-)$ where*

$$F(b+) \equiv \lim_{t \to b+} F(t), F(b-) \equiv \lim_{t \to a-} F(t).$$

Proof: First it is necessary to define the function, μ. This is contained in the following definition.

Definition 7.2.2 *For $A \subseteq \mathbb{R}$,*

$$\mu(A) = \inf \left\{ \sum_{j=1}^{\infty} (F(b_i-) - F(a_i+)) : A \subseteq \cup_{i=1}^{\infty} (a_i, b_i) \right\}$$

In words, you look at all coverings of A with open intervals. For each of these open coverings, you add the "lengths" of the individual open intervals and you take the infimum of all such numbers obtained.

Then 1.) is obvious because if a countable collection of open intervals covers B then it also covers A. Thus the set of numbers obtained for B is smaller than the set of numbers for A. Why is $\mu(\emptyset) = 0$? Pick a point of continuity of F. Such points exist because F is increasing and so it has only countably many points of discontinuity. Let a be this point. Then $\emptyset \subseteq (a-\delta, a+\delta)$ and so $\mu(\emptyset) \leq 2\delta$ for every $\delta > 0$.

Consider 2.). If any $\mu(A_i) = \infty$, there is nothing to prove. The assertion simply is $\infty \leq \infty$. Assume then that $\mu(A_i) < \infty$ for all i. Then for each $m \in \mathbb{N}$ there exists a countable set of open intervals, $\{(a_i^m, b_i^m)\}_{i=1}^{\infty}$ such that

$$\mu(A_m) + \frac{\varepsilon}{2^m} > \sum_{i=1}^{\infty} (F(b_i^m-) - F(a_i^m+)).$$

Then using Theorem 2.3.5 on Page 20,

$$\begin{aligned}\mu\left(\cup_{m=1}^{\infty}A_m\right) &\leq \sum_{im}\left(F\left(b_i^m-\right)-F\left(a_i^m+\right)\right)\\ &= \sum_{m=1}^{\infty}\sum_{i=1}^{\infty}\left(F\left(b_i^m-\right)-F\left(a_i^m+\right)\right)\\ &\leq \sum_{m=1}^{\infty}\mu\left(A_m\right)+\frac{\varepsilon}{2^m}\\ &= \sum_{m=1}^{\infty}\mu\left(A_m\right)+\varepsilon\end{aligned}$$

and since ε is arbitrary, this establishes 2.).

Next consider 3.). By definition, there exists a sequence of open intervals, $\{(a_i,b_i)\}_{i=1}^{\infty}$ whose union contains $[a,b]$ such that

$$\mu\left([a,b]\right)+\varepsilon\geq\sum_{i=1}^{\infty}\left(F\left(b_i-\right)-F\left(a_i+\right)\right)$$

By Theorem 7.1.3, finitely many of these intervals also cover $[a,b]$. It follows there exists finitely many of these intervals, $\{(a_i,b_i)\}_{i=1}^{n}$ which overlap such that $a\in\left(a_1,b_1\right),b_1\in\left(a_2,b_2\right),\cdots,b\in\left(a_n,b_n\right)$. Therefore,

$$\mu\left([a,b]\right)\leq\sum_{i=1}^{n}\left(F\left(b_i-\right)-F\left(a_i+\right)\right)$$

It follows

$$\begin{aligned}\sum_{i=1}^{n}\left(F\left(b_i-\right)-F\left(a_i+\right)\right) &\geq \mu\left([a,b]\right)\\ &\geq \sum_{i=1}^{n}\left(F\left(b_i-\right)-F\left(a_i+\right)\right)-\varepsilon\\ &\geq F\left(b+\right)-F\left(a-\right)-\varepsilon\end{aligned}$$

Since ε is arbitrary, this shows

$$\mu\left([a,b]\right)\geq F\left(b+\right)-F\left(a-\right)$$

but also, from the definition, the following inequality holds for all $\delta>0$.

$$\mu\left([a,b]\right)\leq F\left(\left(b+\delta\right)-\right)-F\left(\left(a-\delta\right)-\right)\leq F\left(b+\delta\right)-F\left(a-\delta\right)$$

Therefore, letting $\delta\to0$ yields

$$\mu\left([a,b]\right)\leq F\left(b+\right)-F\left(a-\right)$$

This establishes 3.).

Consider 4.). For small $\delta>0$,

$$\mu\left([a+\delta,b-\delta]\right)\leq\mu\left((a,b)\right)\leq\mu\left([a,b]\right).$$

Therefore, from 3.) and the definition of μ,

$$F((b-\delta)) - F((a+\delta)) \leq F((b-\delta)+) - F((a+\delta)-)$$
$$= \mu([a+\delta, b-\delta]) \leq \mu((a,b)) \leq F(b-) - F(a+)$$

Now letting δ decrease to 0 it follows

$$F(b-) - F(a+) \leq \mu((a,b)) \leq F(b-) - F(a+)$$

This shows 4.)

Consider 5.). From 3.) and 4.)

$$\begin{aligned} & F(b+) - F((a+\delta)) \\ \leq\ & F(b+) - F((a+\delta)-) \\ =\ & \mu([a+\delta, b]) \leq \mu((a,b]) \\ \leq\ & \mu((a, b+\delta)) = F((b+\delta)-) - F(a+) \\ \leq\ & F(b+\delta) - F(a+). \end{aligned}$$

Now let δ converge to 0 from above to obtain

$$F(b+) - F(a+) = \mu((a,b]) = F(b+) - F(a+).$$

This establishes 5.) and 6.) is entirely similar to 5.). This proves the theorem.

Definition 7.2.3 *Let Ω be a nonempty set. A function mapping $\mathcal{P}(\Omega) \to [0,\infty]$ is called an outer measure if it satisfies the conditions 1.) and 2.) in Theorem 7.2.1.*

7.3 General Outer Measures And Measures

First the general concept of a measure will be presented. Then it will be shown how to get a measure from any outer measure. Using the outer measure just obtained, this yields Lebesgue Stieltjes measure on $\mathbb{R}$. Then an abstract Lebesgue integral and its properties will be presented. After this the theory is specialized to the situation of $\mathbb{R}$ and the outer measure in Theorem 7.2.1. This will yield the Lebesgue Stieltjes integral on $\mathbb{R}$ along with spectacular theorems about its properties. The generalization to Lebesgue integration on $\mathbb{R}^n$ turns out to be very easy.

7.3.1 Measures And Measure Spaces

First here is a definition of a measure.

Definition 7.3.1 *$\mathcal{S} \subseteq \mathcal{P}(\Omega)$ is called a σ algebra , pronounced "sigma algebra", if*

$$\emptyset, \Omega \in \mathcal{S},$$

$$\text{If } E \in \mathcal{S} \text{ then } E^C \in \mathcal{S}$$

and

$$\text{If } E_i \in \mathcal{S}, \text{ for } i = 1, 2, \cdots, \text{ then } \cup_{i=1}^{\infty} E_i \in \mathcal{S}.$$

A function $\mu : \mathcal{S} \to [0,\infty]$ where $\mathcal{S}$ is a σ algebra is called a measure if whenever $\{E_i\}_{i=1}^{\infty} \subseteq \mathcal{S}$ and the E_i are disjoint, then it follows

$$\mu\left(\cup_{j=1}^{\infty} E_j\right) = \sum_{j=1}^{\infty} \mu(E_j).$$

The triple $(\Omega, \mathcal{S}, \mu)$ is often called a measure space. Sometimes people refer to $(\Omega, \mathcal{S})$ as a measurable space, making no reference to the measure. Sometimes $(\Omega, \mathcal{S})$ may also be called a measure space.

Theorem 7.3.2 *Let $\{E_m\}_{m=1}^{\infty}$ be a sequence of measurable sets in a measure space $(\Omega, \mathcal{F}, \mu)$. Then if $\cdots E_n \subseteq E_{n+1} \subseteq E_{n+2} \subseteq \cdots$,*

$$\mu(\cup_{i=1}^{\infty} E_i) = \lim_{n\to\infty} \mu(E_n) \tag{7.1}$$

and if $\cdots E_n \supseteq E_{n+1} \supseteq E_{n+2} \supseteq \cdots$ and $\mu(E_1) < \infty$, then

$$\mu(\cap_{i=1}^{\infty} E_i) = \lim_{n\to\infty} \mu(E_n). \tag{7.2}$$

Stated more succinctly, $E_k \uparrow E$ implies $\mu(E_k) \uparrow \mu(E)$ and $E_k \downarrow E$ with $\mu(E_1) < \infty$ implies $\mu(E_k) \downarrow \mu(E)$.

Proof: First note that $\cap_{i=1}^{\infty} E_i = (\cup_{i=1}^{\infty} E_i^C)^C \in \mathcal{F}$ so $\cap_{i=1}^{\infty} E_i$ is measurable. Also note that for A and B sets of $\mathcal{F}$, $A \setminus B \equiv \left(A^C \cup B\right)^C \in \mathcal{F}$. To show 7.1, note that 7.1 is obviously true if $\mu(E_k) = \infty$ for any k. Therefore, assume $\mu(E_k) < \infty$ for all k. Thus

$$\mu(E_{k+1} \setminus E_k) + \mu(E_k) = \mu(E_{k+1})$$

and so

$$\mu(E_{k+1} \setminus E_k) = \mu(E_{k+1}) - \mu(E_k).$$

Also,

$$\bigcup_{k=1}^{\infty} E_k = E_1 \cup \bigcup_{k=1}^{\infty} (E_{k+1} \setminus E_k)$$

and the sets in the above union are disjoint. Hence

$$\mu(\cup_{i=1}^{\infty} E_i) = \mu(E_1) + \sum_{k=1}^{\infty} \mu(E_{k+1} \setminus E_k) = \mu(E_1)$$

$$+ \sum_{k=1}^{\infty} \mu(E_{k+1}) - \mu(E_k)$$

$$= \mu(E_1) + \lim_{n\to\infty} \sum_{k=1}^{n} \mu(E_{k+1}) - \mu(E_k) = \lim_{n\to\infty} \mu(E_{n+1}).$$

This shows part 7.1.

To verify 7.2,

$$\mu(E_1) = \mu(\cap_{i=1}^{\infty} E_i) + \mu(E_1 \setminus \cap_{i=1}^{\infty} E_i)$$

since $\mu(E_1) < \infty$, it follows $\mu(\cap_{i=1}^{\infty} E_i) < \infty$. Also, $E_1 \setminus \cap_{i=1}^{n} E_i \uparrow E_1 \setminus \cap_{i=1}^{\infty} E_i$ and so by 7.1,

$$\mu(E_1) - \mu(\cap_{i=1}^{\infty} E_i) = \mu(E_1 \setminus \cap_{i=1}^{\infty} E_i) = \lim_{n\to\infty} \mu(E_1 \setminus \cap_{i=1}^{n} E_i)$$

$$= \mu(E_1) - \lim_{n\to\infty} \mu(\cap_{i=1}^{n} E_i) = \mu(E_1) - \lim_{n\to\infty} \mu(E_n),$$

Hence, subtracting $\mu(E_1)$ from both sides,

$$\lim_{n\to\infty} \mu(E_n) = \mu(\cap_{i=1}^{\infty} E_i).$$

This proves the theorem.

The following definition is important.

Definition 7.3.3 *If something happens except for on a set of measure zero, then it is said to happen a.e. "almost everywhere". For example, $\{f_k(x)\}$ is said to converge to $f(x)$ a.e. if there is a set of measure zero, N such that if $x \in N$, then $f_k(x) \to f(x)$.*

7.4 The Borel Sets, Regular Measures

7.4.1 Definition of Regular Measures

It is important to consider the interaction between measures and open and compact sets. This involves the concept of a regular measure.

Definition 7.4.1 *Let Y be a closed subset of X a finite dimensional normed vector space. The closed sets in Y are the intersections of closed sets in X with Y. The open sets in Y are intersections of open sets of X with Y. Now let $\mathcal{F}$ be a σ algebra of sets of Y and let μ be a measure defined on $\mathcal{F}$. Then μ is said to be a regular measure if the following two conditions hold.*

1. *For every $F \in \mathcal{F}$*

$$\mu(F) = \sup\{\mu(K) : K \subseteq F \text{ and } K \text{ is compact}\} \tag{7.3}$$

2. *For every $F \in \mathcal{F}$*

$$\mu(F) = \inf\{\mu(V) : V \supseteq F \text{ and } V \text{ is open in } Y\} \tag{7.4}$$

The first of the above conditions is called inner regularity and the second is called outer regularity.

Proposition 7.4.2 *In the above situation, a set, $K \subseteq Y$ is compact in Y if and only if it is compact in X.*

Proof: If K is compact in X and $K \subseteq Y$, let $\mathcal{U}$ be an open cover of K of sets open in Y. This means $\mathcal{U} = \{Y \cap V : V \in \mathcal{V}\}$ where $\mathcal{V}$ is an open cover of K consisting of sets open in X. Therefore, $\mathcal{V}$ admits a finite subcover, $\{V_1, \cdots, V_m\}$ and consequently, $\{Y \cap V_1, \cdots, Y \cap V_m\}$ is a finite subcover from $\mathcal{U}$. Thus K is compact in Y.

Now suppose K is compact in Y. This means that if $\mathcal{U}$ is an open cover of sets open in Y it admitts a finite subcover. Now let $\mathcal{V}$ be any open cover of K, consisting of sets open in X. Then $\mathcal{U} \equiv \{V \cap Y : V \in \mathcal{V}\}$ is a cover consisting of sets open in Y and by definition, this admitts a finite subcover, $\{Y \cap V_1, \cdots, Y \cap V_m\}$ but this implies $\{V_1, \cdots, V_m\}$ is also a finite subcover consisting of sets of $\mathcal{V}$. This proves the proposition.

7.4.2 The Borel Sets

If Y is a closed subset of X, a normed vector space, denote by $\mathcal{B}(Y)$ the smallest σ algebra of subsets of Y which contains all the open sets of Y. To see such a smallest σ algebra exists, let $\mathfrak{H}$ denote the set of all σ algebras which contain the open sets $\mathcal{P}(Y)$, the set of all subsets of Y is one such σ algebra. Define $\mathcal{B}(Y) \equiv \cap\mathfrak{H}$. Then $\mathcal{B}(Y)$ is a σ algebra because $\emptyset, Y$ are both open sets in Y and so they are in each σ algebra of $\mathfrak{H}$. If $F \in \mathcal{B}(Y)$, then F is a set of every σ algebra of $\mathfrak{H}$ and so F^C is also a set of every σ algebra of $\mathfrak{H}$. Thus $F^C \in \mathcal{B}(Y)$. If $\{F_i\}$ is a sequence of sets of $\mathcal{B}(Y)$, then $\{F_i\}$ is a sequence of sets of every σ algebra of $\mathfrak{H}$ and so $\cup_i F_i$ is a set in every σ algebra of $\mathfrak{H}$ which implies $\cup_i F_i \in \mathcal{B}(Y)$ so $\mathcal{B}(Y)$ is a σ algebra as claimed. From its definition, it is the smallest σ algebra which contains the open sets.

7.4.3 Borel Sets And Regularity

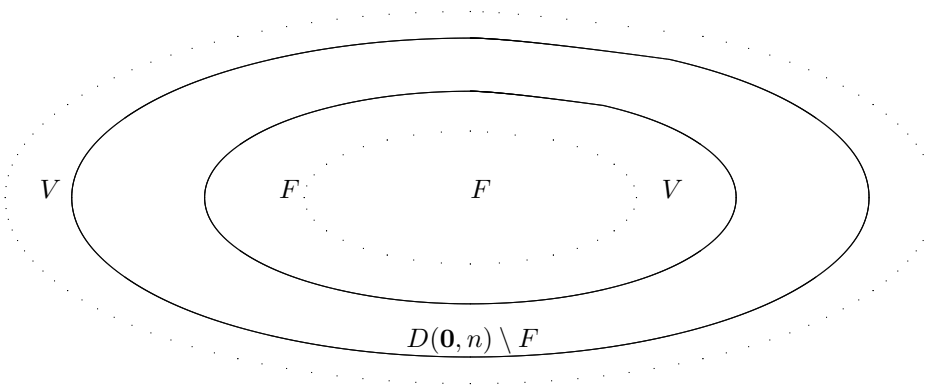
To illustrate how nice the Borel sets are, here are some interesting results about regularity. The first Lemma holds for any σ algebra, not just the Borel sets. Here is some notation which will be used. Let

$$\begin{aligned} S(\mathbf{0},r) &\equiv \{\mathbf{x} \in Y : ||\mathbf{x}|| = r\} \\ D(\mathbf{0},r) &\equiv \{\mathbf{x} \in Y : ||\mathbf{x}|| \leq r\} \\ B(\mathbf{0},r) &\equiv \{\mathbf{x} \in Y : ||\mathbf{x}|| < r\} \end{aligned}$$

Thus $S(\mathbf{0},r)$ is a closed set as is $D(\mathbf{0},r)$ while $B(\mathbf{0},r)$ is an open set. These are closed or open as stated in Y. Since $S(\mathbf{0},r)$ and $D(\mathbf{0},r)$ are intersections of closed sets, Y and a closed set in X, these are also closed in X. Of course $B(\mathbf{0},r)$ might not be open in X. This would happen if Y has empty interior in X for example. However, $S(\mathbf{0},r)$ and $D(\mathbf{0},r)$ are compact.

Lemma 7.4.3 *Let Y be a closed subset of X a finite dimensional normed vector space and let $\mathcal{S}$ be a σ algebra of sets of Y containing the open sets of Y. Suppose μ is a measure defined on $\mathcal{S}$ and suppose also $\mu(K) < \infty$ whenever K is compact. Then if 7.4 holds, so does 7.3.*

Proof: It is desired to show that in this setting outer regularity implies inner regularity. First suppose $F \subseteq D(\mathbf{0},n)$ where $n \in \mathbb{N}$ and $F \in \mathcal{S}$. The following diagram will help to follow the technicalities. In this picture, V is the material between the two dotted curves, F is the inside of the solid curve and $D(\mathbf{0},n)$ is inside the larger solid curve.



The idea is to use outer regularity on $D(\mathbf{0},n) \setminus F$ to come up with V approximating this set as suggested in the picture. Then $V^C \cap D(\mathbf{0},n)$ is a compact set contained in F which approximates F. On the picture, the error is represented by the material between the small dotted curve and the smaller solid curve which is less than the error between V and $D(\mathbf{0},n) \setminus F$ as indicated by the picture. If you need the details, they follow. Otherwise the rest of the proof starts at ♣

Taking complements with respect to Y

$$D(\mathbf{0},n) \setminus F = D(\mathbf{0},n) \cap F^C = \left(D(\mathbf{0},n)^C \cup F\right)^C \in \mathcal{S}$$

because it is given that $\mathcal{S}$ contains the open sets. By 7.4 there exists an open set, $V \supseteq D(\mathbf{0},n) \setminus F$ such that

$$\mu(D(\mathbf{0},n) \setminus F) + \varepsilon > \mu(V). \tag{7.5}$$

Since μ is a measure,

$$\mu(V \setminus (D(\mathbf{0},n) \setminus F)) + \mu(D(\mathbf{0},n) \setminus F) = \mu(V)$$

and so from 7.5

$$\mu(V \setminus (D(\mathbf{0},n) \setminus F)) < \varepsilon \tag{7.6}$$

Note

$$V \setminus (D(\mathbf{0},n) \setminus F) = V \cap \left(D(\mathbf{0},n) \cap F^C\right)^C = \left(V \cap D(\mathbf{0},n)^C\right) \cup (V \cap F)$$

and by 7.6,

$$\mu(V \setminus (D(\mathbf{0},n) \setminus F)) < \varepsilon$$

so in particular,

$$\mu(V \cap F) < \varepsilon.$$

Now

$$V \supseteq D(\mathbf{0},n) \cap F^C$$

and so

$$V^C \subseteq D(\mathbf{0},n)^C \cup F$$

which implies

$$V^C \cap D(\mathbf{0},n) \subseteq F \cap D(\mathbf{0},n) = F$$

Since $F \subseteq D(\mathbf{0},n)$,

$$\begin{aligned} \mu\left(F \setminus \left(V^C \cap D(\mathbf{0},n)\right)\right) &= \mu\left(F \cap \left(V^C \cap D(\mathbf{0},n)\right)^C\right) \\ &= \mu\left((F \cap V) \cup \left(F \cap D(\mathbf{0},n)^C\right)\right) \\ &= \mu(F \cap V) < \varepsilon \end{aligned}$$

showing the compact set, $V^C \cap D(\mathbf{0},n)$ is contained in F and

$$\mu\left(V^C \cap D(\mathbf{0},n)\right) + \varepsilon > \mu(F).$$

♣In the general case where F is only given to be in $\mathcal{S}$, let $F_n = B(\mathbf{0},n) \cap F$. Then by 7.1, if $l < \mu(F)$ is given, then for all ε sufficiently small,

$$l + \varepsilon < \mu(F_n)$$

provided n is large enough. Now it was just shown there exists K a compact subset of F_n such that $\mu(F_n) < \mu(K) + \varepsilon$. Then $K \subseteq F$ and

$$l + \varepsilon < \mu(F_n) < \mu(K) + \varepsilon$$

and so whenever $l < \mu(F)$, it follows there exists K a compact subset of F such that

$$l < \mu(K)$$

and this proves the lemma.

The following is a useful result which will be used in what follows.

Lemma 7.4.4 *Let X be a normed vector space and let S be any nonempty subset of X. Define*

$$\text{dist}(\mathbf{x}, S) \equiv \inf\{||\mathbf{x}-\mathbf{y}|| : \mathbf{y} \in S\}$$

Then

$$|\text{dist}(\mathbf{x}_1, S) - \text{dist}(\mathbf{x}_2, S)| \leq ||\mathbf{x}_1 - \mathbf{x}_2||.$$

Proof: Suppose $\text{dist}(\mathbf{x}_1, S) \geq \text{dist}(\mathbf{x}_2, S)$. Then let $\mathbf{y} \in S$ such that

$$\text{dist}(\mathbf{x}_2, S) + \varepsilon > ||\mathbf{x}_2 - \mathbf{y}||$$

Then

$$\begin{aligned} |\text{dist}(\mathbf{x}_1, S) - \text{dist}(\mathbf{x}_2, S)| &= \text{dist}(\mathbf{x}_1, S) - \text{dist}(\mathbf{x}_2, S) \\ &\leq \text{dist}(\mathbf{x}_1, S) - (||\mathbf{x}_2 - \mathbf{y}|| - \varepsilon) \\ &\leq ||\mathbf{x}_1 - \mathbf{y}|| - ||\mathbf{x}_2 - \mathbf{y}|| + \varepsilon \\ &\leq |||\mathbf{x}_1 - \mathbf{y}|| - ||\mathbf{x}_2 - \mathbf{y}||| + \varepsilon \\ &\leq ||\mathbf{x}_1 - \mathbf{x}_2|| + \varepsilon. \end{aligned}$$

Since ε is arbitrary, this proves the lemma in case $\text{dist}(\mathbf{x}_1, S) \geq \text{dist}(\mathbf{x}_2, S)$. The case where $\text{dist}(\mathbf{x}_2, S) \geq \text{dist}(\mathbf{x}_1, S)$ is entirely similar. This proves the lemma.

The next lemma says that regularity comes free for finite measures defined on the Borel sets. Actually, it only almost says this. The following theorem will say it. This lemma deals with closed in place of compact.

Lemma 7.4.5 *Let μ be a finite measure defined on $\mathcal{B}(Y)$ where Y is a closed subset of X, a finite dimensional normed vector space. Then for every $F \in \mathcal{B}(Y)$,*

$$\mu(F) = \sup\{\mu(K) : K \subseteq F,\ K \text{ is closed}\}$$

$$\mu(F) = \inf\{\mu(V) : V \supseteq F,\ V \text{ is open}\}$$

Proof: For convenience, I will call a measure which satisfies the above two conditions "almost regular". It would be regular if closed were replaced with compact. First note every open set is the countable union of closed sets and every closed set is the countable intersection of open sets. Here is why. Let V be an open set and let

$$K_k \equiv \left\{\mathbf{x} \in V : \text{dist}\left(\mathbf{x}, V^C\right) \geq 1/k\right\}.$$

Then clearly the union of the K_k equals V and each is closed because $x \to \text{dist}(\mathbf{x}, S)$ is always a continuous function whenever S is any nonempty set. Next, for K closed let

$$V_k \equiv \{\mathbf{x} \in Y : \text{dist}(\mathbf{x}, K) < 1/k\}.$$

Clearly the intersection of the V_k equals K because if $\mathbf{x} \notin K$, then since K is closed, $B(\mathbf{x}, r)$ has empty intersection with K and so for k large enough that $1/k < r$, V_k excludes $\mathbf{x}$. Thus the only points in the intersection of the V_k are those in K and in addition each point of K is in this intersection.

Therefore from what was just shown, letting V denote an open set and K a closed set, it follows from Theorem 7.3.2 that

$$\begin{aligned} \mu(V) &= \sup\{\mu(K) : K \subseteq V \text{ and } K \text{ is closed}\} \\ \mu(K) &= \inf\{\mu(V) : V \supseteq K \text{ and } V \text{ is open}\}. \end{aligned}$$

Also since V is open and K is closed,

$$\begin{aligned}\mu(V) &= \inf\{\mu(U) : U \supseteq V \text{ and } V \text{ is open}\}\\ \mu(K) &= \sup\{\mu(L) : L \subseteq K \text{ and } L \text{ is closed}\}\end{aligned}$$

In words, μ is almost regular on open and closed sets. Let

$$\mathcal{F} \equiv \{F \in \mathcal{B}(Y) \text{ such that } \mu \text{ is almost regular on } F\}.$$

Then $\mathcal{F}$ contains the open sets. I want to show $\mathcal{F}$ is a σ algebra and then it will follow $\mathcal{F} = \mathcal{B}(Y)$.

First I will show $\mathcal{F}$ is closed with respect to complements. Let $F \in \mathcal{F}$. Then since μ is finite and F is inner regular, there exists $K \subseteq F$ such that

$$\mu(F \setminus K) = \mu(F) - \mu(K) < \varepsilon.$$

But $K^C \setminus F^C = F \setminus K$ and so

$$\mu\left(K^C \setminus F^C\right) = \mu\left(K^C\right) - \mu\left(F^C\right) < \varepsilon$$

showing that μ is outer regular on F^C. I have just approximated the measure of F^C with the measure of K^C, an open set containing F^C. A similar argument works to show F^C is inner regular. You start with $V \supseteq F$ such that $\mu(V \setminus F) < \varepsilon$, note $F^C \setminus V^C = V \setminus F$, and then conclude $\mu\left(F^C \setminus V^C\right) < \varepsilon$, thus approximating F^C with the closed subset, V^C.

Next I will show $\mathcal{F}$ is closed with respect to taking countable unions. Let $\{F_k\}$ be a sequence of sets in $\mathcal{F}$. Then since $F_k \in \mathcal{F}$, there exist $\{K_k\}$ such that $K_k \subseteq F_k$ and $\mu(F_k \setminus K_k) < \varepsilon/2^{k+1}$. First choose m large enough that

$$\mu\left(\left(\cup_{k=1}^{\infty} F_k\right) \setminus \left(\cup_{k=1}^{m} F_k\right)\right) < \frac{\varepsilon}{2}.$$

Then

$$\begin{aligned}\mu\left(\left(\cup_{k=1}^{m} F_k\right) \setminus \left(\cup_{k=1}^{m} K_k\right)\right) &\leq \mu\left(\cup_{k=1}^{m}\left(F_k \setminus K_k\right)\right)\\ &\leq \sum_{k=1}^{m} \frac{\varepsilon}{2^{k+1}} < \frac{\varepsilon}{2}\end{aligned}$$

and so

$$\begin{aligned}\mu\left(\left(\cup_{k=1}^{\infty} F_k\right) \setminus \left(\cup_{k=1}^{m} K_k\right)\right) &\leq \mu\left(\left(\cup_{k=1}^{\infty} F_k\right) \setminus \left(\cup_{k=1}^{m} F_k\right)\right)\\ &\quad + \mu\left(\left(\cup_{k=1}^{m} F_k\right) \setminus \left(\cup_{k=1}^{m} K_k\right)\right)\\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon\end{aligned}$$

Since μ is outer regular on F_k, there exists V_k such that $\mu(V_k \setminus F_k) < \varepsilon/2^k$. Then

$$\begin{aligned}\mu\left(\left(\cup_{k=1}^{\infty} V_k\right) \setminus \left(\cup_{k=1}^{\infty} F_k\right)\right) &\leq \mu\left(\cup_{k=1}^{\infty}\left(V_k \setminus F_k\right)\right)\\ &\leq \sum_{k=1}^{\infty} \mu\left(V_k \setminus F_k\right)\\ &< \sum_{k=1}^{\infty} \frac{\varepsilon}{2^k} = \varepsilon\end{aligned}$$

and this completes the demonstration that $\mathcal{F}$ is a σ algebra. This proves the lemma.

The next theorem is the main result. It shows regularity is automatic if $\mu(K) < \infty$ for all compact K.

Theorem 7.4.6 *Let μ be a finite measure defined on $\mathcal{B}(Y)$ where Y is a closed subset of X, a finite dimensional normed vector space. Then μ is regular. If μ is not necessarily finite but is finite on compact sets, then μ is regular.*

Proof: From Lemma 7.4.5 μ is outer regular. Now let $F \in \mathcal{B}(Y)$. Then since μ is finite, it follows from Lemma 7.4.5 there exists $H \subseteq F$ such that H is closed, $H \subseteq F$, and

$$\mu(F) < \mu(H) + \varepsilon.$$

Then let $K_k \equiv H \cap \overline{B(\mathbf{0},k)}$. Thus K_k is a closed and bounded, hence compact set and $\cup_{k=1}^{\infty} K_k = H$. Therefore by Theorem 7.3.2, for all k large enough,

$$\begin{aligned} & \mu(F) \\ < \ & \mu(K_k) + \varepsilon \\ < \ & \sup\{\mu(K) : K \subseteq F \text{ and } K \text{ compact}\} + \varepsilon \\ \leq \ & \mu(F) + \varepsilon \end{aligned}$$

Since ε was arbitrary, it follows

$$\sup\{\mu(K) : K \subseteq F \text{ and } K \text{ compact}\} = \mu(F).$$

This establishes μ is regular if μ is finite.

Now suppose it is only known that μ is finite on compact sets. Consider outer regularity. There are at most finitely many $r \in [0,R]$ such that $\mu(S(\mathbf{0},r)) > \delta > 0$. If this were not so, then $\mu(D(\mathbf{0},R)) = \infty$ contrary to the assumption that μ is finite on compact sets. Therefore, there are at most countably many $r \in [0,R]$ such that $\mu(S(\mathbf{0},r)) > 0$. Here is why. Let S_k denote those values of $r \in [0,R]$ such that $\mu(S(\mathbf{0},r)) > 1/k$. Then the values of r such that $\mu(S(\mathbf{0},r)) > 0$ equals $\cup_{m=1}^{\infty} S_m$, a countable union of finite sets which is at most countable.

It follows there are at most countably many $r \in (0,\infty)$ such that $\mu(S(\mathbf{0},r)) > 0$. Therefore, there exists an increasing sequence $\{r_k\}$ such that $\lim_{k\to\infty} r_k = \infty$ and $\mu(S(\mathbf{0},r_k)) = 0$. This is easy to see by noting that $(n, n+1]$ contains uncountably many points and so it contains at least one r such that $\mu(S(\mathbf{0},r)) = 0$.

$$S(\mathbf{0},r) = \cap_{k=1}^{\infty} (B(\mathbf{0}, r+1/k) - D(\mathbf{0}, r-1/k))$$

a countable intersection of open sets which are decreasing as $k \to \infty$. Since $\mu(B(\mathbf{0},r)) < \infty$ by assumption, it follows from Theorem 7.3.2 that for each r_k there exists an open set, $U_k \supseteq S(\mathbf{0},r_k)$ such that

$$\mu(U_k) < \varepsilon/2^{k+1}.$$

Let $\mu(F) < \infty$. There is nothing to show if $\mu(F) = \infty$. Define finite measures, μ_k as follows.

$$\begin{aligned} \mu_1(A) &\equiv \mu(B(\mathbf{0},1) \cap A), \\ \mu_2(A) &\equiv \mu((B(\mathbf{0},2) \setminus D(\mathbf{0},1)) \cap A), \\ \mu_3(A) &\equiv \mu((B(\mathbf{0},3) \setminus D(\mathbf{0},2)) \cap A) \end{aligned}$$

etc. Thus

$$\mu(A) = \sum_{k=1}^{\infty} \mu_k(A)$$

and each μ_k is a finite measure. By the first part there exists an open set V_k such that

$$V_k \supseteq F \cap (B(\mathbf{0},k) \setminus D(\mathbf{0},k-1))$$

and

$$\mu_k(V_k) < \mu_k(F) + \varepsilon/2^{k+1}$$

Without loss of generality $V_k \subseteq (B(\mathbf{0},k) \setminus D(\mathbf{0},k-1))$ since you can take the intersection of V_k with this open set. Thus

$$\mu_k(V_k) = \mu((B(\mathbf{0},k) \setminus D(\mathbf{0},k-1)) \cap V_k) = \mu(V_k)$$

and the V_k are disjoint. Then let $V = \cup_{k=1}^{\infty} V_k$ and $U = \cup_{k=1}^{\infty} U_k$. It follows $V \cup U$ is an open set containing F and

$$\begin{aligned} \mu(F) &= \sum_{k=1}^{\infty} \mu_k(F) > \sum_{k=1}^{\infty} \mu_k(V_k) - \frac{\varepsilon}{2^{k+1}} = \sum_{k=1}^{\infty} \mu(V_k) - \frac{\varepsilon}{2} \\ &= \mu(V) - \frac{\varepsilon}{2} \geq \mu(V) + \mu(U) - \frac{\varepsilon}{2} - \frac{\varepsilon}{2} \geq \mu(V \cup U) - \varepsilon \end{aligned}$$

which shows μ is outer regular. Inner regularity can be obtained from Lemma 7.4.3. Alternatively, you can use the above construction to get it right away. It is easier than the outer regularity.

First assume $\mu(F) < \infty$. By the first part, there exists a compact set,

$$K_k \subseteq F \cap (B(\mathbf{0},k) \setminus D(\mathbf{0},k-1))$$

such that

$$\begin{aligned} \mu_k(K_k) + \varepsilon/2^{k+1} &> \mu_k(F \cap (B(\mathbf{0},k) \setminus D(\mathbf{0},k-1))) \\ &= \mu_k(F) = \mu(F \cap (B(\mathbf{0},k) \setminus D(\mathbf{0},k-1))). \end{aligned}$$

Since K_k is a subset of $F \cap (B(\mathbf{0},k) \setminus D(\mathbf{0},k-1))$ it follows $\mu_k(K_k) = \mu(K_k)$. Therefore,

$$\begin{aligned} \mu(F) &= \sum_{k=1}^{\infty} \mu_k(F) < \sum_{k=1}^{\infty} \mu_k(K_k) + \varepsilon/2^k \\ &< \left(\sum_{k=1}^{\infty} \mu_k(K_k) \right) + \varepsilon/2 < \sum_{k=1}^{N} \mu(K_k) + \varepsilon \end{aligned}$$

provided N is large enough. The K_k are disjoint and so letting $K = \cup_{k=1}^{N} K_k$, this says $K \subseteq F$ and

$$\mu(F) < \mu(K) + \varepsilon.$$

Now consider the case where $\mu(F) = \infty$. If $l < \infty$, it follows from Theorem 7.3.2

$$\mu(F \cap B(\mathbf{0},m)) > l$$

whenever m is large enough. Therefore, letting $\mu_m(A) \equiv \mu(A \cap B(\mathbf{0},m))$, there exists a compact set, $K \subseteq F \cap B(\mathbf{0},m)$ such that

$$\mu(K) = \mu_m(K) > \mu_m(F \cap B(\mathbf{0},m)) = \mu(F \cap B(\mathbf{0},m)) > l$$

This proves the theorem.

7.5 Measures And Outer Measures

7.5.1 Measures From Outer Measures

Earlier an outer measure on $\mathcal{P}(\mathbb{R})$ was constructed. This can be used to obtain a measure defined on $\mathbb{R}$. However, the procedure for doing so is a special case of a general approach due to Caratheodory in about 1918.

Definition 7.5.1 *Let Ω be a nonempty set and let $\mu : \mathcal{P}(\Omega) \to [0,\infty]$ be an outer measure. For $E \subseteq \Omega$, E is μ measurable if for all $S \subseteq \Omega$,*

$$\mu(S) = \mu(S \setminus E) + \mu(S \cap E). \tag{7.7}$$

To help in remembering 7.7, think of a measurable set, E, as a process which divides a given set into two pieces, the part in E and the part not in E as in 7.7. In the Bible, there are several incidents recorded in which a process of division resulted in more stuff than was originally present.[2] Measurable sets are exactly those which are incapable of such a miracle. You might think of the measurable sets as the nonmiraculous sets. The idea is to show that they form a σ algebra on which the outer measure, μ is a measure.

First here is a definition and a lemma.

Definition 7.5.2 *$(\mu\lfloor S)(A) \equiv \mu(S \cap A)$ for all $A \subseteq \Omega$. Thus $\mu\lfloor S$ is the name of a new outer measure, called μ restricted to S.*

The next lemma indicates that the property of measurability is not lost by considering this restricted measure.

Lemma 7.5.3 *If A is μ measurable, then A is $\mu\lfloor S$ measurable.*

Proof: Suppose A is μ measurable. It is desired to to show that for all $T \subseteq \Omega$,

$$(\mu\lfloor S)(T) = (\mu\lfloor S)(T \cap A) + (\mu\lfloor S)(T \setminus A).$$

Thus it is desired to show

$$\mu(S \cap T) = \mu(T \cap A \cap S) + \mu(T \cap S \cap A^C). \tag{7.8}$$

But 7.8 holds because A is μ measurable. Apply Definition 7.5.1 to $S \cap T$ instead of S.

If A is $\mu\lfloor S$ measurable, it does not follow that A is μ measurable. Indeed, if you believe in the existence of non measurable sets, you could let $A = S$ for such a μ non measurable set and verify that S is $\mu\lfloor S$ measurable. In fact there do exist nonmeasurable sets but this is a topic for a more advanced course in analysis and will not be needed in this book.

The next theorem is the main result on outer measures which shows that starting with an outer measure you can obtain a measure.

Theorem 7.5.4 *Let Ω be a set and let μ be an outer measure on $\mathcal{P}(\Omega)$. The collection of μ measurable sets, $\mathcal{S}$, forms a σ algebra and*

$$\text{If } F_i \in \mathcal{S},\ F_i \cap F_j = \emptyset,\ \text{then } \mu(\cup_{i=1}^{\infty} F_i) = \sum_{i=1}^{\infty} \mu(F_i). \tag{7.9}$$

[2] 1 Kings 17, 2 Kings 4, Mathew 14, and Mathew 15 all contain such descriptions. The stuff involved was either oil, bread, flour or fish. In mathematics such things have also been done with sets. In the book by Bruckner Bruckner and Thompson there is an interesting discussion of the Banach Tarski paradox which says it is possible to divide a ball in $\mathbb{R}^3$ into five disjoint pieces and assemble the pieces to form two disjoint balls of the same size as the first. The details can be found in: The Banach Tarski Paradox by Wagon, Cambridge University press. 1985. It is known that all such examples must involve the axiom of choice.

If $\cdots F_n \subseteq F_{n+1} \subseteq \cdots$, then if $F = \cup_{n=1}^{\infty} F_n$ and $F_n \in \mathcal{S}$, it follows that

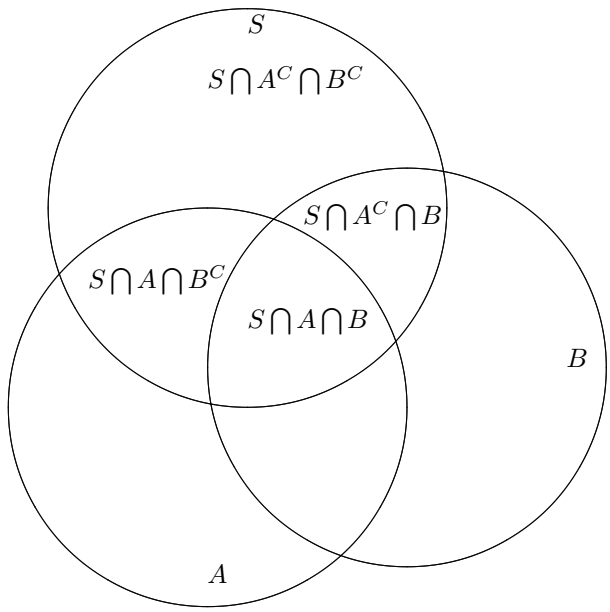
$$\mu(F) = \lim_{n\to\infty} \mu(F_n). \tag{7.10}$$

If $\cdots F_n \supseteq F_{n+1} \supseteq \cdots$, and if $F = \cap_{n=1}^{\infty} F_n$ for $F_n \in \mathcal{S}$ then if $\mu(F_1) < \infty$,

$$\mu(F) = \lim_{n\to\infty} \mu(F_n). \tag{7.11}$$

This measure space is also complete which means that if $\mu(F) = 0$ for some $F \in \mathcal{S}$ then if $G \subseteq F$, it follows $G \in \mathcal{S}$ also.

Proof: First note that $\emptyset$ and Ω are obviously in $\mathcal{S}$. Now suppose $A, B \in \mathcal{S}$. I will show $A \setminus B \equiv A \cap B^C$ is in $\mathcal{S}$. To do so, consider the following picture.



Since μ is subadditive,

$$\mu(S) \leq \mu\left(S \cap A \cap B^C\right) + \mu\left(A \cap B \cap S\right) + \mu\left(S \cap B \cap A^C\right) + \mu\left(S \cap A^C \cap B^C\right).$$

Now using $A, B \in \mathcal{S}$,

$$\begin{aligned} \mu(S) &\leq \mu\left(S \cap A \cap B^C\right) + \mu\left(S \cap A \cap B\right) + \mu\left(S \cap B \cap A^C\right) + \mu\left(S \cap A^C \cap B^C\right) \\ &= \mu\left(S \cap A\right) + \mu\left(S \cap A^C\right) = \mu(S) \end{aligned}$$

It follows equality holds in the above. Now observe, using the picture if you like, that

$$\left(A \cap B \cap S\right) \cup \left(S \cap B \cap A^C\right) \cup \left(S \cap A^C \cap B^C\right) = S \setminus \left(A \setminus B\right)$$

and therefore,

$$\begin{aligned} \mu(S) &= \mu\left(S \cap A \cap B^C\right) + \mu\left(A \cap B \cap S\right) + \mu\left(S \cap B \cap A^C\right) + \mu\left(S \cap A^C \cap B^C\right) \\ &\geq \mu\left(S \cap \left(A \setminus B\right)\right) + \mu\left(S \setminus \left(A \setminus B\right)\right). \end{aligned}$$

Therefore, since S is arbitrary, this shows $A \setminus B \in \mathcal{S}$.

Since $\Omega \in \mathcal{S}$, this shows that $A \in \mathcal{S}$ if and only if $A^C \in \mathcal{S}$. Now if $A, B \in \mathcal{S}$, $A \cup B = (A^C \cap B^C)^C = (A^C \setminus B)^C \in \mathcal{S}$. By induction, if $A_1, \cdots, A_n \in \mathcal{S}$, then so is $\cup_{i=1}^n A_i$. If $A, B \in \mathcal{S}$, with $A \cap B = \emptyset$,

$$\mu(A \cup B) = \mu((A \cup B) \cap A) + \mu((A \cup B) \setminus A) = \mu(A) + \mu(B).$$

By induction, if $A_i \cap A_j = \emptyset$ and $A_i \in \mathcal{S}$,

$$\mu(\cup_{i=1}^n A_i) = \sum_{i=1}^n \mu(A_i). \tag{7.12}$$

Now let $A = \cup_{i=1}^\infty A_i$ where $A_i \cap A_j = \emptyset$ for $i \neq j$.

$$\sum_{i=1}^\infty \mu(A_i) \geq \mu(A) \geq \mu(\cup_{i=1}^n A_i) = \sum_{i=1}^n \mu(A_i).$$

Since this holds for all n, you can take the limit as $n \to \infty$ and conclude,

$$\sum_{i=1}^\infty \mu(A_i) = \mu(A)$$

which establishes 7.9.

Consider part 7.10. Without loss of generality $\mu(F_k) < \infty$ for all k since otherwise there is nothing to show. Suppose $\{F_k\}$ is an increasing sequence of sets of $\mathcal{S}$. Then letting $F_0 \equiv \emptyset$, $\{F_{k+1} \setminus F_k\}_{k=0}^\infty$ is a sequence of disjoint sets of $\mathcal{S}$ since it was shown above that the difference of two sets of $\mathcal{S}$ is in $\mathcal{S}$. Also note that from 7.12

$$\mu(F_{k+1} \setminus F_k) + \mu(F_k) = \mu(F_{k+1})$$

and so if $\mu(F_k) < \infty$, then

$$\mu(F_{k+1} \setminus F_k) = \mu(F_{k+1}) - \mu(F_k).$$

Therefore, letting

$$F \equiv \cup_{k=1}^\infty F_k$$

which also equals

$$\cup_{k=1}^\infty (F_{k+1} \setminus F_k),$$

it follows from part 7.9 just shown that

$$\begin{aligned} \mu(F) &= \sum_{k=1}^\infty \mu(F_{k+1} \setminus F_k) = \lim_{n\to\infty} \sum_{k=1}^n \mu(F_{k+1} \setminus F_k) \\ &= \lim_{n\to\infty} \sum_{k=1}^n \mu(F_{k+1}) - \mu(F_k) = \lim_{n\to\infty} \mu(F_{n+1}). \end{aligned}$$

In order to establish 7.11, let the F_n be as given there. Then, since $(F_1 \setminus F_n)$ increases to $(F_1 \setminus F)$, 7.10 implies

$$\lim_{n\to\infty} (\mu(F_1) - \mu(F_n)) = \mu(F_1 \setminus F).$$

Now $\mu(F_1 \setminus F) + \mu(F) \geq \mu(F_1)$ and so $\mu(F_1 \setminus F) \geq \mu(F_1) - \mu(F)$. Hence

$$\lim_{n\to\infty} (\mu(F_1) - \mu(F_n)) = \mu(F_1 \setminus F) \geq \mu(F_1) - \mu(F)$$

which implies

$$\lim_{n\to\infty}\mu(F_n)\leq\mu(F).$$

But since $F\subseteq F_n$,

$$\mu(F)\leq\lim_{n\to\infty}\mu(F_n)$$

and this establishes 7.11. Note that it was assumed $\mu(F_1)<\infty$ because $\mu(F_1)$ was subtracted from both sides.

It remains to show $\mathcal{S}$ is closed under countable unions. Recall that if $A\in\mathcal{S}$, then $A^C\in\mathcal{S}$ and $\mathcal{S}$ is closed under finite unions. Let $A_i\in\mathcal{S}$, $A=\cup_{i=1}^{\infty}A_i$, $B_n=\cup_{i=1}^{n}A_i$. Then

$$\begin{aligned}\mu(S) &= \mu(S\cap B_n)+\mu(S\setminus B_n) \\ &= (\mu\lfloor S)(B_n)+(\mu\lfloor S)(B_n^C).\end{aligned} \tag{7.13}$$

By Lemma 7.5.3 B_n is $(\mu\lfloor S)$ measurable and so is B_n^C. I want to show $\mu(S)\geq\mu(S\setminus A)+\mu(S\cap A)$. If $\mu(S)=\infty$, there is nothing to prove. Assume $\mu(S)<\infty$. Then apply Parts 7.11 and 7.10 to the outer measure, $\mu\lfloor S$ in 7.13 and let $n\to\infty$. Thus

$$B_n\uparrow A,\ B_n^C\downarrow A^C$$

and this yields $\mu(S)=(\mu\lfloor S)(A)+(\mu\lfloor S)(A^C)=\mu(S\cap A)+\mu(S\setminus A)$.

Therefore $A\in\mathcal{S}$ and this proves Parts 7.9, 7.10, and 7.11.

It only remains to verify the assertion about completeness. Letting G and F be as described above, let $S\subseteq\Omega$. I need to verify

$$\mu(S)\geq\mu(S\cap G)+\mu(S\setminus G)$$

However,

$$\begin{aligned}\mu(S\cap G)+\mu(S\setminus G) &\leq \mu(S\cap F)+\mu(S\setminus F)+\mu(F\setminus G) \\ &= \mu(S\cap F)+\mu(S\setminus F)=\mu(S)\end{aligned}$$

because by assumption, $\mu(F\setminus G)\leq\mu(F)=0$.This proves the theorem.

7.5.2 Completion Of Measure Spaces

Suppose $(\Omega,\mathcal{F},\mu)$ is a measure space. Then it is always possible to enlarge the σ algebra and define a new measure $\overline{\mu}$ on this larger σ algebra such that $\left(\Omega,\overline{\mathcal{F}},\overline{\mu}\right)$ is a complete measure space. Recall this means that if $N\subseteq N'\in\overline{\mathcal{F}}$ and $\overline{\mu}(N')=0$, then $N\in\overline{\mathcal{F}}$. The following theorem is the main result. The new measure space is called the completion of the measure space.

Definition 7.5.5 *A measure space, $(\Omega,\mathcal{F},\mu)$ is called σ finite if there exists a sequence $\{\Omega_n\}\subseteq\mathcal{F}$ such that $\cup_n\Omega_n=\Omega$ and $\mu(\Omega_n)<\infty$.*

For example, if X is a finite dimensional normed vector space and μ is a measure defined on $\mathcal{B}(X)$ which is finite on compact sets, then you could take $\Omega_n=B(\mathbf{0},n)$.

Theorem 7.5.6 *Let $(\Omega,\mathcal{F},\mu)$ be a σ finite measure space. Then there exists a unique measure space, $\left(\Omega,\overline{\mathcal{F}},\overline{\mu}\right)$ satisfying*

1. *$\left(\Omega,\overline{\mathcal{F}},\overline{\mu}\right)$ is a complete measure space.*
2. *$\overline{\mu}=\mu$ on $\mathcal{F}$*

3. $\overline{\mathcal{F}} \supseteq \mathcal{F}$

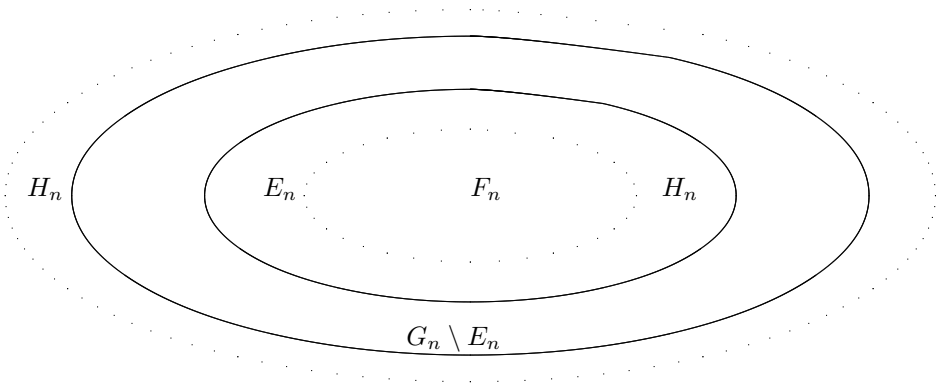
4. *For every* $E \in \overline{\mathcal{F}}$ *there exists* $G \in \mathcal{F}$ *such that* $G \supseteq E$ *and* $\mu(G) = \overline{\mu}(E)$. *In addition to this,*

5. *For every* $E \in \overline{\mathcal{F}}$ *there exists* $F \in \mathcal{F}$ *such that* $F \subseteq E$ *and* $\mu(F) = \overline{\mu}(E)$.

Also for every $E \in \overline{\mathcal{F}}$ *there exist sets* $G, F \in \mathcal{F}$ *such that* $G \supseteq E \supseteq F$ *and*

$$\mu(G \setminus F) = \overline{\mu}(G \setminus F) = 0 \tag{7.14}$$

Proof: First consider the claim about uniqueness. Suppose $(\Omega, \mathcal{F}_1, \nu_1)$ and $(\Omega, \mathcal{F}_2, \nu_2)$ both satisfy 1.) - 4.) and let $E \in \mathcal{F}_1$. Also let $\mu(\Omega_n) < \infty$, $\cdots \Omega_n \subseteq \Omega_{n+1} \cdots$, and $\cup_{n=1}^{\infty} \Omega_n = \Omega$. Define $E_n \equiv E \cap \Omega_n$. Then there exists $G_n \supseteq E_n$ such that $\mu(G_n) = \nu_1(E_n), G_n \in \mathcal{F}$ and $G_n \subseteq \Omega_n$. I claim there exists $F_n \in \mathcal{F}$ such that $G_n \supseteq E_n \supseteq F_n$ and $\mu(G_n \setminus F_n) = 0$. To see this, look at the following diagram.



In this diagram, there exists $H_n \in \mathcal{F}$ containing $G_n \setminus E_n$, represented in the picture as the set between the dotted lines, such that $\mu(H_n) = \overline{\mu}(G_n \setminus E_n)$. Then define $F_n \equiv H_n^C \cap G_n$. This set is in $\mathcal{F}$, is contained in E_n and as shown in the diagram,

$$\overline{\mu}(E_n) - \mu(F_n) \leq \mu(H_n) - \overline{\mu}(G_n \setminus E_n) = 0.$$

Therefore, since $\overline{\mu}$ is a measure,

$$\begin{aligned} \mu(G_n \setminus F_n) &= \overline{\mu}(G_n \setminus E_n) + \overline{\mu}(E_n \setminus F_n) \\ &= \mu(G_n) - \overline{\mu}(E_n) + \overline{\mu}(E_n) - \mu(F_n) = 0 \end{aligned}$$

Then letting $G = \cup_n G_n, F \equiv \cup_n F_n$, it follows $G \supseteq E \supseteq F$ and

$$\begin{aligned} \mu(G \setminus F) &\leq \mu(\cup_n (G_n \setminus F_n)) \\ &\leq \sum_n \mu(G_n \setminus F_n) = 0. \end{aligned}$$

Thus $\nu_i(G \setminus F) = 0$ for $i = 1, 2$. Now $E \setminus F \subseteq G \setminus F$ and since $(\Omega, \mathcal{F}_2, \nu_2)$ is complete, it follows $E \setminus F \in \mathcal{F}_2$. Since $F \in \mathcal{F}_2$, it follows $E = (E \setminus F) \cup F \in \mathcal{F}_2$. Thus $\mathcal{F}_1 \subseteq \mathcal{F}_2$. Similarly $\mathcal{F}_2 \subseteq \mathcal{F}_1$.

Now it only remains to verify $\nu_1 = \nu_2$. Thus let $E \in \mathcal{F}_1 = \mathcal{F}_2$ and let G and F be as just described. Since $\nu_i = \mu$ on $\mathcal{F}$,

$$\begin{aligned} \mu(F) &\leq \nu_1(E) \\ &= \nu_1(E \setminus F) + \nu_1(F) \\ &\leq \nu_1(G \setminus F) + \nu_1(F) \\ &= \nu_1(F) = \mu(F) \end{aligned}$$

Similarly $\nu_2(E) = \mu(F)$. This proves uniqueness. The construction has also verified 7.14.

Next define an outer measure, $\overline{\mu}$ on $\mathcal{P}(\Omega)$ as follows. For $S \subseteq \Omega$,

$$\overline{\mu}(S) \equiv \inf\{\mu(E) : E \in \mathcal{F}\}.$$

Then it is clear $\overline{\mu}$ is increasing. It only remains to verify $\overline{\mu}$ is subadditive. Then let $S = \cup_{i=1}^{\infty} S_i$. If any $\overline{\mu}(S_i) = \infty$, there is nothing to prove so suppose $\overline{\mu}(S_i) < \infty$ for each i. Then there exist $E_i \in \mathcal{F}$ such that $E_i \supseteq S_i$ and

$$\overline{\mu}(S_i) + \varepsilon/2^i > \mu(E_i).$$

Then

$$\begin{aligned} \overline{\mu}(S) &= \overline{\mu}(\cup_i S_i) \\ &\leq \mu(\cup_i E_i) \leq \sum_i \mu(E_i) \\ &\leq \sum_i \left(\overline{\mu}(S_i) + \varepsilon/2^i\right) = \sum_i \overline{\mu}(S_i) + \varepsilon. \end{aligned}$$

Since ε is arbitrary, this verifies $\overline{\mu}$ is subadditive and is an outer measure as claimed.

Denote by $\overline{\mathcal{F}}$ the σ algebra of measurable sets in the sense of Caratheodory. Then it follows from the Caratheodory procedure, Theorem 7.5.4, that $\left(\Omega, \overline{\mathcal{F}}, \overline{\mu}\right)$ is a complete measure space. This verifies 1.

Now let $E \in \mathcal{F}$. Then from the definition of $\overline{\mu}$, it follows

$$\overline{\mu}(E) \equiv \inf\{\mu(F) : F \in \mathcal{F} \text{ and } F \supseteq E\} \leq \mu(E).$$

If $F \supseteq E$ and $F \in \mathcal{F}$, then $\mu(F) \geq \mu(E)$ and so $\mu(E)$ is a lower bound for all such $\mu(F)$ which shows that

$$\overline{\mu}(E) \equiv \inf\{\mu(F) : F \in \mathcal{F} \text{ and } F \supseteq E\} \geq \mu(E).$$

This verifies 2.

Next consider 3. Let $E \in \mathcal{F}$ and let S be a set. I must show

$$\overline{\mu}(S) \geq \overline{\mu}(S \setminus E) + \overline{\mu}(S \cap E).$$

If $\overline{\mu}(S) = \infty$ there is nothing to show. Therefore, suppose $\overline{\mu}(S) < \infty$. Then from the definition of $\overline{\mu}$ there exists $G \supseteq S$ such that $G \in \mathcal{F}$ and $\mu(G) = \overline{\mu}(S)$. Then from the definition of $\overline{\mu}$,

$$\begin{aligned} \overline{\mu}(S) &\leq \overline{\mu}(S \setminus E) + \overline{\mu}(S \cap E) \\ &\leq \mu(G \setminus E) + \mu(G \cap E) \\ &= \mu(G) = \overline{\mu}(S) \end{aligned}$$

This verifies 3.

Claim 4 comes by the definition of $\overline{\mu}$ as used above. The other case is when $\overline{\mu}(S) = \infty$. However, in this case, you can let $G = \Omega$.

It only remains to verify 5. Let the Ω_n be as described above and let $E \in \overline{\mathcal{F}}$ such that $E \subseteq \Omega_n$. By 4 there exists $H \in \mathcal{F}$ such that $H \subseteq \Omega_n$, $H \supseteq \Omega_n \setminus E$, and

$$\mu(H) = \overline{\mu}(\Omega_n \setminus E). \tag{7.15}$$

Then let $F \equiv \Omega_n \cap H^C$. It follows $F \subseteq E$ and

$$\begin{aligned} E \setminus F &= E \cap F^C = E \cap \left(H \cup \Omega_n^C\right) \\ &= E \cap H = H \setminus (\Omega_n \setminus E) \end{aligned}$$

Hence from 7.15

$$\overline{\mu}(E \setminus F) = \overline{\mu}(H \setminus (\Omega_n \setminus E)) = 0.$$

It follows

$$\overline{\mu}(E) = \overline{\mu}(F) = \mu(F).$$

In the case where $E \in \overline{\mathcal{F}}$ is arbitrary, not necessarily contained in some Ω_n, it follows from what was just shown that there exists $F_n \in \mathcal{F}$ such that $F_n \subseteq E \cap \Omega_n$ and

$$\mu(F_n) = \overline{\mu}(E \cap \Omega_n).$$

Letting $F \equiv \cup_n F_n$

$$\overline{\mu}(E \setminus F) \leq \overline{\mu}(\cup_n (E \cap \Omega_n \setminus F_n)) \leq \sum_n \overline{\mu}(E \cap \Omega_n \setminus F_n) = 0.$$

Therefore, $\overline{\mu}(E) = \mu(F)$ and this proves 5. This proves the theorem.

Here is another observation about regularity which follows from the above theorem.

Theorem 7.5.7 *Suppose μ is a regular measure defined on $\mathcal{B}(X)$ where X is a finite dimensional normed vector space. Then denoting by $\left(X, \overline{\mathcal{B}(X)}, \overline{\mu}\right)$ the completion of $(X, \mathcal{B}(X), \mu)$, it follows $\overline{\mu}$ is also regular. Furthermore, if a σ algebra, $\mathcal{F} \supseteq \mathcal{B}(X)$ and $(X, \mathcal{F}, \mu)$ is a complete measure space such that for every $F \in \mathcal{F}$ there exists $G \in \mathcal{B}(X)$ such that $\mu(F) = \mu(G)$ and $G \supseteq F$, then $\mathcal{F} = \overline{\mathcal{B}(X)}$ and $\mu = \overline{\mu}$.*

Proof: Let $F \in \overline{\mathcal{B}(X)}$ with $\overline{\mu}(F) < \infty$. By Theorem 7.5.6 there exists $G \in \mathcal{B}(X)$ such that

$$\overline{\mu}(G) = \mu(G) = \mu(F).$$

Now by regularity of μ there exists an open set, $V \supseteq G \supseteq F$ such that

$$\overline{\mu}(F) + \varepsilon = \mu(G) + \varepsilon > \mu(V) = \overline{\mu}(V)$$

Therefore, $\overline{\mu}$ is outer regular. If $\overline{\mu}(F) = \infty$, there is nothing to show.

Now take $F \in \overline{\mathcal{B}(X)}$. By Theorem 7.5.6 there exists $H \subseteq F$ with $H \in \mathcal{B}(X)$ and $\mu(H) = \overline{\mu}(F)$. If $l < \overline{\mu}(F) = \mu(H)$, it follows from regularity of μ there exists K a compact subset of H such that

$$l < \mu(K) = \overline{\mu}(K)$$

Thus $\overline{\mu}$ is also inner regular. The last assertion follows from the uniqueness part of Theorem 7.5.6 and this proves the theorem.

A repeat of the above argument yields the following corollary.

Corollary 7.5.8 *The conclusion of the above theorem holds for X replaced with Y where Y is a closed subset of X.*

7.6 One Dimensional Lebesgue Stieltjes Measure

Now with these major results about measures, it is time to specialize to the outer measure of Theorem 7.2.1. The next theorem gives Lebesgue Stieltjes measure on $\mathbb{R}$.

Theorem 7.6.1 *Let $\mathcal{S}$ denote the σ algebra of Theorem 7.5.4 applied to the outer measure μ in Theorem 7.2.1 on which μ is a measure. Then every open interval is in $\mathcal{S}$. So are all open and closed sets. Furthermore, if E is any set in $\mathcal{S}$*

$$\mu(E) = \sup\{\mu(K) : K \text{ is a closed and bounded set, } K \subseteq E\} \tag{7.16}$$

$$\mu(E) = \inf\{\mu(V) : V \text{ is an open set, } V \supseteq E\} \tag{7.17}$$

Proof: The first task is to show $(a,b) \in \mathcal{S}$. I need to show that for every $S \subseteq \mathbb{R}$,

$$\mu(S) \geq \mu(S \cap (a,b)) + \mu\left(S \cap (a,b)^C\right) \tag{7.18}$$

Suppose first S is an open interval, (c,d). If (c,d) has empty intersection with (a,b) or is contained in (a,b) there is nothing to prove. The above expression reduces to nothing more than $\mu(S) = \mu(S)$. Suppose next that $(c,d) \supseteq (a,b)$. In this case, the right side of the above reduces to

$$\begin{aligned} & \mu((a,b)) + \mu((c,a] \cup [b,d)) \\ \leq\ & F(b-) - F(a+) + F(a+) - F(c+) + F(d-) - F(b-) \\ =\ & F(d-) - F(c+) = \mu((c,d)) \end{aligned}$$

The only other cases are $c \leq a < d \leq b$ or $a \leq c < d \leq b$. Consider the first of these cases. Then the right side of 7.18 for $S = (c,d)$ is

$$\begin{aligned} \mu((a,d)) + \mu((c,a]) &= F(d-) - F(a+) + F(a+) - F(c+) \\ &= F(d-) - F(c+) = \mu((c,d)) \end{aligned}$$

The last case is entirely similar. Thus 7.18 holds whenever S is an open interval. Now it is clear 7.18 also holds if $\mu(S) = \infty$. Suppose then that $\mu(S) < \infty$ and let

$$S \subseteq \cup_{k=1}^{\infty}(a_k, b_k)$$

such that

$$\mu(S) + \varepsilon > \sum_{k=1}^{\infty}(F(b_k-) - F(a_k+)) = \sum_{k=1}^{\infty}\mu((a_k,b_k)).$$

Then since μ is an outer measure, and using what was just shown,

$$\begin{aligned} & \mu(S \cap (a,b)) + \mu\left(S \cap (a,b)^C\right) \\ \leq\ & \mu(\cup_{k=1}^{\infty}(a_k,b_k) \cap (a,b)) + \mu\left(\cup_{k=1}^{\infty}(a_k,b_k) \cap (a,b)^C\right) \\ \leq\ & \sum_{k=1}^{\infty}\mu((a_k,b_k) \cap (a,b)) + \mu\left((a_k,b_k) \cap (a,b)^C\right) \\ \leq\ & \sum_{k=1}^{\infty}\mu((a_k,b_k)) \leq \mu(S) + \varepsilon. \end{aligned}$$

Since ε is arbitrary, this shows 7.18 holds for any S and so any open interval is in S.

It follows any open set is in $\mathcal{S}$. This follows from Theorem 5.3.10 which implies that if U is open, it is the countable union of disjoint open intervals. Since each of these open intervals is in $\mathcal{S}$ and $\mathcal{S}$ is a σ algebra, their union is also in $\mathcal{S}$. It follows every closed set is in $\mathcal{S}$ also. This is because $\mathcal{S}$ is a σ algebra and if a set is in $\mathcal{S}$ then so is its complement. The closed sets are those which are complements of open sets.

Thus the σ algebra of μ measurable sets, $\mathcal{F}$ includes $\mathcal{B}(\mathbb{R})$. Consider the completion of the measure space, $(\mathbb{R}, \mathcal{B}(\mathbb{R}), \mu)$, $\left(\mathbb{R}, \overline{\mathcal{B}(\mathbb{R})}, \overline{\mu}\right)$. By the uniqueness assertion in Theorem 7.5.6 and the fact that $(\mathbb{R}, \mathcal{F}, \mu)$ is complete, this coincides with $(\mathbb{R}, \mathcal{F}, \mu)$ because the construction of μ implies μ is outer regular and for every $F \in \mathcal{F}$, there exists $G \in \mathcal{B}(\mathbb{R})$ containing F such that $\mu(F) = \mu(G)$. In fact, you can take G to equal a countable intersection of open sets. By Theorem 7.4.6 μ is regular on every set of $\mathcal{B}(\mathbb{R})$, this because μ is finite on compact sets. Therefore, by Theorem 7.5.7 $\mu = \overline{\mu}$ is regular on $\mathcal{F}$ which verifies the last two claims. This proves the theorem.

7.7 Measurable Functions

The integral will be defined on measurable functions which is the next topic considered. It is sometimes convenient to allow functions to take the value $+\infty$. You should think of $+\infty$, usually referred to as ∞ as something out at the right end of the real line and its only importance is the notion of sequences converging to it. $x_n \to \infty$ exactly when for all $l \in \mathbb{R}$, there exists N such that if $n \geq N$, then

$$x_n > l.$$

This is what it means for a sequence to converge to ∞. Don't think of ∞ as a number. It is just a convenient symbol which allows the consideration of some limit operations more simply. Similar considerations apply to $-\infty$ but this value is not of very great interest. In fact the set of most interest for the values of a function, f is the complex numbers or more generally some normed vector space.

Recall the notation,

$$f^{-1}(A) \equiv \{x : f(x) \in A\} \equiv [f(x) \in A]$$

in whatever context the notation occurs.

Lemma 7.7.1 *Let $f : \Omega \to (-\infty, \infty]$ where $\mathcal{F}$ is a σ algebra of subsets of Ω. Then the following are equivalent.*

$$f^{-1}((d, \infty]) \in \mathcal{F} \text{ for all finite } d,$$
$$f^{-1}((-\infty, d)) \in \mathcal{F} \text{ for all finite } d,$$
$$f^{-1}([d, \infty]) \in \mathcal{F} \text{ for all finite } d,$$
$$f^{-1}((-\infty, d]) \in \mathcal{F} \text{ for all finite } d,$$
$$f^{-1}((a, b)) \in \mathcal{F} \text{ for all } a < b, -\infty < a < b < \infty.$$

Proof: First note that the first and the third are equivalent. To see this, observe

$$f^{-1}([d, \infty]) = \cap_{n=1}^{\infty} f^{-1}((d - 1/n, \infty]),$$

and so if the first condition holds, then so does the third.

$$f^{-1}((d, \infty]) = \cup_{n=1}^{\infty} f^{-1}([d + 1/n, \infty]),$$

and so if the third condition holds, so does the first.

Similarly, the second and fourth conditions are equivalent. Now

$$f^{-1}((-\infty,d]) = (f^{-1}((d,\infty]))^C$$

so the first and fourth conditions are equivalent. Thus the first four conditions are equivalent and if any of them hold, then for $-\infty < a < b < \infty$,

$$f^{-1}((a,b)) = f^{-1}((-\infty,b)) \cap f^{-1}((a,\infty]) \in \mathcal{F}.$$

Finally, if the last condition holds,

$$f^{-1}([d,\infty]) = \left(\cup_{k=1}^{\infty} f^{-1}((-k+d,d))\right)^C \in \mathcal{F}$$

and so the third condition holds. Therefore, all five conditions are equivalent. This proves the lemma.

This lemma allows for the following definition of a measurable function having values in $(-\infty,\infty]$.

Definition 7.7.2 *Let $(\Omega,\mathcal{F},\mu)$ be a measure space and let $f:\Omega\to(-\infty,\infty]$. Then f is said to be $\mathcal{F}$ measurable if any of the equivalent conditions of Lemma 7.7.1 hold.*

Theorem 7.7.3 *Let f_n and f be functions mapping Ω to $(-\infty,\infty]$ where $\mathcal{F}$ is a σ algebra of measurable sets of Ω. Then if f_n is measurable, and $f(\omega)=\lim_{n\to\infty} f_n(\omega)$, it follows that f is also measurable. (Pointwise limits of measurable functions are measurable.)*

Proof: The idea is to show $f^{-1}((a,b)) \in \mathcal{F}$. Let $V_m \equiv \left(a+\frac{1}{m}, b-\frac{1}{m}\right)$ and $\overline{V}_m = \left[a+\frac{1}{m}, b-\frac{1}{m}\right]$. Then for all m, $V_m \subseteq (a,b)$ and

$$(a,b) = \cup_{m=1}^{\infty} V_m = \cup_{m=1}^{\infty} \overline{V}_m.$$

Note that $V_m \neq \emptyset$ for all m large enough. Since f is the pointwise limit of f_n,

$$f^{-1}(V_m) \subseteq \{\omega : f_k(\omega) \in V_m \text{ for all } k \text{ large enough}\} \subseteq f^{-1}(\overline{V}_m).$$

You should note that the expression in the middle is of the form

$$\cup_{n=1}^{\infty} \cap_{k=n}^{\infty} f_k^{-1}(V_m).$$

Therefore,

$$\begin{aligned} f^{-1}((a,b)) &= \cup_{m=1}^{\infty} f^{-1}(V_m) \subseteq \cup_{m=1}^{\infty} \cup_{n=1}^{\infty} \cap_{k=n}^{\infty} f_k^{-1}(V_m) \\ &\subseteq \cup_{m=1}^{\infty} f^{-1}(\overline{V}_m) = f^{-1}((a,b)). \end{aligned}$$

It follows $f^{-1}((a,b)) \in \mathcal{F}$ because it equals the expression in the middle which is measurable. This shows f is measurable.

Proposition 7.7.4 *Let $(\Omega,\mathcal{F},\mu)$ be a measure space and let $f:\Omega\to(-\infty,\infty]$. Then f is $\mathcal{F}$ measurable if and only if $f^{-1}(U) \in \mathcal{F}$ whenever U is an open set in $\mathbb{R}$.*

Proof: If $f^{-1}(U) \in \mathcal{F}$ whenever U is an open set in $\mathbb{R}$ then it follows from the last condition of Lemma 7.7.1 that f is measurable. Next suppose f is measurable so this last condition of Lemma 7.7.1 holds. Then by Theorem 5.3.10 if U is any open set in $\mathbb{R}$, it is the countable union of open intervals, $U = \cup_{k=1}^{\infty}(a_k,b_k)$. Hence

$$f^{-1}(U) = \cup_{k=1}^{\infty} f^{-1}((a_k,b_k)) \in \mathcal{F}$$

because $\mathcal{F}$ is a σ algebra.

From this proposition, it follows one can generalize the definition of a measurable function to those which have values in any normed vector space as follows.

Definition 7.7.5 *Let $(\Omega, \mathcal{F}, \mu)$ be a measure space and let $f : \Omega \to X$ where X is a normed vector space. Then f is measurable means $f^{-1}(U) \in \mathcal{F}$ whenever U is an open set in X.*

Now here is an important theorem which shows that you can do lots of things to measurable functions and still have a measurable function.

Theorem 7.7.6 *Let $(\Omega, \mathcal{F}, \mu)$ be a measure space and let X, Y be normed vector spaces and $\mathbf{g} : X \to Y$ continuous. Then if $\mathbf{f} : \Omega \to X$ is $\mathcal{F}$ measurable, it follows $\mathbf{g} \circ \mathbf{f}$ is also $\mathcal{F}$ measurable.*

Proof: From the definition, it suffices to show $(\mathbf{g} \circ \mathbf{f})^{-1}(U) \in \mathcal{F}$ whenever U is an open set in Y. However, since $\mathbf{g}$ is continuous, it follows $\mathbf{g}^{-1}(U)$ is open and so

$$(\mathbf{g} \circ \mathbf{f})^{-1}(U) = \mathbf{f}^{-1}\left(\mathbf{g}^{-1}(U)\right) = \mathbf{f}^{-1}(\text{an open set}) \in \mathcal{F}.$$

This proves the theorem.

This theorem implies for example that if $\mathbf{f}$ is a measurable X valued function, then $||\mathbf{f}||$ is a measurable $\mathbb{R}$ valued function. It also implies that if $\mathbf{f}$ is an X valued function, then if $\{\mathbf{v}_1, \cdots, \mathbf{v}_n\}$ is a basis for X and π_k is the projection onto the k^{th} component, then $\pi_k \circ \mathbf{f}$ is a measurable $\mathbb{F}$ valued function. Does it go the other way? That is, if it is known that $\pi_k \circ \mathbf{f}$ is measurable for each k, does it follow $\mathbf{f}$ is measurable? The following technical lemma is interesting for its own sake.

Lemma 7.7.7 *Let $||\mathbf{x}|| \equiv \max\{|x_i|, i = 1, 2, \cdots, n\}$ for $\mathbf{x} \in \mathbb{F}^n$. Then every set U which is open in $\mathbb{F}^n$ is the countable union of balls of the form $B(\mathbf{x}, r)$ where the open ball is defined in terms of the above norm.*

Proof: By Theorem 5.8.3 if you consider the two normed vector spaces $(\mathbb{F}^n, |\cdot|)$ and $(\mathbb{F}^n, ||\cdot||)$, the identity map is continuous in both directions. Therefore, if a set, U is open with respect to $|\cdot|$ it follows it is open with respect to $||\cdot||$ and the other way around. The other thing to notice is that there exists a countable dense subset of $\mathbb{F}$. The rationals will work if $\mathbb{F} = \mathbb{R}$ and if $\mathbb{F} = \mathbb{C}$, then you use $\mathbb{Q} + i\mathbb{Q}$. Letting D be a countable dense subset of $\mathbb{F}$, D^n is a countable dense subset of $\mathbb{F}^n$. It is countable because it is a finite Cartesian product of countable sets and you can use Theorem 2.1.7 of Page 13 repeatedly. It is dense because if $\mathbf{x} \in \mathbb{F}^n$, then by density of D, there exists $d_j \in D$ such that

$$|d_j - x_j| < \varepsilon$$

then $\mathbf{d} \equiv (d_1, \cdots, d_n)$ is such that $||\mathbf{d} - \mathbf{x}|| < \varepsilon$.

Now consider the set of open balls,

$$\mathcal{B} \equiv \{B(\mathbf{d}, r) : \mathbf{d} \in D^n, r \in \mathbb{Q}\}.$$

This collection of open balls is countable by Theorem 2.1.7 of Page 13. I claim every open set is the union of balls from $\mathcal{B}$. Let U be an open set in $\mathbb{F}^n$ and $\mathbf{x} \in U$. Then there exists $\delta > 0$ such that $B(\mathbf{x}, \delta) \subseteq U$. There exists $\mathbf{d} \in D^n \cap B(\mathbf{x}, \delta/5)$. Then pick rational number $\delta/5 < r < 2\delta/5$. Consider the set of $\mathcal{B}$, $B(\mathbf{d}, r)$. Then $\mathbf{x} \in B(\mathbf{d}, r)$ because $r > \delta/5$. However, it is also the case that $B(\mathbf{d}, r) \subseteq B(\mathbf{x}, \delta)$ because if $\mathbf{y} \in B(\mathbf{d}, r)$ then

$$\begin{aligned} ||\mathbf{y} - \mathbf{x}|| &\leq ||\mathbf{y} - \mathbf{d}|| + ||\mathbf{d} - \mathbf{x}|| \\ &< \frac{2\delta}{5} + \frac{\delta}{5} < \delta. \end{aligned}$$

This proves the lemma.

Corollary 7.7.8 *Let $(\Omega, \mathcal{F}, \mu)$ be a measure space and let X be a normed vector space with basis $\{\mathbf{v}_1, \cdots, \mathbf{v}_n\}$. Let π_k be the k^{th} projection map onto the k^{th} component. Thus*

$$\pi_k \mathbf{x} \equiv x_k \text{ where } \mathbf{x} = \sum_{i=1}^{n} x_i \mathbf{v}_i.$$

Then each $\pi_k \circ \mathbf{f}$ is a measurable $\mathbb{F}$ valued function if and only if $\mathbf{f}$ is a measurable X valued function.

Proof: The if part has already been noted. Suppose that each $\pi_k \circ \mathbf{f}$ is an $\mathbb{F}$ valued measurable function. Let $\mathbf{g} : X \to \mathbb{F}^n$ be given by

$$\mathbf{g}(\mathbf{x}) \equiv (\pi_1 \mathbf{x}, \cdots, \pi_n \mathbf{x}).$$

Thus $\mathbf{g}$ is linear, one to one, and onto. By Theorem 5.8.3 both $\mathbf{g}$ and $\mathbf{g}^{-1}$ are continuous. Therefore, every open set in X is of the form $\mathbf{g}^{-1}(U)$ where U is an open set in $\mathbb{F}^n$. To see this, start with V open set in X. Since $\mathbf{g}^{-1}$ is continuous, $\mathbf{g}(V)$ is open in $\mathbb{F}^n$ and so $V = \mathbf{g}^{-1}(\mathbf{g}(V))$. Therefore, it suffices to show that for every U an open set in $\mathbb{F}^n$,

$$\mathbf{f}^{-1}\left(\mathbf{g}^{-1}(U)\right) = (\mathbf{g} \circ \mathbf{f})^{-1}(U) \in \mathcal{F}.$$

By Lemma 7.7.7 there are countably many open balls of the form $B(\mathbf{x}_j, r_j)$ such that U is equal to the union of these balls. Thus

$$\begin{aligned} (\mathbf{g} \circ \mathbf{f})^{-1}(U) &= (\mathbf{g} \circ \mathbf{f})^{-1}\left(\cup_{k=1}^{\infty} B(\mathbf{x}_k, r_k)\right) \\ &= \cup_{k=1}^{\infty} (\mathbf{g} \circ \mathbf{f})^{-1}(B(\mathbf{x}_k, r_k)) \end{aligned} \tag{7.19}$$

Now from the definition of the norm,

$$B(\mathbf{x}_k, r_k) = \prod_{j=1}^{n} (x_{kj} - \delta, x_{kj} + \delta)$$

and so

$$(\mathbf{g} \circ \mathbf{f})^{-1}(B(\mathbf{x}_k, r_k)) = \cap_{j=1}^{n} (\pi_j \circ \mathbf{f})^{-1}((x_{kj} - \delta, x_{kj} + \delta)) \in \mathcal{F}.$$

It follows 7.19 is the countable union of sets in $\mathcal{F}$ and so it is also in $\mathcal{F}$. This proves the corollary.

Note that if $\{f_i\}_{i=1}^{n}$ are measurable functions defined on $(\Omega, \mathcal{F}, \mu)$ having values in $\mathbb{F}$ then letting $\mathbf{f} \equiv (f_1, \cdots, f_n)$, it follows $\mathbf{f}$ is a measurable $\mathbb{F}^n$ valued function. Now let $\Sigma : \mathbb{F}^n \to \mathbb{F}$ be given by $\Sigma(\mathbf{x}) \equiv \sum_{k=1}^{n} a_k x_k$. Then Σ is linear and so by Theorem 5.8.3 it follows Σ is continuous. Hence by Theorem 7.7.6, $\Sigma(\mathbf{f})$ is an $\mathbb{F}$ valued measurable function. Thus linear combinations of measurable functions are measurable. By similar reasoning, products of measurable functions are measurable. In general, it seems like you can start with a collection of measurable functions and do almost anything you like with them and the result, if it is a function will be measurable. This is in stark contrast to the functions which are generalized Riemann integrable.

The following theorem considers the case of functions which have values in a normed vector space.

Theorem 7.7.9 *Let $\{\mathbf{f}_n\}$ be a sequence of measurable functions mapping Ω to X where X is a normed vector space and $(\Omega, \mathcal{F})$ is a measure space. Suppose also that $\mathbf{f}(\omega) = \lim_{n\to\infty} \mathbf{f}_n(\omega)$ for all $\omega \in \Omega$. Then $\mathbf{f}$ is also a measurable function.*

Proof: It is required to show $\mathbf{f}^{-1}(U)$ is measurable for all U open. Let

$$V_m \equiv \left\{ \mathbf{x} \in U : \text{dist}\left(\mathbf{x}, U^C\right) > \frac{1}{m} \right\}.$$

Thus

$$V_m \subseteq \left\{ \mathbf{x} \in U : \text{dist}\left(\mathbf{x}, U^C\right) \geq \frac{1}{m} \right\}$$

and $V_m \subseteq \overline{V_m} \subseteq V_{m+1}$ and $\cup_m V_m = U$. Then since V_m is open, it follows that if $\mathbf{f}(\omega) \in V_m$ then for all sufficiently large k, it must be the case $\mathbf{f}_k(\omega) \in V_m$ also. That is, $\omega \in \mathbf{f}_k^{-1}(V_m)$ for all sufficiently large k. Thus

$$\mathbf{f}^{-1}(V_m) = \cup_{n=1}^{\infty} \cap_{k=n}^{\infty} \mathbf{f}_k^{-1}(V_m)$$

and so

$$\begin{aligned} \mathbf{f}^{-1}(U) &= \cup_{m=1}^{\infty} \mathbf{f}^{-1}(V_m) \\ &= \cup_{m=1}^{\infty} \cup_{n=1}^{\infty} \cap_{k=n}^{\infty} \mathbf{f}_k^{-1}(V_m) \\ &\subseteq \cup_{m=1}^{\infty} \mathbf{f}^{-1}\left(\overline{V_m}\right) = \mathbf{f}^{-1}(U) \end{aligned}$$

which shows $\mathbf{f}^{-1}(U)$ is measurable. The step from the second to the last line follows because if $\omega \in \cup_{n=1}^{\infty} \cap_{k=n}^{\infty} \mathbf{f}_k^{-1}(V_m)$, this says $\mathbf{f}_k(\omega) \in V_m$ for all k large enough. Therefore, the point of X to which the sequence $\{\mathbf{f}_k(\omega)\}$ converges must be in $\overline{V_m}$ which equals $V_m \cup V_m'$, the limit points of V_m. This proves the theorem.

Now here is a simple observation involving something called simple functions. It uses the following notation.

Notation 7.7.10 *For E a set let $\mathcal{X}_E(\omega)$ be defined by*

$$\mathcal{X}_E(x) = \begin{cases} 1 \text{ if } \omega \in E \\ 0 \text{ if } \omega \notin E \end{cases}$$

Theorem 7.7.11

Let $\mathbf{f} : \Omega \to X$ where X is some normed vector space. Suppose

$$\mathbf{f}(\omega) = \sum_{k=1}^{m} \mathbf{x}_k \mathcal{X}_{A_k}(\omega)$$

where each $\mathbf{x}_k \in X$ and the A_k are disjoint measurable sets. (Such functions are often referred to as simple functions.) Then $\mathbf{f}$ is measurable.

Proof: Letting U be open, $\mathbf{f}^{-1}(U) = \cup\{A_k : \mathbf{x}_k \in U\}$, a finite union of measurable sets.

In the Lebesgue integral, the simple functions play a role similar to step functions in the theory of the Riemann integral. Also there is a fundamental theorem about measurable functions and simple functions which says essentially that the measurable functions are those which are pointwise limits of simple functions.

Theorem 7.7.12

Let $f \geq 0$ be measurable with respect to the measure space $(\Omega, \mathcal{F}, \mu)$. Then there exists a sequence of nonnegative simple functions $\{s_n\}$ satisfying

$$0 \leq s_n(\omega) \tag{7.20}$$

$$\cdots s_n(\omega) \leq s_{n+1}(\omega) \cdots$$

$$f(\omega) = \lim_{n \to \infty} s_n(\omega) \text{ for all } \omega \in \Omega. \tag{7.21}$$

If f is bounded the convergence is actually uniform.

Proof: First note that

$$\begin{aligned} f^{-1}([a,b)) &= f^{-1}((-\infty,a))^C \cap f^{-1}((-\infty,b)) \\ &= \left(f^{-1}((-\infty,a)) \cup f^{-1}((-\infty,b))^C\right)^C \in \mathcal{F}. \end{aligned}$$

Letting $I \equiv \{\omega : f(\omega) = \infty\}$, define

$$t_n(\omega) = \sum_{k=0}^{2^n} \frac{k}{n} \mathcal{X}_{f^{-1}([k/n,(k+1)/n))}(\omega) + n\mathcal{X}_I(\omega).$$

Then $t_n(\omega) \leq f(\omega)$ for all ω and $\lim_{n\to\infty} t_n(\omega) = f(\omega)$ for all ω. This is because $t_n(\omega) = n$ for $\omega \in I$ and if $f(\omega) \in [0, \frac{2^n+1}{n})$, then

$$0 \leq f(\omega) - t_n(\omega) \leq \frac{1}{n}. \tag{7.22}$$

Thus whenever $\omega \notin I$, the above inequality will hold for all n large enough. Let

$$s_1 = t_1,\ s_2 = \max(t_1, t_2),\ s_3 = \max(t_1, t_2, t_3), \cdots.$$

Then the sequence $\{s_n\}$ satisfies 7.20-7.21.

To verify the last claim, note that in this case the term $n\mathcal{X}_I(\omega)$ is not present. Therefore, for all n large enough that $2^n n \geq f(\omega)$ for all ω, 7.22 holds for all ω. Thus the convergence is uniform. This proves the theorem.

7.8 Exercises

1. Let $\mathcal{C}$ be a set whose elements are σ algebras of subsets of Ω. Show $\cap\mathcal{C}$ is a σ algebra also.

2. Let Ω be any set. Show $\mathcal{P}(\Omega)$, the set of all subsets of Ω is a σ algebra. Now let $\mathcal{L}$ denote some subset of $\mathcal{P}(\Omega)$. Consider all σ algebras which contain $\mathcal{L}$. Show the intersection of all these σ algebras which contain $\mathcal{L}$ is a σ algebra containing $\mathcal{L}$ and it is the smallest σ algebra containing $\mathcal{L}$, denoted by $\sigma(\mathcal{L})$. When Ω is a normed vector space, and $\mathcal{L}$ consists of the open sets, $\sigma(\mathcal{L})$ is called the σ algebra of Borel sets.

3. Consider $\Omega = [0,1]$ and let $\mathcal{S}$ denote all subsets of $[0,1], F$ such that either F^C or F is countable. Note the empty set must be countable. Show $\mathcal{S}$ is a σ algebra. (This is a sick σ algebra.) Now let $\mu : \mathcal{S} \to [0,\infty]$ be defined by $\mu(F) = 1$ if F^C is countable and $\mu(F) = 0$ if F is countable. Show μ is a measure on $\mathcal{S}$.

4. Let $\Omega = \mathbb{N}$, the positive integers and let a σ algebra be given by $\mathcal{F} = \mathcal{P}(\mathbb{N})$, the set of all subsets of $\mathbb{N}$. What are the measurable functions having values in $\mathbb{C}$? Let $\mu(E)$ be the number of elements of E where E is a subset of $\mathbb{N}$. Show μ is a measure.

5. Let $\mathcal{F}$ be a σ algebra of subsets of Ω and suppose $\mathcal{F}$ has infinitely many elements. Show that $\mathcal{F}$ is uncountable. **Hint:** You might try to show there exists a countable sequence of disjoint sets of $\mathcal{F}$, $\{A_i\}$. It might be easiest to verify this by contradiction if it doesn't exist rather than a direct construction however, I have seen this done several ways. Once this has been done, you can define a map, θ, from $\mathcal{P}(\mathbb{N})$ into $\mathcal{F}$ which is one to one by $\theta(S) = \cup_{i\in S} A_i$. Then argue $\mathcal{P}(\mathbb{N})$ is uncountable and so $\mathcal{F}$ is also uncountable.

6. A probability space is a measure space, $(\Omega, \mathcal{F}, P)$ where the measure, P has the property that $P(\Omega) = 1$. Such a measure is called a probability measure. Random vectors are measurable functions, $\mathbf{X}$, mapping a probability space, $(\Omega, \mathcal{F}, P)$ to $\mathbb{R}^n$. Thus $\mathbf{X}(\omega) \in \mathbb{R}^n$ for each $\omega \in \Omega$ and P is a probability measure defined on the sets of $\mathcal{F}$, a σ algebra of subsets of Ω. For E a Borel set in $\mathbb{R}^n$, define

$$\mu(E) \equiv P\left(\mathbf{X}^{-1}(E)\right) \equiv \text{ probability that } \mathbf{X} \in E.$$

Show this is a well defined probability measure on the Borel sets of $\mathbb{R}^n$. Thus $\mu(E) = P(\mathbf{X}(\omega) \in E)$. It is called the distribution. Explain why μ must be regular.

7. Suppose $(\Omega, \mathcal{S}, \mu)$ is a measure space which may not be complete. Show that another way to complete the measure space is to define $\overline{\mathcal{S}}$ to consist of all sets of the form E where there exists $F \in \mathcal{S}$ such that $(F \setminus E) \cup (E \setminus F) \subseteq N$ for some $N \in \mathcal{S}$ which has measure zero and then let $\mu(E) = \mu_1(F)$? Explain.

The Abstract Lebesgue Integral

The general Lebesgue integral requires a measure space, $(\Omega, \mathcal{F}, \mu)$ and, to begin with, a nonnegative measurable function.

8.1 Definition For Nonnegative Measurable Functions

First of all, the notation

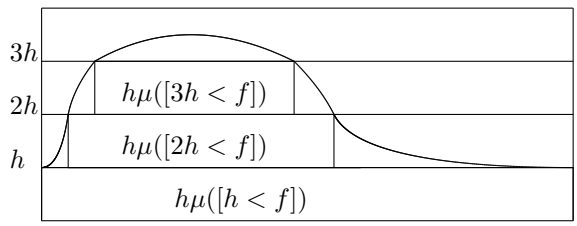
$$[g < f]$$

is short for

$$\{\omega \in \Omega : g(\omega) < f(\omega)\}$$

with other variants of this notation being similar. Also, the convention, $0 \cdot \infty = 0$ will be used to simplify the presentation whenever it is convenient to do so.

The following picture illustrates the idea used to define the Lebesgue integral to be like the area under a curve.



You can see that by following the procedure illustrated in the picture and letting h get smaller, you would expect to obtain better approximations to the area under the curve[1] although all these approximations would likely be too small. Therefore, define

$$\int f d\mu \equiv \sup_{h>0} \sum_{i=1}^{\infty} h\mu([ih < f])$$

Lemma 8.1.1 *The following inequality holds.*

$$\sum_{i=1}^{\infty} h\mu([ih < f]) \leq \sum_{i=1}^{\infty} \frac{h}{2}\mu\left(\left[i\frac{h}{2} < f\right]\right).$$

[1]Note the difference between this picture and the one usually drawn in calculus courses where the little rectangles are upright rather than on their sides. This illustrates a fundamental philosophical difference between the Riemann and the Lebesgue integrals. With the Riemann integral intervals are measured. With the Lebesgue integral, it is inverse images of intervals which are measured.

Also, it suffices to consider only h smaller than a given positive number in the above definition of the integral.

Proof: Let $N \in \mathbb{N}$.

$$\sum_{i=1}^{2N} \frac{h}{2}\mu\left(\left[i\frac{h}{2} < f\right]\right) = \sum_{i=1}^{2N} \frac{h}{2}\mu\left([ih < 2f]\right)$$

$$= \sum_{i=1}^{N} \frac{h}{2}\mu\left([(2i-1)\,h < 2f]\right) + \sum_{i=1}^{N} \frac{h}{2}\mu\left([(2i)\,h < 2f]\right)$$

$$= \sum_{i=1}^{N} \frac{h}{2}\mu\left(\left[\frac{(2i-1)}{2}h < f\right]\right) + \sum_{i=1}^{N} \frac{h}{2}\mu\left([ih < f]\right)$$

$$\geq \sum_{i=1}^{N} \frac{h}{2}\mu\left([ih < f]\right) + \sum_{i=1}^{N} \frac{h}{2}\mu\left([ih < f]\right) = \sum_{i=1}^{N} h\mu\left([ih < f]\right).$$

Now letting $N \to \infty$ yields the claim of the lemma.

To verify the last claim, suppose $M < \int f d\mu$ and let $\delta > 0$ be given. Then there exists $h > 0$ such that

$$M < \sum_{i=1}^{\infty} h\mu\left([ih < f]\right) \leq \int f d\mu.$$

By the first part of this lemma,

$$M < \sum_{i=1}^{\infty} \frac{h}{2}\mu\left(\left[i\frac{h}{2} < f\right]\right) \leq \int f d\mu$$

and continuing to apply the first part,

$$M < \sum_{i=1}^{\infty} \frac{h}{2^n}\mu\left(\left[i\frac{h}{2^n} < f\right]\right) \leq \int f d\mu.$$

Choose n large enough that $h/2^n < \delta$. It follows

$$M < \sup_{\delta > h > 0} \sum_{i=1}^{\infty} h\mu\left([ih < f]\right) \leq \int f d\mu.$$

Since M is arbitrary, this proves the last claim.

8.2 The Lebesgue Integral For Nonnegative Simple Functions

Now the Lebesgue integral for a nonnegative function has been defined, what does it do to a nonnegative simple function. Recall a nonnegative simple function is one of the form

$$s(\omega) = \sum_{i=1}^{n} c_i \mathcal{X}_{E_i}(\omega)$$

where the c_i are each nonnegative real numbers.

Note that by taking the union of some of the E_i, you can assume the numbers, c_i are the distinct values of s. Simple functions are important because it will turn out to be very easy to take their integrals as shown in the following lemma.

Lemma 8.2.1 *Let $s(\omega)=\sum_{i=1}^{p} a_i \mathcal{X}_{E_i}(\omega)$ be a nonnegative simple function with the a_i the distinct non zero values of s. Then*

$$\int s d\mu = \sum_{i=1}^{p} a_i \mu(E_i). \tag{8.1}$$

Also, for any nonnegative measurable function, f, if $\lambda \geq 0$, then

$$\int \lambda f d\mu = \lambda \int f d\mu. \tag{8.2}$$

Proof: Consider 8.1 first. Without loss of generality, you can assume $0 < a_1 < a_2 < \cdots < a_p$ and that $\mu(E_i) < \infty$. Otherwise, both sides will equal ∞. Let $\varepsilon > 0$ be given and let δ_1 be small enough that

$$\delta_1 \sum_{i=1}^{p} \mu(E_i) < \varepsilon.$$

Pick $\delta < \delta_1$ such that for $h < \delta$ it is also true that

$$h < \frac{1}{2}\min(a_1, a_2 - a_1, a_3 - a_2, \cdots, a_n - a_{n-1}).$$

That is, h is smaller than half the distance between any two successive values of s. Then for $0 < h < \delta$, it follows from Theorem 2.3.5 on Page 20 about double sums of nonnegative terms,

$$\begin{aligned}
\sum_{k=1}^{\infty} h\mu([s > kh]) &= \sum_{k=1}^{\infty} h \sum_{i=k}^{\infty} \mu([ih < s \leq (i+1)h]) \\
&= \sum_{i=1}^{\infty} \sum_{k=1}^{i} h\mu([ih < s \leq (i+1)h]) \\
&= \sum_{i=1}^{\infty} ih\mu([ih < s \leq (i+1)h]).
\end{aligned} \tag{8.3}$$

Because of the choice of h there exist positive integers, i_k such that $i_1 < i_2 < \cdots, < i_p$ and

$$\begin{aligned}
i_1 h &< a_1 \leq (i_1+1)h < \cdots < i_2 h < a_2 < \\
&< (i_2+1)h < \cdots < i_p h < a_p \leq (i_p+1)h
\end{aligned}$$

Then in the sum of 8.3 the only terms which are nonzero are those for which $i \in \{i_1, i_2 \cdots, i_p\}$. From the above, you see that

$$\mu([i_k h < s \leq (i_k+1)h]) = \mu(E_k).$$

Therefore,

$$\sum_{k=1}^{\infty} h\mu([s > kh]) = \sum_{k=1}^{p} i_k h \mu(E_k).$$

It follows that for all h this small,

$$\begin{aligned}
0 &< \sum_{k=1}^{p} a_k \mu(E_k) - \sum_{k=1}^{\infty} h\mu([s > kh]) \\
&= \sum_{k=1}^{p} a_k \mu(E_k) - \sum_{k=1}^{p} i_k h \mu(E_k) \leq h \sum_{k=1}^{p} \mu(E_k) < \varepsilon.
\end{aligned}$$

Taking the inf for h this small and using Lemma 8.1.1,

$$\begin{aligned} 0 &\leq \sum_{k=1}^{p} a_k \mu(E_k) - \sup_{\delta > h > 0} \sum_{k=1}^{\infty} h\mu([s > kh]) \\ &= \sum_{k=1}^{p} a_k \mu(E_k) - \int s d\mu \leq \varepsilon. \end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, this proves the first part.

To verify 8.2 Note the formula is obvious if $\lambda = 0$ because then $[ih < \lambda f] = \emptyset$ for all $i > 0$. Assume $\lambda > 0$. Then

$$\begin{aligned} \int \lambda f d\mu &\equiv \sup_{h>0} \sum_{i=1}^{\infty} h\mu([ih < \lambda f]) \\ &= \sup_{h>0} \sum_{i=1}^{\infty} h\mu([ih/\lambda < f]) \\ &= \sup_{h>0} \lambda \sum_{i=1}^{\infty} (h/\lambda)\, \mu([i\,(h/\lambda) < f]) \\ &= \lambda \int f d\mu. \end{aligned}$$

This proves the lemma.

Lemma 8.2.2 *Let the nonnegative simple function, s be defined as*

$$s(\omega) = \sum_{i=1}^{n} c_i \mathcal{X}_{E_i}(\omega)$$

where the c_i are not necessarily distinct but the E_i are disjoint. It follows that

$$\int s = \sum_{i=1}^{n} c_i \mu(E_i).$$

Proof: Let the values of s be $\{a_1, \cdots, a_m\}$. Therefore, since the E_i are disjoint, each a_i equal to one of the c_j. Let $A_i \equiv \cup \{E_j : c_j = a_i\}$. Then from Lemma 8.2.1 it follows that

$$\begin{aligned} \int s &= \sum_{i=1}^{m} a_i \mu(A_i) = \sum_{i=1}^{m} a_i \sum_{\{j : c_j = a_i\}} \mu(E_j) \\ &= \sum_{i=1}^{m} \sum_{\{j : c_j = a_i\}} c_j \mu(E_j) = \sum_{i=1}^{n} c_i \mu(E_i). \end{aligned}$$

This proves the lemma.

Note that $\int s$ could equal $+\infty$ if $\mu(A_k) = \infty$ and $a_k > 0$ for some k, but $\int s$ is well defined because $s \geq 0$. Recall that $0 \cdot \infty = 0$.

Lemma 8.2.3 *If $a, b \geq 0$ and if s and t are nonnegative simple functions, then*

$$\int as + bt = a \int s + b \int t.$$

Proof: Let

$$s(\omega) = \sum_{i=1}^{n} \alpha_i \mathcal{X}_{A_i}(\omega),\ t(\omega) = \sum_{i=1}^{m} \beta_j \mathcal{X}_{B_j}(\omega)$$

where α_i are the distinct values of s and the β_j are the distinct values of t. Clearly $as + bt$ is a nonnegative simple function because it has finitely many values on measurable sets In fact,

$$(as + bt)(\omega) = \sum_{j=1}^{m} \sum_{i=1}^{n} (a\alpha_i + b\beta_j) \mathcal{X}_{A_i \cap B_j}(\omega)$$

where the sets $A_i \cap B_j$ are disjoint and measurable. By Lemma 8.2.2,

$$\begin{aligned} \int as + bt &= \sum_{j=1}^{m} \sum_{i=1}^{n} (a\alpha_i + b\beta_j) \mu(A_i \cap B_j) \\ &= \sum_{i=1}^{n} a \sum_{j=1}^{m} \alpha_i \mu(A_i \cap B_j) + b \sum_{j=1}^{m} \sum_{i=1}^{n} \beta_j \mu(A_i \cap B_j) \\ &= a \sum_{i=1}^{n} \alpha_i \mu(A_i) + b \sum_{j=1}^{m} \beta_j \mu(B_j) \\ &= a \int s + b \int t. \end{aligned}$$

This proves the lemma.

8.3 The Monotone Convergence Theorem

The following is called the monotone convergence theorem. This theorem and related convergence theorems are the reason for using the Lebesgue integral.

Theorem 8.3.1 *(Monotone Convergence theorem) Let f have values in $[0, \infty]$ and suppose $\{f_n\}$ is a sequence of nonnegative measurable functions having values in $[0, \infty]$ and satisfying*

$$\lim_{n \to \infty} f_n(\omega) = f(\omega) \text{ for each } \omega.$$

$$\cdots f_n(\omega) \leq f_{n+1}(\omega) \cdots$$

Then f is measurable and

$$\int f d\mu = \lim_{n \to \infty} \int f_n d\mu.$$

Proof: From Lemmas 2.3.3 and 2.3.4, see Page 20,

$$\begin{aligned}
\int f d\mu &\equiv \sup_{h>0} \sum_{i=1}^{\infty} h\mu\left([ih < f]\right) \\
&= \sup_{h>0} \sup_{k} \sum_{i=1}^{k} h\mu\left([ih < f]\right) \\
&= \sup_{h>0} \sup_{k} \sup_{m} \sum_{i=1}^{k} h\mu\left([ih < f_m]\right) \\
&= \sup_{m} \sup_{h>0} \sum_{i=1}^{\infty} h\mu\left([ih < f_m]\right) \\
&\equiv \sup_{m} \int f_m d\mu \\
&= \lim_{m\to\infty} \int f_m d\mu.
\end{aligned}$$

The third equality follows from the observation that

$$\lim_{m\to\infty} \mu\left([ih < f_m]\right) = \mu\left([ih < f]\right)$$

which follows from Theorem 7.3.2 since the sets, $[ih < f_m]$ are increasing in m and their union equals $[ih < f]$. This proves the theorem.

To illustrate what goes wrong without the Lebesgue integral, consider the following example.

Example 8.3.2 *Let $\{r_n\}$ denote the rational numbers in $[0,1]$ and let*

$$f_n(t) \equiv \begin{cases} 1 \text{ if } t \notin \{r_1, \cdots, r_n\} \\ 0 \text{ otherwise} \end{cases}$$

Then $f_n(t) \uparrow f(t)$ where f is the function which is one on the rationals and zero on the irrationals. Each f_n is Riemann integrable (why?) but f is not Riemann integrable. Therefore, you can't write $\int f dx = \lim_{n\to\infty} \int f_n dx$.

A meta-mathematical observation related to this type of example is this. If you can choose your functions, you don't need the Lebesgue integral. The Riemann Darboux integral is just fine. It is when you can't choose your functions and they come to you as pointwise limits that you really need the superior Lebesgue integral or at least something more general than the Riemann integral. The Riemann integral is entirely adequate for evaluating the seemingly endless lists of boring problems found in calculus books.

8.4 Other Definitions

To review and summarize the above, if $f \geq 0$ is measurable,

$$\int f d\mu \equiv \sup_{h>0} \sum_{i=1}^{\infty} h\mu\left([f > ih]\right) \tag{8.4}$$

another way to get the same thing for $\int f d\mu$ is to take an increasing sequence of nonnegative simple functions, $\{s_n\}$ with $s_n(\omega) \to f(\omega)$ and then by monotone convergence theorem,

$$\int f d\mu = \lim_{n\to\infty} \int s_n$$

where if $s_n(\omega) = \sum_{j=1}^m c_i \mathcal{X}_{E_i}(\omega)$,

$$\int s_n d\mu = \sum_{i=1}^m c_i \mu(E_i).$$

Similarly this also shows that for such nonnegative measurable function,

$$\int f d\mu = \sup\left\{\int s : 0 \le s \le f,\ s \text{ simple}\right\}$$

Here is an equivalent definition of the integral of a nonnegative measurable function. The fact it is well defined has been discussed above.

Definition 8.4.1 *For s a nonnegative simple function,*

$$s(\omega) = \sum_{k=1}^n c_k \mathcal{X}_{E_k}(\omega), \int s = \sum_{k=1}^n c_k \mu(E_k).$$

For f a nonnegative measurable function,

$$\int f d\mu = \sup\left\{\int s : 0 \le s \le f,\ s \text{ simple}\right\}.$$

8.5 Fatou's Lemma

The next theorem, known as Fatou's lemma is another important theorem which justifies the use of the Lebesgue integral.

Theorem 8.5.1 *(Fatou's lemma) Let f_n be a nonnegative measurable function with values in $[0,\infty]$. Let $g(\omega) = \liminf_{n\to\infty} f_n(\omega)$. Then g is measurable and*

$$\int g d\mu \le \liminf_{n\to\infty} \int f_n d\mu.$$

In other words,

$$\int \left(\liminf_{n\to\infty} f_n\right) d\mu \le \liminf_{n\to\infty} \int f_n d\mu$$

Proof: Let $g_n(\omega) = \inf\{f_k(\omega) : k \ge n\}$. Then

$$\begin{aligned} g_n^{-1}([a,\infty]) &= \cap_{k=n}^\infty f_k^{-1}([a,\infty]) \\ &= \left(\cup_{k=n}^\infty f_k^{-1}([a,\infty])^C\right)^C \in \mathcal{F}. \end{aligned}$$

Thus g_n is measurable by Lemma 7.7.1. Also $g(\omega) = \lim_{n\to\infty} g_n(\omega)$ so g is measurable because it is the pointwise limit of measurable functions. Now the functions g_n form an increasing sequence of nonnegative measurable functions so the monotone convergence theorem applies. This yields

$$\int g d\mu = \lim_{n\to\infty} \int g_n d\mu \le \liminf_{n\to\infty} \int f_n d\mu.$$

The last inequality holding because

$$\int g_n d\mu \le \int f_n d\mu.$$

(Note that it is not known whether $\lim_{n\to\infty} \int f_n d\mu$ exists.) This proves the Theorem.

8.6 The Righteous Algebraic Desires Of The Lebesgue Integral

The monotone convergence theorem shows the integral wants to be linear. This is the essential content of the next theorem.

Theorem 8.6.1 *Let f, g be nonnegative measurable functions and let a, b be nonnegative numbers. Then $af + bg$ is measurable and*

$$\int (af + bg)\, d\mu = a \int f d\mu + b \int g d\mu. \tag{8.5}$$

Proof: By Theorem 7.7.12 on Page 168 there exist increasing sequences of nonnegative simple functions, $s_n \to f$ and $t_n \to g$. Then $af + bg$, being the pointwise limit of the simple functions $as_n + bt_n$, is measurable. Now by the monotone convergence theorem and Lemma 8.2.3,

$$\begin{aligned} \int (af + bg)\, d\mu &= \lim_{n\to\infty} \int as_n + bt_n d\mu \\ &= \lim_{n\to\infty} \left(a \int s_n d\mu + b \int t_n d\mu \right) \\ &= a \int f d\mu + b \int g d\mu. \end{aligned}$$

This proves the theorem.

As long as you are allowing functions to take the value $+\infty$, you cannot consider something like $f + (-g)$ and so you can't very well expect a satisfactory statement about the integral being linear until you restrict yourself to functions which have values in a vector space. This is discussed next.

8.7 The Lebesgue Integral, L^1

The functions considered here have values in $\mathbb{C}$, a vector space.

Definition 8.7.1 *Let $(\Omega, \mathcal{S}, \mu)$ be a measure space and suppose $f : \Omega \to \mathbb{C}$. Then f is said to be measurable if both $\operatorname{Re} f$ and $\operatorname{Im} f$ are measurable real valued functions.*

Definition 8.7.2 *A complex simple function will be a function which is of the form*

$$s(\omega) = \sum_{k=1}^{n} c_k \mathcal{X}_{E_k}(\omega)$$

where $c_k \in \mathbb{C}$ and $\mu(E_k) < \infty$. For s a complex simple function as above, define

$$I(s) \equiv \sum_{k=1}^{n} c_k \mu(E_k).$$

Lemma 8.7.3 *The definition, 8.7.2 is well defined. Furthermore, I is linear on the vector space of complex simple functions. Also the triangle inequality holds,*

$$|I(s)| \le I(|s|).$$

Proof: Suppose $\sum_{k=1}^n c_k \mathcal{X}_{E_k}(\omega) = 0$. Does it follow that $\sum_k c_k \mu(E_k) = 0$? The supposition implies

$$\sum_{k=1}^n \operatorname{Re} c_k \mathcal{X}_{E_k}(\omega) = 0, \ \sum_{k=1}^n \operatorname{Im} c_k \mathcal{X}_{E_k}(\omega) = 0. \tag{8.6}$$

Choose λ large and positive so that $\lambda + \operatorname{Re} c_k \geq 0$. Then adding $\sum_k \lambda \mathcal{X}_{E_k}$ to both sides of the first equation above,

$$\sum_{k=1}^n (\lambda + \operatorname{Re} c_k) \mathcal{X}_{E_k}(\omega) = \sum_{k=1}^n \lambda \mathcal{X}_{E_k}$$

and by Lemma 8.2.3 on Page 174, it follows upon taking $\int$ of both sides that

$$\sum_{k=1}^n (\lambda + \operatorname{Re} c_k) \mu(E_k) = \sum_{k=1}^n \lambda \mu(E_k)$$

which implies $\sum_{k=1}^n \operatorname{Re} c_k \mu(E_k) = 0$. Similarly, $\sum_{k=1}^n \operatorname{Im} c_k \mu(E_k) = 0$ and so $\sum_{k=1}^n c_k \mu(E_k) = 0$. Thus if

$$\sum_j c_j \mathcal{X}_{E_j} = \sum_k d_k \mathcal{X}_{F_k}$$

then $\sum_j c_j \mathcal{X}_{E_j} + \sum_k (-d_k) \mathcal{X}_{F_k} = 0$ and so the result just established verifies $\sum_j c_j \mu(E_j) - \sum_k d_k \mu(F_k) = 0$ which proves I is well defined.

That I is linear is now obvious. It only remains to verify the triangle inequality.

Let s be a simple function,

$$s = \sum_j c_j \mathcal{X}_{E_j}$$

Then pick $\theta \in \mathbb{C}$ such that $\theta I(s) = |I(s)|$ and $|\theta| = 1$. Then from the triangle inequality for sums of complex numbers,

$$\begin{aligned} |I(s)| &= \theta I(s) = I(\theta s) = \sum_j \theta c_j \mu(E_j) \\ &= \left| \sum_j \theta c_j \mu(E_j) \right| \leq \sum_j |\theta c_j| \mu(E_j) = I(|s|). \end{aligned}$$

This proves the lemma.

With this lemma, the following is the definition of $L^1(\Omega)$.

Definition 8.7.4 *$f \in L^1(\Omega)$ means there exists a sequence of complex simple functions, $\{s_n\}$ such that*

$$\begin{gathered} s_n(\omega) \to f(\omega) \text{ for all } \omega \in \Omega \\ \lim_{m,n\to\infty} I(|s_n - s_m|) = \lim_{n,m\to\infty} \textstyle\int |s_n - s_m| \, d\mu = 0 \end{gathered} \tag{8.7}$$

Then

$$I(f) \equiv \lim_{n\to\infty} I(s_n). \tag{8.8}$$

Lemma 8.7.5 *Definition 8.7.4 is well defined.*

Proof: There are several things which need to be verified. First suppose 8.7. Then by Lemma 8.7.3

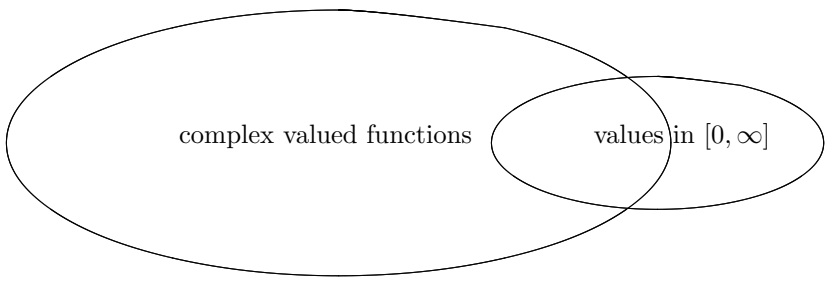
$$|I(s_n) - I(s_m)| = |I(s_n - s_m)| \leq I(|s_n - s_m|)$$

and for m, n large enough this last is given to be small so $\{I(s_n)\}$ is a Cauchy sequence in $\mathbb{C}$ and so it converges. This verifies the limit in 8.8 at least exists. It remains to consider another sequence $\{t_n\}$ having the same properties as $\{s_n\}$ and verifying $I(f)$ determined by this other sequence is the same. By Lemma 8.7.3 and Fatou's lemma, Theorem 8.5.1 on Page 177,

$$\begin{aligned} |I(s_n) - I(t_n)| &\leq I(|s_n - t_n|) = \int |s_n - t_n| \, d\mu \\ &\leq \int |s_n - f| + |f - t_n| \, d\mu \\ &\leq \liminf_{k\to\infty} \int |s_n - s_k| \, d\mu + \liminf_{k\to\infty} \int |t_n - t_k| \, d\mu < \varepsilon \end{aligned}$$

whenever n is large enough. Since ε is arbitrary, this shows the limit from using the t_n is the same as the limit from using s_n. This proves the lemma.

Consider the following picture. I have just given a definition of an integral for functions having values in $\mathbb{C}$. However, $[0, \infty) \subseteq \mathbb{C}$.



What if f has values in $[0, \infty)$? Earlier $\int f d\mu$ was defined for such functions and now $I(f)$ has been defined. Are they the same? If so, I can be regarded as an extension of $\int d\mu$ to a larger class of functions.

Lemma 8.7.6 *Suppose f has values in $[0, \infty)$ and $f \in L^1(\Omega)$. Then f is measurable and*

$$I(f) = \int f d\mu.$$

Proof: Since f is the pointwise limit of a sequence of complex simple functions, $\{s_n\}$ having the properties described in Definition 8.7.4, it follows

$$f(\omega) = \lim_{n\to\infty} \operatorname{Re} s_n(\omega)$$

and so f is measurable. Also it is always the case that if a, b are real numbers,

$$\left|a^+ - b^+\right| \leq |a - b|$$

and so

$$\int \left| (\operatorname{Re} s_n)^+ - (\operatorname{Re} s_m)^+ \right| d\mu \leq \int |\operatorname{Re} s_n - \operatorname{Re} s_m| \, d\mu \leq \int |s_n - s_m| \, d\mu$$

where $x^+ \equiv \frac{1}{2}(|x| + x)$, the positive part of the real number, x. [2]Thus there is no loss of generality in assuming $\{s_n\}$ is a sequence of complex simple functions having values in $[0, \infty)$. Then since for such complex simple functions, $I(s) = \int s d\mu$,

$$\left| I(f) - \int f d\mu \right| \leq |I(f) - I(s_n)| + \left| \int s_n d\mu - \int f d\mu \right| < \varepsilon + \int |s_n - f| \, d\mu$$

whenever n is large enough. But by Fatou's lemma, Theorem 8.5.1 on Page 177, the last term is no larger than

$$\lim_{k \to \infty} \inf \int |s_n - s_k| \, d\mu < \varepsilon$$

whenever n is large enough. Since ε is arbitrary, this shows $I(f) = \int f d\mu$ as claimed.

As explained above, I can be regarded as an extension of $\int d\mu$ so from now on, the usual symbol, $\int d\mu$ will be used. It is now easy to verify $\int d\mu$ is linear on $L^1(\Omega)$.

8.8 Approximation With Simple Functions

The next theorem says the integral as defined above is linear and also gives a way to compute the integral in terms of real and imaginary parta. In addition, functions in L^1 can be approximated with simple functions.

Theorem 8.8.1 *$\int d\mu$ is linear on $L^1(\Omega)$ and $L^1(\Omega)$ is a complex vector space. If $f \in L^1(\Omega)$, then $\operatorname{Re} f, \operatorname{Im} f$, and $|f|$ are all in $L^1(\Omega)$. Furthermore, for $f \in L^1(\Omega)$,*

$$\int f d\mu = \int (\operatorname{Re} f)^+ d\mu - \int (\operatorname{Re} f)^- d\mu + i \left(\int (\operatorname{Im} f)^+ d\mu - \int (\operatorname{Im} f)^- d\mu \right),$$

and the triangle inequality holds,

$$\left| \int f d\mu \right| \leq \int |f| \, d\mu$$

Also for every $f \in L^1(\Omega)$, for every $\varepsilon > 0$ there exists a simple function s such that

$$\int |f - s| \, d\mu < \varepsilon.$$

Proof: First it is necessary to verify that $L^1(\Omega)$ is really a vector space because it makes no sense to speak of linear maps without having these maps defined on a vector space. Let f, g be in $L^1(\Omega)$ and let $a, b \in \mathbb{C}$. Then let $\{s_n\}$ and $\{t_n\}$ be sequences of complex simple functions associated with f and g respectively as described in Definition 8.7.4. Consider $\{as_n + bt_n\}$, another sequence of complex simple functions. Then $as_n(\omega) + bt_n(\omega) \to af(\omega) + bg(\omega)$ for each ω. Also, from Lemma 8.7.3

$$\int |as_n + bt_n - (as_m + bt_m)| \, d\mu \leq |a| \int |s_n - s_m| \, d\mu + |b| \int |t_n - t_m| \, d\mu$$

[2]The negative part of the real number x is defined to be $x^- \equiv \frac{1}{2}(|x| - x)$. Thus $|x| = x^+ + x^-$ and $x = x^+ - x^-$. .

and the sum of the two terms on the right converge to zero as $m, n \to \infty$. Thus $af + bg \in L^1(\Omega)$. Also

$$\begin{aligned}\int (af + bg)\, d\mu &\equiv \lim_{n\to\infty} \int (as_n + bt_n)\, d\mu \\ &= \lim_{n\to\infty} \left(a \int s_n d\mu + b \int t_n d\mu \right) \\ &= a \lim_{n\to\infty} \int s_n d\mu + b \lim_{n\to\infty} \int t_n d\mu \\ &= a \int f d\mu + b \int g d\mu.\end{aligned}$$

If $\{s_n\}$ is a sequence of complex simple functions described in Definition 8.7.4 corresponding to f, then $\{|s_n|\}$ is a sequence of complex simple functions satisfying the conditions of Definition 8.7.4 corresponding to $|f|$. This is because $|s_n(\omega)| \to |f(\omega)|$ and

$$\int ||s_n| - |s_m||\, d\mu \leq \int |s_m - s_n|\, d\mu$$

with this last expression converging to 0 as $m, n \to \infty$. Thus $|f| \in L^1(\Omega)$. Also, by similar reasoning, $\{\operatorname{Re} s_n\}$ and $\{\operatorname{Im} s_n\}$ correspond to $\operatorname{Re} f$ and $\operatorname{Im} f$ respectively in the manner described by Definition 8.7.4 showing that $\operatorname{Re} f$ and $\operatorname{Im} f$ are in $L^1(\Omega)$. Now

$$(\operatorname{Re} f)^+ = \frac{1}{2}(|\operatorname{Re} f| + \operatorname{Re} f)$$

and

$$(\operatorname{Re} f)^- = \frac{1}{2}(|\operatorname{Re} f| - \operatorname{Re} f)$$

so both of these functions are in $L^1(\Omega)$. Similar formulas establish that $(\operatorname{Im} f)^+$ and $(\operatorname{Im} f)^-$ are in $L^1(\Omega)$.

The formula follows from the observation that

$$f = (\operatorname{Re} f)^+ - (\operatorname{Re} f)^- + i\left((\operatorname{Im} f)^+ - (\operatorname{Im} f)^-\right)$$

and the fact shown first that $f \to \int f d\mu$ is linear.

To verify the triangle inequality, let $\{s_n\}$ be complex simple functions for f as in Definition 8.7.4. Then

$$\left|\int f d\mu\right| = \lim_{n\to\infty} \left|\int s_n d\mu\right| \leq \lim_{n\to\infty} \int |s_n|\, d\mu = \int |f|\, d\mu.$$

Now the last assertion follows from the definition. There exists a sequence of simple functions $\{s_n\}$ converging pointwise to f such that for all m, n large enough,

$$\frac{\varepsilon}{2} > \int |s_n - s_m|\, d\mu$$

Fix m and let $n \to \infty$. By Fatou's lemma

$$\varepsilon > \frac{\varepsilon}{2} \geq \liminf_{n\to\infty} \int |s_n - s_m|\, d\mu \geq \int |f - s_m|\, d\mu.$$

Let $s = s_m$. This proves the theorem.

Now here is an equivalent description of $L^1(\Omega)$ which is the version which will be used more often than not.

Corollary 8.8.2 *Let $(\Omega, \mathcal{S}, \mu)$ be a measure space and let $f : \Omega \to \mathbb{C}$. Then $f \in L^1(\Omega)$ if and only if f is measurable and $\int |f| \, d\mu < \infty$.*

Proof: Suppose $f \in L^1(\Omega)$. Then from Definition 8.7.4, it follows both real and imaginary parts of f are measurable. Just take real and imaginary parts of s_n and observe the real and imaginary parts of f are limits of the real and imaginary parts of s_n respectively. By Theorem 8.8.1 this shows the only if part.

The more interesting part is the if part. Suppose then that f is measurable and $\int |f| \, d\mu < \infty$. Suppose first that f has values in $[0, \infty)$. It is necessary to obtain the sequence of complex simple functions. By Theorem 7.7.12, there exists an increasing sequence of nonnegative simple functions, $\{s_n\}$ such that $s_n(\omega) \uparrow f(\omega)$. Then by the monotone convergence theorem,

$$\lim_{n\to\infty} \int (2f - (f - s_n)) \, d\mu = \int 2f d\mu$$

and so

$$\lim_{n\to\infty} \int (f - s_n) \, d\mu = 0.$$

Letting m be large enough, it follows $\int (f - s_m) \, d\mu < \varepsilon$ and so if $n > m$

$$\int |s_m - s_n| \, d\mu \leq \int |f - s_m| \, d\mu < \varepsilon.$$

Therefore, $f \in L^1(\Omega)$ because $\{s_n\}$ is a suitable sequence.

The general case follows from considering positive and negative parts of real and imaginary parts of f. These are each measurable and nonnegative and their integrals are finite so each is in $L^1(\Omega)$ by what was just shown. Since

$$f = \operatorname{Re} f^+ - \operatorname{Re} f^- + i\left(\operatorname{Im} f^+ - \operatorname{Im} f^-\right),$$

it follows $f \in L^1(\Omega)$. This proves the corollary.

One of the major theorems in this theory is the dominated convergence theorem. Before presenting it, here is a technical lemma about lim sup and lim inf.

Lemma 8.8.3 *Let $\{a_n\}$ be a sequence in $[-\infty, \infty]$. Then $\lim_{n\to\infty} a_n$ exists if and only if*

$$\liminf_{n\to\infty} a_n = \limsup_{n\to\infty} a_n$$

and in this case, the limit equals the common value of these two numbers.

Proof: Suppose first $\lim_{n\to\infty} a_n = a \in \mathbb{R}$. Then, letting $\varepsilon > 0$ be given, $a_n \in (a - \varepsilon, a + \varepsilon)$ for all n large enough, say $n \geq N$. Therefore, both $\inf\{a_k : k \geq n\}$ and $\sup\{a_k : k \geq n\}$ are contained in $[a - \varepsilon, a + \varepsilon]$ whenever $n \geq N$. It follows $\limsup_{n\to\infty} a_n$ and $\liminf_{n\to\infty} a_n$ are both in $[a - \varepsilon, a + \varepsilon]$, showing

$$\left|\liminf_{n\to\infty} a_n - \limsup_{n\to\infty} a_n\right| < 2\varepsilon.$$

Since ε is arbitrary, the two must be equal and they both must equal a. Next suppose $\lim_{n\to\infty} a_n = \infty$. Then if $l \in \mathbb{R}$, there exists N such that for $n \geq N$,

$$l \leq a_n$$

and therefore, for such n,

$$l \leq \inf\{a_k : k \geq n\} \leq \sup\{a_k : k \geq n\}$$

and this shows, since l is arbitrary that

$$\lim \inf_{n\to\infty} a_n = \lim \sup_{n\to\infty} a_n = \infty.$$

The case for $-\infty$ is similar.

Conversely, suppose $\lim\inf_{n\to\infty} a_n = \lim\sup_{n\to\infty} a_n = a$. Suppose first that $a \in \mathbb{R}$. Then, letting $\varepsilon > 0$ be given, there exists N such that if $n \geq N$,

$$\sup\{a_k : k \geq n\} - \inf\{a_k : k \geq n\} < \varepsilon$$

therefore, if $k, m > N$, and $a_k > a_m$,

$$|a_k - a_m| = a_k - a_m \leq \sup\{a_k : k \geq n\} - \inf\{a_k : k \geq n\} < \varepsilon$$

showing that $\{a_n\}$ is a Cauchy sequence. Therefore, it converges to $a \in \mathbb{R}$, and as in the first part, the $\lim\inf$ and $\lim\sup$ both equal a. If $\lim\inf_{n\to\infty} a_n = \lim\sup_{n\to\infty} a_n = \infty$, then given $l \in \mathbb{R}$, there exists N such that for $n \geq N$,

$$\inf_{n>N} a_n > l.$$

Therefore, $\lim_{n\to\infty} a_n = \infty$. The case for $-\infty$ is similar. This proves the lemma.

8.9 The Dominated Convergence Theorem

The dominated convergence theorem is one of the most important theorems in the theory of the integral. It is one of those big theorems which justifies the study of the Lebesgue integral.

Theorem 8.9.1 *(Dominated Convergence theorem) Let $f_n \in L^1(\Omega)$ and suppose*

$$f(\omega) = \lim_{n\to\infty} f_n(\omega),$$

and there exists a measurable function g, with values in $[0,\infty]$,[3] such that

$$|f_n(\omega)| \leq g(\omega) \text{ and } \int g(\omega)d\mu < \infty.$$

Then $f \in L^1(\Omega)$ and

$$0 = \lim_{n\to\infty} \int |f_n - f|\, d\mu = \lim_{n\to\infty} \left| \int f d\mu - \int f_n d\mu \right|$$

Proof: f is measurable by Theorem 7.7.3. Since $|f| \leq g$, it follows that

$$f \in L^1(\Omega) \text{ and } |f - f_n| \leq 2g.$$

By Fatou's lemma (Theorem 8.5.1),

$$\begin{aligned} \int 2g d\mu &\leq \lim \inf_{n\to\infty} \int 2g - |f - f_n| d\mu \\ &= \int 2g d\mu - \lim \sup_{n\to\infty} \int |f - f_n| d\mu. \end{aligned}$$

[3]Note that, since g is allowed to have the value ∞, it is not known that $g \in L^1(\Omega)$.

Subtracting $\int 2gd\mu$,

$$0 \leq -\limsup_{n\to\infty} \int |f - f_n| d\mu.$$

Hence

$$\begin{aligned} 0 &\geq \limsup_{n\to\infty} \left(\int |f - f_n| d\mu \right) \\ &\geq \liminf_{n\to\infty} \left(\int |f - f_n| d\mu \right) \geq \left| \int f d\mu - \int f_n d\mu \right| \geq 0. \end{aligned}$$

This proves the theorem by Lemma 8.8.3 because the lim sup and lim inf are equal.

Corollary 8.9.2 *Suppose $f_n \in L^1(\Omega)$ and $f(\omega) = \lim_{n\to\infty} f_n(\omega)$. Suppose also there exist measurable functions, g_n, g with values in $[0,\infty]$ such that $\lim_{n\to\infty} \int g_n d\mu = \int g d\mu$, $g_n(\omega) \to g(\omega)$ μ a.e. and both $\int g_n d\mu$ and $\int g d\mu$ are finite. Also suppose $|f_n(\omega)| \leq g_n(\omega)$. Then*

$$\lim_{n\to\infty} \int |f - f_n| \, d\mu = 0.$$

Proof: It is just like the above. This time $g + g_n - |f - f_n| \geq 0$ and so by Fatou's lemma,

$$\int 2g d\mu - \limsup_{n\to\infty} \int |f - f_n| \, d\mu =$$

$$\begin{aligned} &\liminf_{n\to\infty} \int (g_n + g) \, d\mu - \limsup_{n\to\infty} \int |f - f_n| \, d\mu \\ = \; &\liminf_{n\to\infty} \int ((g_n + g) - |f - f_n|) \, d\mu \geq \int 2g d\mu \end{aligned}$$

and so $-\limsup_{n\to\infty} \int |f - f_n| \, d\mu \geq 0$. Thus

$$\begin{aligned} 0 &\geq \limsup_{n\to\infty} \left(\int |f - f_n| d\mu \right) \\ &\geq \liminf_{n\to\infty} \left(\int |f - f_n| d\mu \right) \geq \left| \int f d\mu - \int f_n d\mu \right| \geq 0. \end{aligned}$$

This proves the corollary.

Definition 8.9.3

Let E be a measurable subset of Ω.

$$\int_E f d\mu \equiv \int f \mathcal{X}_E d\mu.$$

If $L^1(E)$ is written, the σ algebra is defined as

$$\{E \cap A : A \in \mathcal{F}\}$$

and the measure is μ restricted to this smaller σ algebra. Clearly, if $f \in L^1(\Omega)$, then

$$f\mathcal{X}_E \in L^1(E)$$

and if $f \in L^1(E)$, then letting $\tilde{f}$ be the 0 extension of f off of E, it follows $\tilde{f} \in L^1(\Omega)$.

8.10 Approximation With $C_c(Y)$

Let $(Y, \mathcal{F}, \mu)$ be a measure space where Y is a closed subset of X a finite dimensional normed vector space and $\mathcal{F} \supseteq \mathcal{B}(Y)$, the Borel sets in Y. Suppose also that $\mu(K) < \infty$ whenever K is a compact set in Y. By Theorem 7.4.6 it follows μ is regular. This regularilty of μ implies an important approximation result valid for any $f \in L^1(Y)$. It turns out that in this situation, for all $\varepsilon > 0$, there exists g a continuous function defined on Y with g equal to 0 outside some compact set and

$$\int |f - g| \, d\mu < \varepsilon.$$

Definition 8.10.1 *Let $\mathbf{f} : X \to Y$ where X is a normed vector space. Then the support of $\mathbf{f}$, denoted by $\operatorname{spt}(\mathbf{f})$ is the closure of the set where $\mathbf{f}$ is not equal to zero. Thus*

$$\operatorname{spt}(\mathbf{f}) \equiv \overline{\{\mathbf{x} : \mathbf{f}(\mathbf{x}) \neq \mathbf{0}\}}$$

Also, if U is an open set, $\mathbf{f} \in C_c(U)$ means $\mathbf{f}$ is continuous on U and $\operatorname{spt}(\mathbf{f}) \subseteq U$. Similarly $\mathbf{f} \in C_c^m(U)$ if $\mathbf{f}$ has m continuous derivatives and $\operatorname{spt}(\mathbf{f}) \subseteq U$ and $\mathbf{f} \in C_c^\infty(U)$ if $\operatorname{spt}(\mathbf{f}) \subseteq U$ and $\mathbf{f}$ has continuous derivatives of every order on U.

Lemma 8.10.2 *Let Y be a closed subset of X a finite dimensional normed vector space. Let $K \subseteq V$ where K is compact in Y and V is open in Y. Then there exists a continuous function $f : Y \to [0,1]$ such that $\operatorname{spt}(f) \subseteq V$, $f(\mathbf{x}) = 1$ for all $\mathbf{x} \in K$. If $(Y, \mathcal{F}, \mu)$ is a measure space with $\mathcal{F} \supseteq \mathcal{B}(Y)$ and $\mu(K) < \infty$, for every compact K, then if $\mu(E) < \infty$ where $E \in \mathcal{F}$, there exists a sequence of functions in $C_c(Y)$ $\{f_k\}$ such that*

$$\lim_{k\to\infty} \int_Y |f_k(\mathbf{x}) - \mathcal{X}_E(\mathbf{x})| \, d\mu = 0.$$

Proof: For each $\mathbf{x} \in K$, there exists $r_{\mathbf{x}}$ such that

$$D(\mathbf{x}, r_{\mathbf{x}}) \equiv \{\mathbf{y} \in Y : ||\mathbf{x} - \mathbf{y}|| \leq r_{\mathbf{x}}\} \subseteq V.$$

Since K is compact, there are finitely many balls, $\{B(\mathbf{x}_k, r_{\mathbf{x}_k})\}_{k=1}^m$ which cover K. Let $W = \cup_{k=1}^m B(\mathbf{x}_k, r_{\mathbf{x}_k})$. Since there are only finitely many of these,

$$\overline{W} = \cup_{k=1}^m D(\mathbf{x}, r_{\mathbf{x}})$$

and $\overline{W}$ is a compact subset of V because it is closed and bounded, being the finite union of closed and bounded sets. Now define

$$f(\mathbf{x}) \equiv \frac{\operatorname{dist}(\mathbf{x}, W^C)}{\operatorname{dist}(\mathbf{x}, W^C) + \operatorname{dist}(\mathbf{x}, K)}$$

The denominator is never equal to 0 because if $\operatorname{dist}(\mathbf{x}, K) = 0$ then since K is closed, $\mathbf{x} \in K$ and so since $K \subseteq W$, an open set, $\operatorname{dist}(\mathbf{x}, W^C) > 0$. Therefore, f is continuous. When $\mathbf{x} \in K$, $f(\mathbf{x}) = 1$. If $\mathbf{x} \notin W$, then $f(\mathbf{x}) = 0$ and so $\operatorname{spt}(f) \subseteq \overline{W} \subseteq V$. In the above situation the following notation is often used.

$$K \prec f \prec V. \tag{8.9}$$

It remains to prove the last assertion. By Theorem 7.4.6, μ is regular and so there exist compact sets, $\{K_k\}$ and open sets $\{V_k\}$ such that $V_k \supseteq V_{k+1}$, $K_k \subseteq K_{k+1}$ for all k, and

$$K_k \subseteq E \subseteq V_k, \ \mu(V_k \setminus K_k) < 2^{-k}.$$

From the first part of the lemma, there exists a sequence $\{f_k\}$ such that

$$K_k \prec f_k \prec V_k.$$

Then $f_k(\mathbf{x})$ converges to $\mathcal{X}_E(\mathbf{x})$ a.e. because if convergence fails to take place, then $\mathbf{x}$ must be in infinitely many of the sets $V_k \setminus K_k$. Thus $\mathbf{x}$ is in

$$\cap_{m=1}^{\infty} \cup_{k=m}^{\infty} V_k \setminus K_k$$

and for each p

$$\begin{aligned} \mu\left(\cap_{m=1}^{\infty} \cup_{k=m}^{\infty} V_k \setminus K_k\right) &\leq \mu\left(\cup_{k=p}^{\infty} V_k \setminus K_k\right) \\ &\leq \sum_{k=p}^{\infty} \mu\left(V_k \setminus K_k\right) \\ &< \sum_{k=p}^{\infty} \frac{1}{2^k} \leq 2^{-(p-1)} \end{aligned}$$

Now the functions are all bounded above by 1 and below by 0 and are equal to zero off V_1, a set of finite measure so by the dominated convergence theorem,

$$\lim_{k\to\infty} \int \left|\mathcal{X}_E(\mathbf{x}) - f_k(\mathbf{x})\right| d\mu = 0,$$

the dominating function being $\mathcal{X}_E(\mathbf{x}) + \mathcal{X}_{V_1}(\mathbf{x})$. This proves the lemma.

With this lemma, here is an important major theorem.

Theorem 8.10.3 *Let Y be a closed subset of X a finite dimensional normed vector space. Let $(Y, \mathcal{F}, \mu)$ be a measure space with $\mathcal{F} \supseteq \mathcal{B}(Y)$ and $\mu(K) < \infty$, for every compact K in Y. Let $f \in L^1(Y)$ and let $\varepsilon > 0$ be given. Then there exists $g \in C_c(Y)$ such that*

$$\int_Y \left|f(\mathbf{x}) - g(\mathbf{x})\right| d\mu < \varepsilon.$$

Proof: By considering separately the positive and negative parts of the real and imaginary parts of f it suffices to consider only the case where $f \geq 0$. Then by Theorem 7.7.12 and the monotone convergence theorem, there exists a simple function,

$$s(\mathbf{x}) \equiv \sum_{m=1}^{p} c_m \mathcal{X}_{E_m}(\mathbf{x}),\ s(\mathbf{x}) \leq f(\mathbf{x})$$

such that

$$\int \left|f(\mathbf{x}) - s(\mathbf{x})\right| d\mu < \varepsilon/2.$$

By Lemma 8.10.2, there exists $\{h_{mk}\}_{k=1}^{\infty}$ be functions in $C_c(Y)$ such that

$$\lim_{k\to\infty} \int_Y \left|\mathcal{X}_{E_m} - f_{mk}\right| d\mu = 0.$$

Let

$$g_k(\mathbf{x}) \equiv \sum_{m=1}^{p} c_m h_{mk}.$$

Thus for k large enough,

$$\begin{aligned}\int |s(\mathbf{x}) - g_k(\mathbf{x})|\, d\mu &= \int \left| \sum_{m=1}^{p} c_m \left(\mathcal{X}_{E_m} - h_{mk}\right) \right| d\mu \\ &\leq \sum_{m=1}^{p} c_m \int |\mathcal{X}_{E_m} - h_{mk}|\, d\mu < \varepsilon/2\end{aligned}$$

Thus for k this large,

$$\begin{aligned}\int |f(\mathbf{x}) - g_k(\mathbf{x})|\, d\mu &\leq \int |f(\mathbf{x}) - s(\mathbf{x})|\, d\mu + \int |s(\mathbf{x}) - g_k(\mathbf{x})|\, d\mu \\ &< \varepsilon/2 + \varepsilon/2 = \varepsilon.\end{aligned}$$

This proves the theorem.

People think of this theorem as saying that f is approximated by g_k in $L^1(Y)$. It is customary to consider functions in $L^1(Y)$ as vectors and the norm of such a vector is given by

$$||f||_1 \equiv \int |f(\mathbf{x})|\, d\mu.$$

You should verify this mostly satisfies the axioms of a norm. The problem comes in asserting $f = 0$ if $||f|| = 0$ which strictly speaking is false. However, the other axioms of a norm do hold.

8.11 The One Dimensional Lebesgue Integral

Let F be an increasing function defined on $\mathbb{R}$. Let μ be the Lebesgue Stieltjes measure defined in Theorems 7.6.1 and 7.2.1. The conclusions of these theorems are reviewed here.

Theorem 8.11.1 *Let F be an increasing function defined on $\mathbb{R}$, an integrator function. There exists a function $\mu : \mathcal{P}(\mathbb{R}) \to [0, \infty]$ which satisfies the following properties.*

1. *If $A \subseteq B$, then $0 \leq \mu(A) \leq \mu(B), \mu(\emptyset) = 0$.*
2. *$\mu(\cup_{k=1}^{\infty} A_i) \leq \sum_{i=1}^{\infty} \mu(A_i)$*
3. *$\mu([a,b]) = F(b+) - F(a-)$,*
4. *$\mu((a,b)) = F(b-) - F(a+)$*
5. *$\mu((a,b]) = F(b+) - F(a+)$*
6. *$\mu([a,b)) = F(b-) - F(a-)$ where*

$$F(b+) \equiv \lim_{t \to b+} F(t), F(b-) \equiv \lim_{t \to a-} F(t).$$

There also exists a σ algebra $\mathcal{S}$ of measurable sets on which μ is a measure which contains the open sets and also satisfies the regularity conditions,

$$\mu(E) = \sup\{\mu(K) : K \text{ is a closed and bounded set, } K \subseteq E\} \tag{8.10}$$

$$\mu(E) = \inf\{\mu(V) : V \text{ is an open set, } V \supseteq E\} \tag{8.11}$$

whenever E is a set in $\mathcal{S}$.

The Lebesgue integral taken with respect to this measure, is called the Lebesgue Stieltjes integral. Note that any real valued continuous function is measurable with respect to $\mathcal{S}$. This is because if f is continuous, inverse images of open sets are open and open sets are in $\mathcal{S}$. Thus f is measurable because $f^{-1}((a,b)) \in \mathcal{S}$. Similarly if f has complex values this argument applied to its real and imaginary parts yields the conclusion that f is measurable.

For f a continuous function, how does the Lebesgue Stieltjes integral compare with the Darboux Stieltjes integral? To answer this question, here is a technical lemma.

Lemma 8.11.2 *Let D be a countable subset of $\mathbb{R}$ and suppose $a, b \notin D$. Also suppose f is a continuous function defined on $[a,b]$. Then there exists a sequence of functions $\{s_n\}$ of the form*

$$s_n(x) \equiv \sum_{k=1}^{m_n} f\left(z_{k-1}^n\right) \mathcal{X}_{[z_{k-1}^n, z_k^n)}(x)$$

such that each $z_k^n \notin D$ and

$$\sup\{|s_n(x) - f(x)| : x \in [a,b]\} < 1/n.$$

Proof: First note that D contains no intervals. To see this let $D = \{d_k\}_{k=1}^{\infty}$. If D has an interval of length 2ε, let I_k be an interval centered at d_k which has length $\varepsilon/2^k$. Therefore, the sum of the lengths of these intervals is no more than

$$\sum_{k=1}^{\infty} \frac{\varepsilon}{2^k} = \varepsilon.$$

Thus D cannot contain an interval of length 2ε. Since ε is arbitrary, D cannot contain any interval.

Since f is continuous, it follows from Theorem 5.4.2 on Page 90 that f is uniformly continuous. Therefore, there exists $\delta > 0$ such that if $|x - y| \leq 3\delta$, then

$$|f(x) - f(y)| < 1/n$$

Now let $\{x_0, \cdots, x_{m_n}\}$ be a partition of $[a,b]$ such that $|x_i - x_{i-1}| < \delta$ for each i. For $k = 1, 2, \cdots, m_n - 1$, let $z_k^n \notin D$ and $|z_k^n - x_k| < \delta$. Then

$$\left|z_k^n - z_{k-1}^n\right| \leq |z_k^n - x_k| + |x_k - x_{k-1}| + \left|x_{k-1} - z_{k-1}^n\right| < 3\delta.$$

It follows that for each $x \in [a,b]$

$$\left|\sum_{k=1}^{m_n} f\left(z_{k-1}^n\right) \mathcal{X}_{[z_{k-1}^n, z_k^n)}(x) - f(x)\right| < 1/n.$$

This proves the lemma.

Proposition 8.11.3 *Let f be a continuous function defined on $\mathbb{R}$. Also let F be an increasing function defined on $\mathbb{R}$. Then whenever c, d are not points of discontinuity of F and $[a,b] \supseteq [c,d]$,*

$$\int_a^b f \mathcal{X}_{[c,d]} dF = \int f d\mu$$

Here μ is the Lebesgue Stieltjes measure defined above.

Proof: Since F is an increasing function it can have only countably many discontinuities. The reason for this is that the only kind of discontinuity it can have is where $F(x+) > F(x-)$. Now since F is increasing, the intervals $(F(x-), F(x+))$ for x a point of discontinuity are disjoint and so since each must contain a rational number and the rational numbers are countable, and therefore so are these intervals.

Let D denote this countable set of discontinuities of F. Then if $l, r \notin D, [l,r] \subseteq [a,b]$, it follows quickly from the definition of the Darboux Stieltjes integral that

$$\begin{aligned}\int_a^b \mathcal{X}_{[l,r)} dF &= F(r) - F(l) = F(r-) - F(l-) \\ &= \mu([l,r)) = \int \mathcal{X}_{[l,r)} d\mu.\end{aligned}$$

Now let $\{s_n\}$ be the sequence of step functions of Lemma 8.11.2 such that these step functions converge uniformly to f on $[c,d]$ Then

$$\left| \int \left(\mathcal{X}_{[c,d]} f - \mathcal{X}_{[c,d]} s_n \right) d\mu \right| \leq \int \left| \mathcal{X}_{[c,d]} (f - s_n) \right| d\mu \leq \frac{1}{n} \mu([c,d])$$

and

$$\left| \int_a^b \left(\mathcal{X}_{[c,d]} f - \mathcal{X}_{[c,d]} s_n \right) dF \right| \leq \int_a^b \mathcal{X}_{[c,d]} |f - s_n| \, dF < \frac{1}{n} (F(b) - F(a)).$$

Also if s_n is given by the formula of Lemma 8.11.2,

$$\begin{aligned}\int \mathcal{X}_{[c,d]} s_n d\mu &= \int \sum_{k=1}^{m_n} f\left(z_{k-1}^n\right) \mathcal{X}_{[z_{k-1}^n, z_k^n)} d\mu \\ &= \sum_{k=1}^{m_n} \int f\left(z_{k-1}^n\right) \mathcal{X}_{[z_{k-1}^n, z_k^n)} d\mu \\ &= \sum_{k=1}^{m_n} f\left(z_{k-1}^n\right) \mu\left([z_{k-1}^n, z_k^n)\right) \\ &= \sum_{k=1}^{m_n} f\left(z_{k-1}^n\right) \left(F(z_k^n -) - F\left(z_{k-1}^n -\right)\right) \\ &= \sum_{k=1}^{m_n} f\left(z_{k-1}^n\right) \left(F(z_k^n) - F\left(z_{k-1}^n\right)\right) \\ &= \sum_{k=1}^{m_n} \int_a^b f\left(z_{k-1}^n\right) \mathcal{X}_{[z_{k-1}^n, z_k^n)} dF = \int_a^b s_n dF.\end{aligned}$$

Therefore,

$$\begin{aligned}\left| \int \mathcal{X}_{[c,d]} f d\mu - \int_a^b \mathcal{X}_{[c,d]} f dF \right| &\leq \left| \int \mathcal{X}_{[c,d]} f d\mu - \int \mathcal{X}_{[c,d]} s_n d\mu \right| \\ &\quad + \left| \int \mathcal{X}_{[c,d]} s_n d\mu - \int_a^b s_n dF \right| + \left| \int_a^b s_n dF - \int_a^b \mathcal{X}_{[c,d]} f dF \right| \\ &\leq \frac{1}{n} \mu([c,d]) + \frac{1}{n} (F(b) - F(a))\end{aligned}$$

and since n is arbitrary, this shows

$$\int f d\mu - \int_a^b f dF = 0.$$

This proves the theorem.

In particular, in the special case where F is continuous and f is continuous,

$$\int_a^b f dF = \int \mathcal{X}_{[a,b]} f d\mu.$$

Thus, if $F(x) = x$ so the Darboux Stieltjes integral is the usual integral from calculus,

$$\int_a^b f(t)\, dt = \int \mathcal{X}_{[a,b]} f d\mu$$

where μ is the measure which comes from $F(x) = x$ as described above. This measure is often denoted by m. Thus when f is continuous

$$\int_a^b f(t)\, dt = \int \mathcal{X}_{[a,b]} f dm$$

and so there is no problem in writing

$$\int_a^b f(t)\, dt$$

for either the Lebesgue or the Riemann integral. Furthermore, when f is continuous, you can compute the Lebesgue integral by using the fundamental theorem of calculus because in this case, the two integrals are equal.

8.12 Exercises

1. Let $\Omega = \mathbb{N} = \{1, 2, \cdots\}$. Let $\mathcal{F} = \mathcal{P}(\mathbb{N})$, the set of all subsets of $\mathbb{N}$, and let $\mu(S) =$ number of elements in S. Thus $\mu(\{1\}) = 1 = \mu(\{2\})$, $\mu(\{1, 2\}) = 2$, etc. Show $(\Omega, \mathcal{F}, \mu)$ is a measure space. It is called counting measure. What functions are measurable in this case? For a nonnegative function, f defined on $\mathbb{N}$, show

$$\int_{\mathbb{N}} f d\mu = \sum_{k=1}^{\infty} f(k)$$

What do the monotone convergence and dominated convergence theorems say about this example?

2. For the measure space of Problem 1, give an example of a sequence of nonnegative measurable functions $\{f_n\}$ converging pointwise to a function f, such that inequality is obtained in Fatou's lemma.

3. For a measurable nonnegative function, f, the integral was defined as

$$\sup_{\delta > h > 0} \sum_{i=1}^{\infty} h\mu([f > ih])$$

Show this is the same as

$$\int_0^\infty \mu\left([f > t]\right) dt$$

where this integral is just the improper Riemann integral defined by

$$\int_0^\infty \mu\left([f > t]\right) dt = \lim_{R\to\infty} \int_0^R \mu\left([f > t]\right) dt.$$

4. Using the Problem 3, show that for s a nonnegative simple function, $s(\omega) = \sum_{i=1}^n c_i \mathcal{X}_{E_i}(\omega)$ where $0 < c_1 < c_2 \cdots < c_n$ and the sets, E_k are disjoint,

$$\int s d\mu = \sum_{i=1}^n c_i \mu\left(E_i\right).$$

Give an easy proof of this, much easier than the one presented earlier.

5. If $(\Omega, \mathcal{F}, \mu)$ is a measure space and $f \geq 0$ is measurable, show that if $g(\omega) = f(\omega)$ a.e. ω and $g \geq 0$, then $\int g d\mu = \int f d\mu$. Show that if $f, g \in L^1(\Omega)$ and $g(\omega) = f(\omega)$ a.e. then $\int g d\mu = \int f d\mu$.

6. An algebra $\mathcal{A}$ of subsets of Ω is a subset of the power set such that Ω is in the algebra and for $A, B \in \mathcal{A}$, $A \setminus B$ and $A \cup B$ are both in $\mathcal{A}$. Let $\mathcal{C} \equiv \{E_i\}_{i=1}^\infty$ be a countable collection of sets and let $\Omega_1 \equiv \cup_{i=1}^\infty E_i$. Show there exists an algebra of sets, $\mathcal{A}$, such that $\mathcal{A} \supseteq \mathcal{C}$ and $\mathcal{A}$ is countable. Note the difference between this problem and Problem 5. **Hint:** Let $\mathcal{C}_1$ denote all finite unions of sets of $\mathcal{C}$ and Ω_1. Thus $\mathcal{C}_1$ is countable. Now let $\mathcal{B}_1$ denote all complements with respect to Ω_1 of sets of $\mathcal{C}_1$. Let $\mathcal{C}_2$ denote all finite unions of sets of $\mathcal{B}_1 \cup \mathcal{C}_1$. Continue in this way, obtaining an increasing sequence $\mathcal{C}_n$, each of which is countable. Let

$$\mathcal{A} \equiv \cup_{i=1}^\infty \mathcal{C}_i.$$

7. Let $\mathcal{A} \subseteq \mathcal{P}(\Omega)$ where $\mathcal{P}(\Omega)$ denotes the set of all subsets of Ω. Let $\sigma(\mathcal{A})$ denote the intersection of all σ algebras which contain $\mathcal{A}$, one of these being $\mathcal{P}(\Omega)$. Show $\sigma(\mathcal{A})$ is also a σ algebra.

8. We say a function g mapping a normed vector space, Ω to a normed vector space is Borel measurable if whenever U is open, $g^{-1}(U)$ is a Borel set. (The Borel sets are those sets in the smallest σ algebra which contains the open sets.) Let $f : \Omega \to X$ and let $g : X \to Y$ where X is a normed vector space and Y equals $\mathbb{C}, \mathbb{R}$, or $(-\infty, \infty]$ and $\mathcal{F}$ is a σ algebra of sets of Ω. Suppose f is measurable and g is Borel measurable. Show $g \circ f$ is measurable.

9. Let $(\Omega, \mathcal{F}, \mu)$ be a measure space. Define $\overline{\mu} : \mathcal{P}(\Omega) \to [0, \infty]$ by

$$\overline{\mu}(A) = \inf\{\mu(B) : B \supseteq A,\ B \in \mathcal{F}\}.$$

Show $\overline{\mu}$ satisfies

$$\begin{aligned} \overline{\mu}(\emptyset) &= 0, \text{ if } A \subseteq B,\ \overline{\mu}(A) \leq \overline{\mu}(B), \\ \overline{\mu}(\cup_{i=1}^\infty A_i) &\leq \sum_{i=1}^\infty \overline{\mu}(A_i),\ \mu(A) = \overline{\mu}(A) \text{ if } A \in \mathcal{F}. \end{aligned}$$

If $\overline{\mu}$ satisfies these conditions, it is called an outer measure. This shows every measure determines an outer measure on the power set.

10. Let $\{E_i\}$ be a sequence of measurable sets with the property that

$$\sum_{i=1}^{\infty} \mu(E_i) < \infty.$$

Let $S = \{\omega \in \Omega \text{ such that } \omega \in E_i \text{ for infinitely many values of } i\}$. Show $\mu(S) = 0$ and S is measurable. This is part of the Borel Cantelli lemma. **Hint:** Write S in terms of intersections and unions. Something is in S means that for every n there exists $k > n$ such that it is in E_k. Remember the tail of a convergent series is small.

11. $\uparrow$ Let $\{f_n\}, f$ be measurable functions with values in $\mathbb{C}$. $\{f_n\}$ converges in measure if

$$\lim_{n\to\infty} \mu(x \in \Omega : |f(x) - f_n(x)| \geq \varepsilon) = 0$$

for each fixed $\varepsilon > 0$. Prove the theorem of F. Riesz. If f_n converges to f in measure, then there exists a subsequence $\{f_{n_k}\}$ which converges to f a.e. **Hint:** Choose n_1 such that

$$\mu(x : |f(x) - f_{n_1}(x)| \geq 1) < 1/2.$$

Choose $n_2 > n_1$ such that

$$\mu(x : |f(x) - f_{n_2}(x)| \geq 1/2) < 1/2^2,$$

$n_3 > n_2$ such that

$$\mu(x : |f(x) - f_{n_3}(x)| \geq 1/3) < 1/2^3,$$

etc. Now consider what it means for $f_{n_k}(x)$ to fail to converge to $f(x)$. Then use Problem 10.

12. Suppose (Ω, μ) is a finite measure space ($\mu(\Omega) < \infty$) and $\mathfrak{S} \subseteq L^1(\Omega)$. Then $\mathfrak{S}$ is said to be uniformly integrable if for every $\varepsilon > 0$ there exists $\delta > 0$ such that if E is a measurable set satisfying $\mu(E) < \delta$, then

$$\int_E |f| \, d\mu < \varepsilon$$

for all $f \in \mathfrak{S}$. Show $\mathfrak{S}$ is uniformly integrable and bounded in $L^1(\Omega)$ if there exists an increasing function h which satisfies

$$\lim_{t\to\infty} \frac{h(t)}{t} = \infty, \ \sup\left\{\int_\Omega h(|f|) \, d\mu : f \in \mathfrak{S}\right\} < \infty.$$

$\mathfrak{S}$ is bounded if there is some number, M such that

$$\int |f| \, d\mu \leq M$$

for all $f \in \mathfrak{S}$.

13. Let $(\Omega, \mathcal{F}, \mu)$ be a measure space and suppose $f, g : \Omega \to (-\infty, \infty]$ are measurable. Prove the sets

$$\{\omega : f(\omega) < g(\omega)\} \text{ and } \{\omega : f(\omega) = g(\omega)\}$$

are measurable. **Hint:** The easy way to do this is to write

$$\{\omega : f(\omega) < g(\omega)\} = \cup_{r\in\mathbb{Q}} [f < r] \cap [g > r].$$

Note that $l(x, y) = x - y$ is not continuous on $(-\infty, \infty]$ so the obvious idea doesn't work.

14. Let $\{f_n\}$ be a sequence of real or complex valued measurable functions. Let

$$S = \{\omega : \{f_n(\omega)\} \text{ converges}\}.$$

Show S is measurable. **Hint:** You might try to exhibit the set where f_n converges in terms of countable unions and intersections using the definition of a Cauchy sequence.

15. Suppose $u_n(t)$ is a differentiable function for $t \in (a,b)$ and suppose that for $t \in (a,b)$,

$$|u_n(t)|,\ |u_n'(t)| < K_n$$

where $\sum_{n=1}^{\infty} K_n < \infty$. Show

$$(\sum_{n=1}^{\infty} u_n(t))' = \sum_{n=1}^{\infty} u_n'(t).$$

Hint: This is an exercise in the use of the dominated convergence theorem and the mean value theorem.

16. Let E be a countable subset of $\mathbb{R}$. Show $m(E) = 0$. **Hint:** Let the set be $\{e_i\}_{i=1}^{\infty}$ and let e_i be the center of an open interval of length $\varepsilon/2^i$.

17. $\uparrow$ If S is an uncountable set of irrational numbers, is it necessary that S has a rational number as a limit point? **Hint:** Consider the proof of Problem 16 when applied to the rational numbers. (This problem was shown to me by Lee Erlebach.)

18. Suppose $\{f_n\}$ is a sequence of nonnegative measurable functions defined on a measure space, $(\Omega, \mathcal{S}, \mu)$. Show that

$$\int \sum_{k=1}^{\infty} f_k d\mu = \sum_{k=1}^{\infty} \int f_k d\mu.$$

Hint: Use the monotone convergence theorem along with the fact the integral is linear.

19. The integral $\int_{-\infty}^{\infty} f(t)\,dt$ will denote the Lebesgue integral taken with respect to one dimensional Lebesgue measure as discussed earlier. Show that for $\alpha > 0, t \to e^{-\alpha t^2}$ is in $L^1(\mathbb{R})$. The gamma function is defined for $x > 0$ as

$$\Gamma(x) \equiv \int_0^{\infty} e^{-t} t^{x-1} dt$$

Show $t \to e^{-t}t^{x-1}$ is in $L^1(\mathbb{R})$ for all $x > 0$. Also show that

$$\Gamma(x+1) = x\Gamma(x),\ \Gamma(1) = 1.$$

How does $\Gamma(n)$ for n an integer compare with $(n-1)!$?

20. This problem outlines a treatment of Stirling's formula which is a very useful approximation to $n!$ based on a section in [33]. It is an excellent application of the monotone convergence theorem. Follow and justify the following steps using the convergence theorems for the Lebesgue integral as needed. Here $x > 0$.

$$\Gamma(x+1) = \int_0^{\infty} e^{-t} t^x dt$$

First change the variables letting $t = x(1+u)$ to get $\Gamma(x+1) =$

$$e^{-x}x^{x+1}\int_{-1}^{\infty}\left(e^{-u}(1+u)\right)^{x}du$$

Next make the change of variables $u = s\sqrt{\frac{2}{x}}$ to obtain $\Gamma(x+1) =$

$$\sqrt{2}e^{-x}x^{x+(1/2)}\int_{-\sqrt{\frac{x}{2}}}^{\infty}\left(e^{-s\sqrt{\frac{2}{x}}}\left(1+s\sqrt{\frac{2}{x}}\right)\right)^{x}ds$$

The integrand is increasing in x. This is most easily seen by taking ln of the integrand and then taking the derivative with respect to x. This derivative is positive. Next show the limit of the integrand as $x \to \infty$ is e^{-s^2}. This isn't too bad if you take ln and then use L'Hospital's rule. Consider the integral. Explain why it must be increasing in x. Next justify the following assertion. Remember the monotone convergence theorem applies to a sequence of functions.

$$\lim_{x\to\infty}\int_{-\sqrt{\frac{x}{2}}}^{\infty}\left(e^{-s\sqrt{\frac{2}{x}}}\left(1+s\sqrt{\frac{2}{x}}\right)\right)^{x}ds = \int_{-\infty}^{\infty}e^{-s^2}ds$$

Now Stirling's formula is

$$\lim_{x\to\infty}\frac{\Gamma(x+1)}{\sqrt{2}e^{-x}x^{x+(1/2)}} = \int_{-\infty}^{\infty}e^{-s^2}ds$$

where this last improper integral equals a well defined constant (why?). It is very easy, when you know something about multiple integrals of functions of more than one variable to verify this constant is $\sqrt{\pi}$ but the necessary mathematical machinery has not yet been presented. It can also be done through much more difficult arguments in the context of functions of only one variable. See [33] for these clever arguments.

21. To show you the power of Stirling's formula, find whether the series

$$\sum_{n=1}^{\infty}\frac{n!e^n}{n^n}$$

converges. The ratio test falls flat but you can try it if you like. Now explain why, if n is large enough

$$n! \geq \frac{1}{2}\left(\int_{-\infty}^{\infty}e^{-s^2}ds\right)\sqrt{2}e^{-n}n^{n+(1/2)} \equiv c\sqrt{2}e^{-n}n^{n+(1/2)}.$$

Use this.

22. The Riemann integral is only defined for functions which are bounded which are also defined on a bounded interval. If either of these two criteria are not satisfied, then the integral is not the Riemann integral. Suppose f is Riemann integrable on a bounded interval, $[a,b]$. Show that it must also be Lebesgue integrable with respect to one dimensional Lebesgue measure and the two integrals coincide.

23. Give a theorem in which the improper Riemann integral coincides with a suitable Lebesgue integral. (There are many such situations just find one.)

24. Note that $\int_0^\infty \frac{\sin x}{x} dx$ is a valid improper Riemann integral defined by

$$\lim_{R\to\infty} \int_0^R \frac{\sin x}{x} dx$$

but this function, $\sin x/x$ is not in $L^1([0,\infty))$. Why?

25. Let f be a nonnegative strictly decreasing function defined on $[0,\infty)$. For $0 \le y \le f(0)$, let $f^{-1}(y) = x$ where $y \in [f(x+), f(x-)]$. (Draw a picture. f could have jump discontinuities.) Show that f^{-1} is nonincreasing and that

$$\int_0^\infty f(t)\, dt = \int_0^{f(0)} f^{-1}(y)\, dy.$$

Hint: Use the distribution function description.

26. Consider the following nested sequence of compact sets, $\{P_n\}$. We let $P_1 = [0,1]$, $P_2 = \left[0, \frac{1}{3}\right] \cup \left[\frac{2}{3}, 1\right]$, etc. To go from P_n to P_{n+1}, delete the open interval which is the middle third of each closed interval in P_n. Let $P = \cap_{n=1}^\infty P_n$. Since P is the intersection of nested nonempty compact sets, it follows from advanced calculus that $P \neq \emptyset$. Show $m(P) = 0$. Show there is a one to one onto mapping of $[0,1]$ to P. The set P is called the Cantor set. Thus, although P has measure zero, it has the same number of points in it as $[0,1]$ in the sense that there is a one to one and onto mapping from one to the other. **Hint:** There are various ways of doing this last part but the most enlightenment is obtained by exploiting the construction of the Cantor set.

27. ↑ Consider the sequence of functions defined in the following way. Let $f_1(x) = x$ on $[0,1]$. To get from f_n to f_{n+1}, let $f_{n+1} = f_n$ on all intervals where f_n is constant. If f_n is nonconstant on $[a,b]$, let $f_{n+1}(a) = f_n(a)$, $f_{n+1}(b) = f_n(b)$, f_{n+1} is piecewise linear and equal to $\frac{1}{2}(f_n(a) + f_n(b))$ on the middle third of $[a,b]$. Sketch a few of these and you will see the pattern. The process of modifying a nonconstant section of the graph of this function is illustrated in the following picture.

Show $\{f_n\}$ converges uniformly on $[0,1]$. If $f(x) = \lim_{n\to\infty} f_n(x)$, show that $f(0) = 0$, $f(1) = 1$, f is continuous, and $f'(x) = 0$ for all $x \notin P$ where P is the Cantor set. This function is called the Cantor function.It is a very important example to remember. Note it has derivative equal to zero a.e. and yet it succeeds in climbing from 0 to 1. Thus

$$\int_0^1 f'(t)\, dt = 0 \neq f(1) - f(0).$$

Is this somehow contradictory to the fundamental theorem of calculus? **Hint:** This isn't too hard if you focus on getting a careful estimate on the difference between two successive functions in the list considering only a typical small interval in which the change takes place. The above picture should be helpful.

28. Let $m(W) > 0$, W is measurable, $W \subseteq [a,b]$. Show there exists a nonmeasurable subset of W. **Hint**: Let $x \sim y$ if $x - y \in \mathbb{Q}$. Observe that $\sim$ is an equivalence relation on $\mathbb{R}$. See Definition 2.1.9 on Page 15 for a review of this terminology. Let $\mathcal{C}$ be the set of equivalence classes and let $\mathcal{D} \equiv \{C \cap W : C \in \mathcal{C} \text{ and } C \cap W \neq \emptyset\}$. By the

axiom of choice, there exists a set, A, consisting of exactly one point from each of the nonempty sets which are the elements of $\mathcal{D}$. Show

$$W \subseteq \cup_{r\in\mathbb{Q}} A + r \tag{a.}$$

$$A + r_1 \cap A + r_2 = \emptyset \text{ if } r_1 \neq r_2,\ r_i \in \mathbb{Q}. \tag{b.}$$

Observe that since $A \subseteq [a,b]$, then $A + r \subseteq [a-1, b+1]$ whenever $|r| < 1$. Use this to show that if $m(A) = 0$, or if $m(A) > 0$ a contradiction results.Show there exists some set, S such that $\overline{m}(S) < \overline{m}(S \cap A) + \overline{m}(S \setminus A)$ where $\overline{m}$ is the outer measure determined by m.

29. ↑ This problem gives a very interesting example found in the book by McShane [30]. Let $g(x) = x + f(x)$ where f is the strange function of Problem 27. Let P be the Cantor set of Problem 26. Let $[0,1] \setminus P = \cup_{j=1}^{\infty} I_j$ where I_j is open and $I_j \cap I_k = \emptyset$ if $j \neq k$. These intervals are the connected components of the complement of the Cantor set. Show $m(g(I_j)) = m(I_j)$ so

$$m(g(\cup_{j=1}^{\infty} I_j)) = \sum_{j=1}^{\infty} m(g(I_j)) = \sum_{j=1}^{\infty} m(I_j) = 1.$$

Thus $m(g(P)) = 1$ because $g([0,1]) = [0,2]$. By Problem 28 there exists a set, $A \subseteq g(P)$ which is non measurable. Define $\phi(x) = \mathcal{X}_A(g(x))$. Thus $\phi(x) = 0$ unless $x \in P$. Tell why ϕ is measurable. (Recall $m(P) = 0$ and Lebesgue measure is complete.) Now show that $\mathcal{X}_A(y) = \phi(g^{-1}(y))$ for $y \in [0,2]$. Tell why g^{-1} is continuous but $\phi \circ g^{-1}$ is not measurable. (This is an example of measurable $\circ$ continuous $\neq$ measurable.) Show there exist Lebesgue measurable sets which are not Borel measurable. **Hint**: The function, ϕ is Lebesgue measurable. Now show that Borel $\circ$ measurable = measurable.

30. If A is $m\lfloor S$ measurable, it does not follow that A is m measurable. Give an example to show this is the case.

31. If f is a nonnegative Lebesgue measurable function, show there exists g a Borel measurable function such that $g(x) = f(x)$ a.e.

The Lebesgue Integral For Functions Of n Variables

9.1 π Systems

The approach to n dimensional Lebesgue measure will be based on a very elegant idea due to Dynkin.

Definition 9.1.1 *Let Ω be a set and let $\mathcal{K}$ be a collection of subsets of Ω. Then $\mathcal{K}$ is called a π system if $\emptyset, \Omega \in \mathcal{K}$ and whenever $A, B \in \mathcal{K}$, it follows $A \cap B \in \mathcal{K}$.*

For example, if $\mathbb{R}^n = \Omega$, an example of a π system would be the set of all open sets. Another example would be sets of the form $\prod_{k=1}^n A_k$ where A_k is a Lebesgue measurable set.

The following is the fundamental lemma which shows these π systems are useful.

Lemma 9.1.2 *Let $\mathcal{K}$ be a π system of subsets of Ω, a set. Also let $\mathcal{G}$ be a collection of subsets of Ω which satisfies the following three properties.*

1. *$\mathcal{K} \subseteq \mathcal{G}$*
2. *If $A \in \mathcal{G}$, then $A^C \in \mathcal{G}$*
3. *If $\{A_i\}_{i=1}^\infty$ is a sequence of disjoint sets from $\mathcal{G}$ then $\cup_{i=1}^\infty A_i \in \mathcal{G}$.*

Then $\mathcal{G} \supseteq \sigma(\mathcal{K})$, where $\sigma(\mathcal{K})$ is the smallest σ algebra which contains $\mathcal{K}$.

Proof: First note that if

$$\mathcal{H} \equiv \{\mathcal{G} : 1 \text{ - } 3 \text{ all hold}\}$$

then $\cap\mathcal{H}$ yields a collection of sets which also satisfies 1 - 3. Therefore, I will assume in the argument that $\mathcal{G}$ is the smallest collection of sets satisfying 1 - 3, the intersection of all such collections. Let $A \in \mathcal{K}$ and define

$$\mathcal{G}_A \equiv \{B \in \mathcal{G} : A \cap B \in \mathcal{G}\}.$$

I want to show $\mathcal{G}_A$ satisfies 1 - 3 because then it must equal $\mathcal{G}$ since $\mathcal{G}$ is the smallest collection of subsets of Ω which satisfies 1 - 3. This will give the conclusion that for $A \in \mathcal{K}$ and $B \in \mathcal{G}$, $A \cap B \in \mathcal{G}$. This information will then be used to show that if $A, B \in \mathcal{G}$ then $A \cap B \in \mathcal{G}$. From this it will follow very easily that $\mathcal{G}$ is a σ algebra which will imply it contains $\sigma(\mathcal{K})$. Now here are the details of the argument.

Since $\mathcal{K}$ is given to be a π system, $\mathcal{K} \subseteq \mathcal{G}_A$. Property 3 is obvious because if $\{B_i\}$ is a sequence of disjoint sets in $\mathcal{G}_A$, then

$$A \cap \cup_{i=1}^{\infty} B_i = \cup_{i=1}^{\infty} A \cap B_i \in \mathcal{G}$$

because $A \cap B_i \in \mathcal{G}$ and the property 3 of $\mathcal{G}$.

It remains to verify Property 2 so let $B \in \mathcal{G}_A$. I need to verify that $B^C \in \mathcal{G}_A$. In other words, I need to show that $A \cap B^C \in \mathcal{G}$. However,

$$A \cap B^C = \left(A^C \cup (A \cap B)\right)^C \in \mathcal{G}$$

Here is why. Since $B \in \mathcal{G}_A$, $A \cap B \in \mathcal{G}$ and since $A \in \mathcal{K} \subseteq \mathcal{G}$ it follows $A^C \in \mathcal{G}$. It follows the union of the disjoint sets, A^C and $(A \cap B)$ is in $\mathcal{G}$ and then from 2 the complement of their union is in $\mathcal{G}$. Thus $\mathcal{G}_A$ satisfies 1 - 3 and this implies since $\mathcal{G}$ is the smallest such, that $\mathcal{G}_A \supseteq \mathcal{G}$. However, $\mathcal{G}_A$ is constructed as a subset of $\mathcal{G}$ and so $\mathcal{G} = \mathcal{G}_A$. This proves that for every $B \in \mathcal{G}$ and $A \in \mathcal{K}$, $A \cap B \in \mathcal{G}$. Now pick $B \in \mathcal{G}$ and consider

$$\mathcal{G}_B \equiv \{A \in \mathcal{G} : A \cap B \in \mathcal{G}\}.$$

I just proved $\mathcal{K} \subseteq \mathcal{G}_B$. The other arguments are identical to show $\mathcal{G}_B$ satisfies 1 - 3 and is therefore equal to $\mathcal{G}$. This shows that whenever $A, B \in \mathcal{G}$ it follows $A \cap B \in \mathcal{G}$.

This implies $\mathcal{G}$ is a σ algebra. To show this, all that is left is to verify $\mathcal{G}$ is closed under countable unions because then it follows $\mathcal{G}$ is a σ algebra. Let $\{A_i\} \subseteq \mathcal{G}$. Then let $A_1' = A_1$ and

$$\begin{aligned} A_{n+1}' &\equiv A_{n+1} \setminus (\cup_{i=1}^{n} A_i) \\ &= A_{n+1} \cap \left(\cap_{i=1}^{n} A_i^C\right) \\ &= \cap_{i=1}^{n} \left(A_{n+1} \cap A_i^C\right) \in \mathcal{G} \end{aligned}$$

because finite intersections of sets of $\mathcal{G}$ are in $\mathcal{G}$. Since the A_i' are disjoint, it follows

$$\cup_{i=1}^{\infty} A_i = \cup_{i=1}^{\infty} A_i' \in \mathcal{G}$$

Therefore, $\mathcal{G} \supseteq \sigma(\mathcal{K})$ because it is a σ algebra which contains $\mathcal{K}$ and this proves the Lemma.

9.2 n Dimensional Lebesgue Measure And Integrals

9.2.1 Iterated Integrals

Let m denote one dimensional Lebesgue measure. That is, it is the Lebesgue Stieltjes measure which comes from the integrator function, $F(x) = x$. Also let the σ algebra of measurable sets be denoted by $\mathcal{F}$. Recall this σ algebra contained the open sets. Also from the construction given above,

$$m([a,b]) = m((a,b)) = b - a$$

Definition 9.2.1 *Let f be a function of n variables and consider the symbol*

$$\int \cdots \int f(x_1, \cdots, x_n)\, dx_{i_1} \cdots dx_{i_n}. \tag{9.1}$$

where $(i_1, \cdots, i_n)$ is a permutation of the integers $\{1, 2, \cdots, n\}$. The symbol means to first do the Lebesgue integral

$$\int f(x_1, \cdots, x_n)\, dx_{i_1}$$

yielding a function of the other $n-1$ variables given above. Then you do

$$\int \left(\int f(x_1, \cdots, x_n)\, dx_{i_1} \right) dx_{i_2}$$

and continue this way. The iterated integral is said to make sense if the process just described makes sense at each step. Thus, to make sense, it is required

$$x_{i_1} \to f(x_1, \cdots, x_n)$$

can be integrated. Either the function has values in $[0, \infty]$ and is measurable or it is a function in L^1. Then it is required

$$x_{i_2} \to \int f(x_1, \cdots, x_n)\, dx_{i_1}$$

can be integrated and so forth. The symbol in 9.1 is called an iterated integral.

With the above explanation of iterated integrals, it is now time to define n dimensional Lebesgue measure.

9.2.2 n Dimensional Lebesgue Measure And Integrals

With the Lemma about π systems given above and the monotone convergence theorem, it is possible to give a very elegant and fairly easy definition of the Lebesgue integral of a function of n real variables. This is done in the following proposition.

Proposition 9.2.2 *There exists a σ algebra of sets of $\mathbb{R}^n$ which contains the open sets, $\mathcal{F}^n$ and a measure m_n defined on this σ algebra such that if $f : \mathbb{R}^n \to [0, \infty)$ is measurable with respect to $\mathcal{F}^n$ then for any permutation $(i_1, \cdots, i_n)$ of $\{1, \cdots, n\}$ it follows*

$$\int_{\mathbb{R}^n} f dm_n = \int \cdots \int f(x_1, \cdots, x_n)\, dx_{i_1} \cdots dx_{i_n} \tag{9.2}$$

In particular, this implies that if A_i is Lebesgue measurable for each $i = 1, \cdots, n$ then

$$m_n \left(\prod_{i=1}^n A_i \right) = \prod_{i=1}^n m(A_i).$$

Proof: Define a π system as

$$\mathcal{K} \equiv \left\{ \prod_{i=1}^n A_i : A_i \text{ is Lebesgue measurable} \right\}$$

Also let $R_p \equiv [-p, p]^n$, the n dimensional rectangle having sides $[-p, p]$. A set $F \subseteq \mathbb{R}^n$ will be said to satisfy property $\mathcal{P}$ if for every $p \in \mathbb{N}$ and any two permutations of $\{1, 2, \cdots, n\}$, $(i_1, \cdots, i_n)$ and $(j_1, \cdots, j_n)$ the two iterated integrals

$$\int \cdots \int \mathcal{X}_{R_p \cap F} dx_{i_1} \cdots dx_{i_n}, \int \cdots \int \mathcal{X}_{R_p \cap F} dx_{j_1} \cdots dx_{j_n}$$

make sense and are equal. Now define $\mathcal{G}$ to be those subsets of $\mathbb{R}^n$ which have property $\mathcal{P}$.

Thus $\mathcal{K} \subseteq \mathcal{G}$ because if $(i_1, \cdots, i_n)$ is any permutation of $\{1, 2, \cdots, n\}$ and

$$A = \prod_{i=1}^n A_i \in \mathcal{K}$$

then

$$\int \cdots \int \mathcal{X}_{R_p \cap A} dx_{i_1} \cdots dx_{i_n} = \prod_{i=1}^{n} m\left([-p,p] \cap A_i\right).$$

Now suppose $F \in \mathcal{G}$ and let $(i_1, \cdots, i_n)$ and $(j_1, \cdots, j_n)$ be two permutations. Then

$$R_p = \left(R_p \cap F^C\right) \cup \left(R_p \cap F\right)$$

and so

$$\int \cdots \int \mathcal{X}_{R_p \cap F^C} dx_{i_1} \cdots dx_{i_n} = \int \cdots \int \left(\mathcal{X}_{R_p} - \mathcal{X}_{R_p \cap F}\right) dx_{i_1} \cdots dx_{i_n}$$

Since $R_p \in \mathcal{G}$ the iterated integrals on the right and hence on the left make sense. Then continuing with the expression on the right and using that $F \in \mathcal{G}$, it equals

$$\begin{aligned} & (2p)^n - \int \cdots \int \mathcal{X}_{R_p \cap F} dx_{i_1} \cdots dx_{i_n} \\ = \; & (2p)^n - \int \cdots \int \mathcal{X}_{R_p \cap F} dx_{j_1} \cdots dx_{j_n} \\ = \; & \int \cdots \int \left(\mathcal{X}_{R_p} - \mathcal{X}_{R_p \cap F}\right) dx_{j_1} \cdots dx_{j_n} \\ = \; & \int \cdots \int \mathcal{X}_{R_p \cap F^C} dx_{j_1} \cdots dx_{j_n} \end{aligned}$$

which shows that if $F \in \mathcal{G}$ then so is F^C.

Next suppose $\{F_i\}_{i=1}^{\infty}$ is a sequence of disjoint sets in $\mathcal{G}$. Let $F = \cup_{i=1}^{\infty} F_i$. I need to show $F \in \mathcal{G}$. Since the sets are disjoint,

$$\int \cdots \int \mathcal{X}_{R_p \cap F} dx_{i_1} \cdots dx_{i_n} = \int \cdots \int \sum_{k=1}^{\infty} \mathcal{X}_{R_p \cap F_k} dx_{i_1} \cdots dx_{i_n}$$

$$= \int \cdots \int \lim_{N \to \infty} \sum_{k=1}^{N} \mathcal{X}_{R_p \cap F_k} dx_{i_1} \cdots dx_{i_n}$$

Do the iterated integrals make sense? Note that the iterated integral makes sense for $\sum_{k=1}^{N} \mathcal{X}_{R_p \cap F_k}$ as the integrand because it is just a finite sum of functions for which the iterated integral makes sense. Therefore,

$$x_{i_1} \to \sum_{k=1}^{\infty} \mathcal{X}_{R_p \cap F_k}(\mathbf{x})$$

is measurable and by the monotone convergence theorem,

$$\int \sum_{k=1}^{\infty} \mathcal{X}_{R_p \cap F_k}(\mathbf{x})\, dx_{i_1} = \lim_{N \to \infty} \int \sum_{k=1}^{N} \mathcal{X}_{R_p \cap F_k} dx_{i_1}$$

Now each of the functions,

$$x_{i_2} \to \int \sum_{k=1}^{N} \mathcal{X}_{R_p \cap F_k} dx_{i_1}$$

is measurable and so the limit of these functions,

$$\int \sum_{k=1}^{\infty} \mathcal{X}_{R_p \cap F_k}(\mathbf{x})\, dx_{i_1}$$

is also measurable. Therefore, one can do another integral to this function. Continuing this way using the monotone convergence theorem, it follows the iterated integral makes sense. The same reasoning shows the iterated integral makes sense for any other permutation.

Now applying the monotone convergence theorem as needed,

$$\int \cdots \int \mathcal{X}_{R_p \cap F} dx_{i_1} \cdots dx_{i_n} = \int \cdots \int \sum_{k=1}^{\infty} \mathcal{X}_{R_p \cap F_k} dx_{i_1} \cdots dx_{i_n}$$

$$\begin{aligned} &= \int \cdots \int \lim_{N \to \infty} \sum_{k=1}^{N} \mathcal{X}_{R_p \cap F_k} dx_{i_1} \cdots dx_{i_n} \\ &= \int \cdots \int \lim_{N \to \infty} \sum_{k=1}^{N} \int \mathcal{X}_{R_p \cap F_k} dx_{i_1} \cdots dx_{i_n} \end{aligned}$$

$$\begin{aligned} &= \int \cdots \lim_{N \to \infty} \sum_{k=1}^{N} \int \int \mathcal{X}_{R_p \cap F_k} dx_{i_1} \cdots dx_{i_n} \cdots \\ &= \lim_{N \to \infty} \sum_{k=1}^{N} \int \cdots \int \mathcal{X}_{R_p \cap F_k} dx_{i_1} \cdots dx_{i_n} \\ &= \lim_{N \to \infty} \sum_{k=1}^{N} \int \cdots \int \mathcal{X}_{R_p \cap F_k} dx_{j_1} \cdots dx_{j_n} \end{aligned}$$

the last step holding because each $F_k \in \mathcal{G}$. Then repeating the steps above in the opposite order, this equals

$$\int \cdots \int \sum_{k=1}^{\infty} \mathcal{X}_{R_p \cap F_k} dx_{j_1} \cdots dx_{j_n} = \int \cdots \int \mathcal{X}_{R_p \cap F} dx_{j_1} \cdots dx_{j_n}$$

Thus $F \in \mathcal{G}$. By Lemma 9.1.2 $\mathcal{G} \supseteq \sigma(\mathcal{K})$.

Let $\mathcal{F}^n = \sigma(\mathcal{K})$. Each set of the form $\prod_{k=1}^{n} U_k$ where U_k is an open set is in $\mathcal{K}$. Also every open set in $\mathbb{R}^n$ is a countable union of open sets of this form. This follows from Lemma 7.7.7 on Page 166. Therefore, every open set is in $\mathcal{F}^n$.

For $F \in \mathcal{F}^n$ define

$$m_n(F) \equiv \lim_{p \to \infty} \int \cdots \int \mathcal{X}_{R_p \cap F} dx_{j_1} \cdots dx_{j_n}$$

where $(j_1, \cdots, j_n)$ is a permutation of $\{1, \cdots, n\}$. It doesn't matter which one. It was shown above they all give the same result. I need to verify m_n is a measure. Let $\{F_k\}$ be a sequence of disjoint sets of $\mathcal{F}^n$.

$$m_n(\cup_{k=1}^{\infty} F_k) = \lim_{p \to \infty} \int \cdots \int \sum_{k=1}^{\infty} \mathcal{X}_{R_p \cap F_k} dx_{j_1} \cdots dx_{j_n}.$$

Using the monotone convergence theorem repeatedly as in the first part of the argument, this equals

$$\sum_{k=1}^{\infty} \lim_{p \to \infty} \int \cdots \int \mathcal{X}_{R_p \cap F_k} dx_{j_1} \cdots dx_{j_n} \equiv \sum_{k=1}^{\infty} m_n(F_k).$$

Thus m_n is a measure. Now letting A_k be a Lebesgue measurable set,

$$\begin{aligned} m_n\left(\prod_{k=1}^n A_k\right) &= \lim_{p\to\infty}\int\cdots\int\prod_{k=1}^n \mathcal{X}_{[-p,p]\cap A_k}(x_k)\,dx_{j_1}\cdots dx_{j_n} \\ &= \lim_{p\to\infty}\prod_{k=1}^n m([-p,p]\cap A_k) = \prod_{k=1}^n m(A_k). \end{aligned}$$

It only remains to prove 9.2.

It was shown above that for $F\in\mathcal{F}$ it follows

$$\int_{\mathbb{R}^n}\mathcal{X}_F dm_n = \lim_{p\to\infty}\int\cdots\int\mathcal{X}_{R_p\cap F}dx_{j_1}\cdots dx_{j_n}$$

Applying the monotone convergence theorem repeatedly on the right, this yields that the iterated integral makes sense and

$$\int_{\mathbb{R}^n}\mathcal{X}_F dm_n = \int\cdots\int\mathcal{X}_F dx_{j_1}\cdots dx_{j_n}$$

It follows 9.2 holds for every nonnegative simple function in place of f because these are just linear combinations of functions, $\mathcal{X}_F$. Now taking an increasing sequence of nonnegative simple functions, $\{s_k\}$ which converges to a measurable nonnegative function f

$$\begin{aligned} \int_{\mathbb{R}^n} f dm_n &= \lim_{k\to\infty}\int_{\mathbb{R}^n} s_k dm_n \\ &= \lim_{k\to\infty}\int\cdots\int s_k dx_{j_1}\cdots dx_{j_n} \\ &= \int\cdots\int f dx_{j_1}\cdots dx_{j_n} \end{aligned}$$

This proves the proposition.

9.2.3 Fubini's Theorem

Formula 9.2 is often called Fubini's theorem. So is the following theorem. In general, people tend to refer to theorems about the equality of iterated integrals as Fubini's theorem and in fact Fubini did produce such theorems but so did Tonelli and some of these theorems presented here and above should be called Tonelli's theorem.

Theorem 9.2.3 *Let m_n be defined in Proposition 9.2.2 on the σ algebra of sets $\mathcal{F}^n$ given there. Suppose $f\in L^1(\mathbb{R}^n)$. Then if $(i_1,\cdots,i_n)$ is any permutation of $\{1,\cdots,n\}$,*

$$\int_{\mathbb{R}^n} f dm_n = \int\cdots\int f(\mathbf{x})\,dx_{i_1}\cdots dx_{i_n}.$$

In particular, iterated integrals for any permutation of $\{1,\cdots,n\}$ are all equal.

Proof: It suffices to prove this for f having real values because if this is shown the general case is obtained by taking real and imaginary parts. Since $f\in L^1(\mathbb{R}^n)$,

$$\int_{\mathbb{R}^n}|f|\,dm_n<\infty$$

and so both $\frac{1}{2}(|f|+f)$ and $\frac{1}{2}(|f|-f)$ are in $L^1(\mathbb{R}^n)$ and are each nonnegative. Hence from Proposition 9.2.2,

$$\begin{aligned}\int_{\mathbb{R}^n} f dm_n &= \int_{\mathbb{R}^n} \left[\frac{1}{2}(|f|+f) - \frac{1}{2}(|f|-f)\right] dm_n \\ &= \int_{\mathbb{R}^n} \frac{1}{2}(|f|+f)\, dm_n - \int_{\mathbb{R}^n} \frac{1}{2}(|f|-f)\, dm_n\end{aligned}$$

$$\begin{aligned}&= \int\cdots\int \frac{1}{2}(|f(\mathbf{x})|+f(\mathbf{x}))\, dx_{i_1}\cdots dx_{i_n} \\ &\quad - \int\cdots\int \frac{1}{2}(|f(\mathbf{x})|-f(\mathbf{x}))\, dx_{i_1}\cdots dx_{i_n}\end{aligned}$$

$$\begin{aligned}&= \int\cdots\int \frac{1}{2}(|f(\mathbf{x})|+f(\mathbf{x})) - \frac{1}{2}(|f(\mathbf{x})|-f(\mathbf{x}))\, dx_{i_1}\cdots dx_{i_n} \\ &= \int\cdots\int f(\mathbf{x})\, dx_{i_1}\cdots dx_{i_n}\end{aligned}$$

This proves the theorem.

The following corollary is a convenient way to verify the hypotheses of the above theorem.

Corollary 9.2.4 *Suppose f is measurable with respect to $\mathcal{F}^n$ and suppose for some permutation, $(i_1,\cdots,i_n)$*

$$\int\cdots\int |f(\mathbf{x})|\, dx_{i_1}\cdots dx_{i_n} < \infty$$

Then $f \in L^1(\mathbb{R}^n)$.

Proof: By Proposition 9.2.2,

$$\int_{\mathbb{R}^n} |f|\, dm_n = \int\cdots\int |f(\mathbf{x})|\, dx_{i_1}\cdots dx_{i_n} < \infty$$

and so f is in $L^1(\mathbb{R}^n)$ by Corollary 8.8.2. This proves the corollary.

The following theorem is a summary of the above specialized to Borel sets along with an assertion about regularity.

Theorem 9.2.5 *Let $\mathcal{B}(\mathbb{R}^n)$ be the Borel sets on $\mathbb{R}^n$. There exists a measure m_n defined on $\mathcal{B}(\mathbb{R}^n)$ such that if f is a nonnegative Borel measurable function,*

$$\int_{\mathbb{R}^n} f dm_n = \int\cdots\int f(\mathbf{x})\, dx_{i_1}\cdots dx_{i_n} \tag{9.3}$$

whenever $(i_1,\cdots,i_n)$ is a permutation of $\{1.\cdots,n\}$. If $f \in L^1(\mathbb{R}^n)$ and f is Borel measurable, then the above equation holds for f and all integrals make sense. If f is Borel measurable and for some $(i_1,\cdots,i_n)$ a permutation of $\{1.\cdots,n\}$

$$\int\cdots\int |f(\mathbf{x})|\, dx_{i_1}\cdots dx_{i_n} < \infty,$$

then $f \in L^1(\mathbb{R}^n)$. The measure m_n is both inner and outer regular on the Borel sets. That is, if $E \in \mathcal{B}(\mathbb{R}^n)$,

$$m_n(E) = \sup\{m_n(K) : K \subseteq E \text{ and } K \text{ is compact}\}$$

$$m_n(E) = \inf\{m_n(V) : V \supseteq E \text{ and } V \text{ is open}\}.$$

Also if A_k is a Borel set in $\mathbb{R}$ then

$$\prod_{k=1}^{n} A_k$$

is a Borel set in $\mathbb{R}^n$ and

$$m_n\left(\prod_{k=1}^{n} A_k\right) = \prod_{k=1}^{n} m(A_k).$$

Proof: Most of it was shown earlier since $\mathcal{B}(\mathbb{R}^n) \subseteq \mathcal{F}^n$. The two assertions about regularity follow from observing that m_n is finite on compact sets and then using Theorem 7.4.6. It remains to show the assertion about the product of Borel sets. If each A_k is open, there is nothing to show because the result is an open set. Suppose then that whenever $A_1, \cdots, A_m, m \leq n$ are open, the product, $\prod_{k=1}^{n} A_k$ is a Borel set. Let $\mathcal{K}$ be the open sets in $\mathbb{R}$ and let $\mathcal{G}$ be those Borel sets such that if $A_m \in \mathcal{G}$ it follows $\prod_{k=1}^{n} A_k$ is Borel. Then $\mathcal{K}$ is a π system and is contained in $\mathcal{G}$. Now suppose $F \in \mathcal{G}$. Then

$$\begin{aligned} &\left(\prod_{k=1}^{m-1} A_k \times F \times \prod_{k=m+1}^{n} A_k\right) \cup \left(\prod_{k=1}^{m-1} A_k \times F^C \times \prod_{k=m+1}^{n} A_k\right) \\ = &\left(\prod_{k=1}^{m-1} A_k \times \mathbb{R} \times \prod_{k=m+1}^{n} A_k\right) \end{aligned}$$

and by assumption this is of the form

$$B \cup A = D.$$

where B, A are disjoint and B and D are Borel. Therefore, $A = D \setminus B$ which is a Borel set. Thus $\mathcal{G}$ is closed with respect to complements. If $\{F_i\}$ is a sequence of disjoint elements of $\mathcal{G}$

$$\left(\prod_{k=1}^{m-1} A_k \times \cup_i F_i \times \prod_{k=m+1}^{n} A_k\right) = \cup_{i=1}^{\infty}\left(\prod_{k=1}^{m-1} A_k \times F_i \times \prod_{k=m+1}^{n} A_k\right)$$

which is a countable union of Borel sets and is therefore, Borel. Hence $\mathcal{G}$ is also closed with respect to countable unions of disjoint sets. Thus by the Lemma on π systems $\mathcal{G} \supseteq \sigma(\mathcal{K}) = \mathcal{B}(\mathbb{R})$ and this shows that A_m can be any Borel set. Thus the assertion about the product is true if only $A_1, \cdots, A_{m-1}$ are open while the rest are Borel. Continuing this way shows the assertion remains true for each A_i being Borel. Now the final formula about the measure of a product follows from 9.3.

$$\begin{aligned} \int_{\mathbb{R}^n} \mathcal{X}_{\prod_{k=1}^{n} A_k} dm_n &= \int \cdots \int \mathcal{X}_{\prod_{k=1}^{n} A_k}(\mathbf{x})\, dx_1 \cdots dx_n \\ &= \int \cdots \int \prod_{k=1}^{n} \mathcal{X}_{A_k}(x_k)\, dx_1 \cdots dx_n = \prod_{k=1}^{n} m(A_k). \end{aligned}$$

This proves the theorem.

Of course iterated integrals can often be used to compute the Lebesgue integral. Sometimes the iterated integral taken in one order will allow you to compute the Lebesgue integral and it does not work well in the other order. Here is a simple example.

Example 9.2.6 *Find the iterated integral*

$$\int_0^1 \int_x^1 \frac{\sin(y)}{y} dydx$$

Notice the limits. The iterated integral equals

$$\int_{\mathbb{R}^2} \mathcal{X}_A(x,y) \frac{\sin(y)}{y} dm_2$$

where

$$A = \{(x,y) : x \leq y \text{ where } x \in [0,1]\}$$

Fubini's theorem can be applied because the function $(x,y) \to \sin(y)/y$ is continuous except at $y = 0$ and can be redefined to be continuous there. The function is also bounded so

$$(x,y) \to \mathcal{X}_A(x,y) \frac{\sin(y)}{y}$$

clearly is in $L^1(\mathbb{R}^2)$. Therefore,

$$\begin{aligned} \int_{\mathbb{R}^2} \mathcal{X}_A(x,y) \frac{\sin(y)}{y} dm_2 &= \int\int \mathcal{X}_A(x,y) \frac{\sin(y)}{y} dxdy \\ &= \int_0^1 \int_0^y \frac{\sin(y)}{y} dxdy \\ &= \int_0^1 \sin(y)\, dy = 1 - \cos(1) \end{aligned}$$

9.3 Exercises

1. Find $\int_0^2 \int_0^{6-2z} \int_{\frac{1}{2}x}^{3-z} (3-z)\cos\left(y^2\right)\, dy\, dx\, dz$.
2. Find $\int_0^1 \int_0^{18-3z} \int_{\frac{1}{3}x}^{6-z} (6-z)\exp\left(y^2\right)\, dy\, dx\, dz$.
3. Find $\int_0^2 \int_0^{24-4z} \int_{\frac{1}{4}y}^{6-z} (6-z)\exp\left(x^2\right)\, dx\, dy\, dz$.
4. Find $\int_0^1 \int_0^{12-4z} \int_{\frac{1}{4}y}^{3-z} \frac{\sin x}{x}\, dx\, dy\, dz$.
5. Find $\int_0^{20} \int_0^1 \int_{\frac{1}{5}y}^{5-z} \frac{\sin x}{x}\, dx\, dz\, dy + \int_{20}^{25} \int_0^{5-\frac{1}{5}y} \int_{\frac{1}{5}y}^{5-z} \frac{\sin x}{x}\, dx\, dz\, dy$. **Hint:** You might try doing it in the order, $dy\, dx\, dz$
6. Explain why for each $t > 0, x \to e^{-tx}$ is a function in $L^1(\mathbb{R})$ and

$$\int_0^\infty e^{-tx} dx = \frac{1}{t}.$$

Thus

$$\int_0^R \frac{\sin(t)}{t} dt = \int_0^R \int_0^\infty \sin(t)\, e^{-tx} dxdt$$

Now explain why you can change the order of integration in the above iterated integral. Then compute what you get. Next pass to a limit as $R \to \infty$ and show

$$\int_0^\infty \frac{\sin(t)}{t} dt = \frac{1}{2}\pi$$

7. Explain why $\int_a^\infty f(t)\,dt \equiv \lim_{r\to\infty}\int_a^r f(t)\,dt$ whenever $f \in L^1(a,\infty)$; that is $f\mathcal{X}_{[a,\infty)} \in L^1(\mathbb{R})$.

8. $B(p,q) = \int_0^1 x^{p-1}(1-x)^{q-1}dx, \Gamma(p) = \int_0^\infty e^{-t}t^{p-1}dt$ for $p,q>0$. The first of these is called the beta function, while the second is the gamma function. Show a.) $\Gamma(p+1) = p\Gamma(p)$; b.) $\Gamma(p)\Gamma(q) = B(p,q)\Gamma(p+q)$.Explain why the gamma function makes sense for any $p>0$.

9. Let $f(y) = g(y) = |y|^{-1/2}$if $y \in (-1,0)\cup(0,1)$ and $f(y) = g(y) = 0$ if $y \notin (-1,0)\cup(0,1)$. For which values of x does it make sense to write the integral $\int_{\mathbb{R}} f(x-y)\,g(y)\,dy$?

10. Let $\{a_n\}$ be an increasing sequence of numbers in $(0,1)$ which converges to 1. Let g_n be a nonnegative function which equals zero outside (a_n, a_{n+1}) such that $\int g_n dx = 1$. Now for $(x,y) \in [0,1)\times[0,1)$ define

$$f(x,y) \equiv \sum_{k=1}^{\infty} g_n(y)\,(g_n(x) - g_{n+1}(x)).$$

Explain why this is actually a finite sum for each such (x,y) so there are no convergence questions in the infinite sum. Explain why f is a continuous function on $[0,1)\times[0,1)$. You can extend f to equal zero off $[0,1)\times[0,1)$ if you like. Show the iterated integrals exist but are not equal. In fact, show

$$\int_0^1\int_0^1 f(x,y)\,dydx = 1 \neq 0 = \int_0^1\int_0^1 f(x,y)\,dxdy.$$

Does this example contradict the Fubini theorem? Explain why or why not.

9.4 Lebesgue Measure On $\mathbb{R}^n$

The σ algebra of Lebesgue measurable sets is larger than the above σ algebra of Borel sets or of the earlier σ algebra which came from an application of the π system lemma. It is convenient to use this larger σ algebra, especially when considering change of variables formulas, although it is certainly true that one can do most interesting theorems with the Borel sets only. However, it is in some ways easier to consider the more general situation and this will be done next.

Definition 9.4.1 *The completion of* $(\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n), m_n)$ *is the Lebesgue measure space. Denote this by* $(\mathbb{R}^n, \mathcal{F}_n, m_n)$.

Thus for each $E \in \mathcal{F}_n$,

$$m_n(E) = \inf\{m_n(F) : F \supseteq E \text{ and } F \in \mathcal{B}(\mathbb{R}^n)\}$$

It follows that for each $E \in \mathcal{F}_n$ there exists $F \in \mathcal{B}(\mathbb{R}^n)$ such that $F \supseteq E$ and

$$m_n(E) = m_n(F).$$

Theorem 9.4.2 *m_n is regular on $\mathcal{F}_n$. In fact, if $E \in \mathcal{F}_n$ there exist sets in $\mathcal{B}(\mathbb{R}^n)$, F, G such that*

$$F \subseteq E \subseteq G,$$

F is a countable union of compact sets, and G is a countable intersection of open sets and

$$m_n(G \setminus F) = 0.$$

If A_k is Lebesgue measurable then $\prod_{k=1}^n A_k \in \mathcal{F}_n$ and

$$m_n\left(\prod_{k=1}^n A_k\right) = \prod_{k=1}^n m(A_k).$$

In addition to this, m_n is translation invariant. This means that if $E \in \mathcal{F}_n$ and $\mathbf{x} \in \mathbb{R}^n$, then

$$m_n(\mathbf{x}+E) = m_n(E).$$

The expression $\mathbf{x}+E$ means $\{\mathbf{x}+\mathbf{e} : \mathbf{e} \in E\}$.

Proof: m_n is regular on $\mathcal{B}(\mathbb{R}^n)$ by Theorem 7.4.6 because it is finite on compact sets. Then from Theorem 7.5.7, it follows m_n is regular on $\mathcal{F}_n$. This is because the regularity of m_n on the Borel sets and the definition of the completion of of a measure given there implies the uniqueness part of Theorem 7.5.6 can be obtained to conclude

$$(\mathbb{R}^n, \mathcal{F}_n, m_n) = \left(\mathbb{R}^n, \overline{\mathcal{B}(\mathbb{R}^n)}, \overline{m_n}\right)$$

Now for $E \in \mathcal{F}_n$, let

$$E_m \equiv (B(\mathbf{0}, m) \setminus B(\mathbf{0}, m-1)) \cap E.$$

It follows from regularity there exists a sequence of open sets, $\{V_{mk}\}_{k=1}^\infty$ such that

$$V_{mk} \supseteq E_m$$

and

$$m_n(V_{mk}) - m_n(E_m) = m_n(V_{mk} \setminus E_m) < \left(2^{-k}\right)\left(2^{-m}\right).$$

Then

$$E = \cup_{m=1}^\infty E_m \subseteq \cup_{m=1}^\infty V_{mk} \equiv V_k$$

and

$$\left(\cap_{l=1}^k V_l\right) \setminus E \subseteq \cup_{m=1}^\infty V_{mk} \setminus E$$

so

$$m_n\left(\left(\cap_{l=1}^k V_l\right) \setminus E\right) \leq \sum_{m=1}^\infty m_n(V_{mk} \setminus E) < 2^{-k} \sum_{m=1}^\infty 2^{-m} \leq 2^{-k}.$$

Let $G \equiv \cap_{k=1}^\infty \cap_{l=1}^k V_l$. Then from the above, and passing to the limit, it follows

$$m_n(G \setminus E) = 0.$$

To obtain F a countable union of compact sets contained in E such that $m_n(E \setminus F) = 0$, let $K_{mk} \subseteq E_m$ such that $m_n(E_m \setminus K_{mk}) < \left(2^{-k}\right)(2^{-m})$.

$$m_n(E \setminus \cup_{m=1}^\infty K_{mk}) \leq m_n(\cup_{m=1}^\infty (E_m \setminus K_{mk})) < 2^{-k}$$

Let $F = \cup_{k=1}^\infty \cup_{m=1}^\infty K_{mk}$. Then

$$m_n(E \setminus F) \leq m_n(E \setminus \cup_{m=1}^\infty K_{mk}) < 2^{-k}$$

for every k and so $m_n(E \setminus F) = 0$.

Consider the next assertion about the measure of a Cartesian product. By regularity of m there exists $B_k, C_k \in \mathcal{B}(\mathbb{R}^n)$ such that $B_k \supseteq A_k \supseteq C_k$ and $m(B_k) = m(A_k) = m(C_k)$. In fact, you can have B_k equal a countable intersection of open sets and C_k a countable union of compact sets. Then

$$\begin{aligned}\prod_{k=1}^{n} m(A_k) &= \prod_{k=1}^{n} m(C_k) \leq m_n\left(\prod_{k=1}^{n} C_k\right)\\ &\leq m_n\left(\prod_{k=1}^{n} A_k\right) \leq m_n\left(\prod_{k=1}^{n} B_k\right)\\ &= \prod_{k=1}^{n} m(B_k) = \prod_{k=1}^{n} m(A_k).\end{aligned}$$

It remains to prove the claim about the measure being translation invariant.

Let $\mathcal{K}$ denote all sets of the form

$$\prod_{k=1}^{n} U_k$$

where each U_k is an open set in $\mathbb{R}$. Thus $\mathcal{K}$ is a π system.

$$\mathbf{x} + \prod_{k=1}^{n} U_k = \prod_{k=1}^{n} (x_k + U_k)$$

which is also a finite Cartesian product of finitely many open sets. Also,

$$\begin{aligned}m_n\left(\mathbf{x} + \prod_{k=1}^{n} U_k\right) &= m_n\left(\prod_{k=1}^{n} (x_k + U_k)\right)\\ &= \prod_{k=1}^{n} m(x_k + U_k)\\ &= \prod_{k=1}^{n} m(U_k) = m_n\left(\prod_{k=1}^{n} U_k\right)\end{aligned}$$

The step to the last line is obvious because an arbitrary open set in $\mathbb{R}$ is the disjoint union of open intervals and the lengths of these intervals are unchanged when they are slid to another location.

Now let $\mathcal{G}$ denote those Borel sets E with the property that for each $p \in \mathbb{N}$

$$m_n(\mathbf{x} + E \cap (-p,p)^n) = m_n(E \cap (-p,p)^n)$$

and the set, $\mathbf{x} + E \cap (-p,p)^n$ is a Borel set. Thus $\mathcal{K} \subseteq \mathcal{G}$. If $E \in \mathcal{G}$ then

$$\left(\mathbf{x} + E^C \cap (-p,p)^n\right) \cup (\mathbf{x} + E \cap (-p,p)^n) = \mathbf{x} + (-p,p)^n$$

which implies $\mathbf{x} + E^C \cap (-p,p)^n$ is a Borel set since it equals a difference of two Borel sets. Now consider the following.

$$\begin{aligned}&m_n\left(\mathbf{x} + E^C \cap (-p,p)^n\right) + m_n(E \cap (-p,p)^n)\\ =\ &m_n\left(\mathbf{x} + E^C \cap (-p,p)^n\right) + m_n(\mathbf{x} + E \cap (-p,p)^n)\\ =\ &m_n(\mathbf{x} + (-p,p)^n) = m_n((-p,p)^n)\end{aligned}$$

$$= m_n\left(E^C \cap (-p,p)^n\right) + m_n\left(E \cap (-p,p)^n\right)$$

which shows

$$m_n\left(\mathbf{x} + E^C \cap (-p,p)^n\right) = m_n\left(E^C \cap (-p,p)^n\right)$$

showing that $E^C \in \mathcal{G}$.

If $\{E_k\}$ is a sequence of disjoint sets of $\mathcal{G}$,

$$m_n\left(\mathbf{x} + \cup_{k=1}^{\infty} E_k \cap (-p,p)^n\right) = m_n\left(\cup_{k=1}^{\infty}\mathbf{x} + E_k \cap (-p,p)^n\right)$$

Now the sets $\{\mathbf{x} + E_k \cap (-p,p)^n\}$ are also disjoint and so the above equals

$$\begin{aligned}\sum_k m_n\left(\mathbf{x} + E_k \cap (-p,p)^n\right) &= \sum_k m_n\left(E_k \cap (-p,p)^n\right)\\ &= m_n\left(\cup_{k=1}^{\infty} E_k \cap (-p,p)^n\right)\end{aligned}$$

Thus $\mathcal{G}$ is also closed with respect to countable disjoint unions. It follows from the lemma on π systems that $\mathcal{G} \supseteq \sigma(\mathcal{K})$. But from Lemma 7.7.7 on Page 166, every open set is a countable union of sets of $\mathcal{K}$ and so $\sigma(\mathcal{K})$ contains the open sets. Therefore, $\mathcal{B}(\mathbb{R}^n) \supseteq \mathcal{G} \supseteq \sigma(\mathcal{K}) \supseteq \mathcal{B}(\mathcal{K})$ which shows $\mathcal{G} = \mathcal{B}(\mathbb{R}^n)$.

I have just shown that for every $E \in \mathcal{B}(\mathbb{R}^n)$, and any $p \in \mathbb{N}$,

$$m_n\left(\mathbf{x} + E \cap (-p,p)^n\right) = m_n\left(E \cap (-p,p)^n\right)$$

Taking the limit as $p \to \infty$ yields

$$m_n(\mathbf{x} + E) = m_n(E).$$

This proves translation invariance on Borel sets.

Now suppose $m_n(S) = 0$ so that S is a set of measure zero. From outer regularity, there exists a Borel set, F such that $F \supseteq S$ and $m_n(F) = 0$. Therefore from what was just shown,

$$m_n(\mathbf{x} + S) \leq m_n(\mathbf{x} + F) = m_n(F) = m_n(S) = 0$$

which shows that if $m_n(S) = 0$ then so does $m_n(\mathbf{x} + S)$. Let F be any set of $\mathcal{F}_n$. By regularity, there exists $E \supseteq F$ where $E \in \mathcal{B}(\mathbb{R}^n)$ and $m_n(E \setminus F) = 0$. Then

$$\begin{aligned}m_n(F) = m_n(E) = m_n(\mathbf{x} + E) = m_n(\mathbf{x} + (E \setminus F) \cup F)\\ = m_n(\mathbf{x} + E \setminus F) + m_n(\mathbf{x} + F) = m_n(\mathbf{x} + F).\end{aligned}$$

This proves the theorem.

9.5 Mollifiers

From Theorem 8.10.3, every function in $L^1(\mathbb{R}^n)$ can be approximated by one in $C_c(\mathbb{R}^n)$ but even more incredible things can be said. In fact, you can approximate an arbitrary function in $L^1(\mathbb{R}^n)$ with one which is infinitely differentiable having compact support. This is very important in partial differential equations. I am just giving a short introduction to this concept here. Consider the following example.

Example 9.5.1 *Let $U = B(\mathbf{z}, 2r)$*

$$\psi(\mathbf{x}) = \begin{cases} \exp\left[\left(|\mathbf{x}-\mathbf{z}|^2 - r^2\right)^{-1}\right] & \text{if } |\mathbf{x}-\mathbf{z}| < r,\\ 0 \text{ if } |\mathbf{x}-\mathbf{z}| \geq r. \end{cases}$$

Then a little work shows $\psi \in C_c^{\infty}(U)$. The following also is easily obtained.

You show this by verifying the partial derivatives all exist and are continuous. The only place this is hard is when $|\mathbf{x}-\mathbf{z}| = r$. It is left as an exercise. You might consider a simpler example,

$$f(x) = \begin{cases} e^{-1/x^2} \text{ if } x \neq 0 \\ 0 \text{ if } x = 0 \end{cases}$$

and reduce the above to a consideration of something like this simpler case.

Lemma 9.5.2 *Let U be any open set. Then $C_c^\infty(U) \neq \emptyset$.*

Proof: Pick $\mathbf{z} \in U$ and let r be small enough that $B(\mathbf{z}, 2r) \subseteq U$. Then let $\psi \in C_c^\infty(B(\mathbf{z}, 2r)) \subseteq C_c^\infty(U)$ be the function of the above example.

Definition 9.5.3 *Let $U = \{\mathbf{x} \in \mathbb{R}^n : |\mathbf{x}| < 1\}$. A sequence $\{\psi_m\} \subseteq C_c^\infty(U)$ is called a mollifier if*

$$\psi_m(\mathbf{x}) \geq 0,\ \psi_m(\mathbf{x}) = 0,\ \text{if } |\mathbf{x}| \geq \frac{1}{m},$$

and $\int \psi_m(\mathbf{x}) = 1$. Sometimes it may be written as $\{\psi_\varepsilon\}$ where ψ_ε satisfies the above conditions except $\psi_\varepsilon(\mathbf{x}) = 0$ if $|\mathbf{x}| \geq \varepsilon$. In other words, ε takes the place of $1/m$ and in everything that follows $\varepsilon \to 0$ instead of $m \to \infty$.

$\int f(\mathbf{x},\mathbf{y}) dm_n(\mathbf{y})$ will mean $\mathbf{x}$ is fixed and the function $\mathbf{y} \to f(\mathbf{x},\mathbf{y})$ is being integrated. To make the notation more familiar, dx is written instead of $dm_n(x)$.

Example 9.5.4 *Let*

$$\psi \in C_c^\infty(B(0,1)) \ \ (B(0,1) = \{\mathbf{x} : |\mathbf{x}| < 1\})$$

with $\psi(\mathbf{x}) \geq 0$ and $\int \psi dm = 1$. Let $\psi_m(\mathbf{x}) = c_m \psi(m\mathbf{x})$ where c_m is chosen in such a way that $\int \psi_m dm = 1$.

Definition 9.5.5 *A function, f, is said to be in $L^1_{loc}(\mathbb{R}^n)$ if f is Lebesgue measurable and if $|f|\mathcal{X}_K \in L^1(\mathbb{R}^n)$ for every compact set, K. If $f \in L^1_{loc}(\mathbb{R}^n)$, and $g \in C_c(\mathbb{R}^n)$,*

$$f * g(\mathbf{x}) \equiv \int f(\mathbf{y}) g(\mathbf{x}-\mathbf{y}) dy.$$

This is called the convolution of f and g.

The following lemma will be useful in what follows. It says that one of these very unregular functions in $L^1_{loc}(\mathbb{R}^n)$ is smoothed out by convolving with a mollifier.

Lemma 9.5.6 *Let $f \in L^1_{loc}(\mathbb{R}^n)$, and $g \in C_c^\infty(\mathbb{R}^n)$. Then $f * g$ is an infinitely differentiable function.*

Proof: Consider the difference quotient for calculating a partial derivative of $f * g$.

$$\frac{f * g(\mathbf{x} + t\mathbf{e}_j) - f * g(\mathbf{x})}{t} = \int f(\mathbf{y}) \frac{g(\mathbf{x} + t\mathbf{e}_j - \mathbf{y}) - g(\mathbf{x}-\mathbf{y})}{t} d\mu(y).$$

Using the fact that $g \in C_c^\infty(\mathbb{R}^n)$, the quotient,

$$\frac{g(\mathbf{x} + t\mathbf{e}_j - \mathbf{y}) - g(\mathbf{x}-\mathbf{y})}{t},$$

is uniformly bounded. To see this easily, use Theorem 6.4.2 on Page 114 to get the existence of a constant, M depending on

$$\max\{||Dg(\mathbf{x})|| : \mathbf{x} \in \mathbb{R}^n\}$$

such that

$$|g(\mathbf{x}+t\mathbf{e}_j-\mathbf{y})-g(\mathbf{x}-\mathbf{y})| \leq M|t|$$

for any choice of $\mathbf{x}$ and $\mathbf{y}$. Therefore, there exists a dominating function for the integrand of the above integral which is of the form $C|f(\mathbf{y})|\mathcal{X}_K$ where K is a compact set depending on the support of g. It follows from the dominated convergence theorem the limit of the difference quotient above passes inside the integral as $t \to 0$ and so

$$\frac{\partial}{\partial x_j}(f*g)(\mathbf{x}) = \int f(\mathbf{y})\frac{\partial}{\partial x_j}g(\mathbf{x}-\mathbf{y})\,d\mu(y).$$

Now letting $\frac{\partial}{\partial x_j}g$ play the role of g in the above argument, a repeat of the above reasoning shows partial derivatives of all orders exist. A similar use of the dominated convergence theorem shows all these partial derivatives are also continuous. This proves the lemma.

Theorem 9.5.7 *Let K be a compact subset of an open set, U. Then there exists a function, $h \in C_c^\infty(U)$, such that $h(\mathbf{x}) = 1$ for all $\mathbf{x} \in K$ and $h(\mathbf{x}) \in [0,1]$ for all $\mathbf{x}$.*

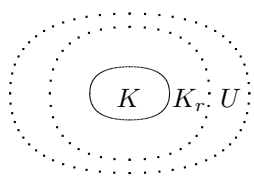
Proof: Let $r > 0$ be small enough that $K + B(\mathbf{0}, 3r) \subseteq U$. The symbol, $K + B(\mathbf{0}, 3r)$ means

$$\{\mathbf{k}+\mathbf{x} : \mathbf{k} \in K \text{ and } \mathbf{x} \in B(\mathbf{0}, 3r)\}.$$

Thus this is simply a way to write

$$\cup\{B(\mathbf{k}, 3r) : \mathbf{k} \in K\}.$$

Think of it as fattening up the set, K. Let $K_r = K + B(0,r)$. A picture of what is happening follows.



Consider $\mathcal{X}_{K_r} * \psi_m$ where ψ_m is a mollifier. Let m be so large that $\frac{1}{m} < r$. Then from the definition of what is meant by a convolution, and using that ψ_m has support in $B\left(\mathbf{0}, \frac{1}{m}\right)$, $\mathcal{X}_{K_r} * \psi_m = 1$ on K and its support is in $K + B(\mathbf{0}, 3r)$. Now using Lemma 9.5.6, $\mathcal{X}_{K_r} * \psi_m$ is also infinitely differentiable. Therefore, let $h = \mathcal{X}_{K_r} * \psi_m$.

The following is the remarkable theorem mentioned above. First, here is some notation.

Definition 9.5.8 *Let $\mathbf{g}$ be a function defined on a vector space. Then $\mathbf{g}_\mathbf{y}(\mathbf{x}) \equiv \mathbf{g}(\mathbf{x}-\mathbf{y})$.*

Theorem 9.5.9 *$C_c^\infty(\mathbb{R}^n)$ is dense in $L^1(\mathbb{R}^n)$. Here the measure is Lebesgue measure.*

Proof: Let $f \in L^1(\mathbb{R}^n)$ and let $\varepsilon > 0$ be given. Choose $g \in C_c(\mathbb{R}^n)$ such that

$$\int |g - f| \, dm_n < \varepsilon/2$$

This can be done by using Theorem 8.10.3. Now let

$$g_m(\mathbf{x}) = g * \psi_m(\mathbf{x}) \equiv \int g(\mathbf{x} - \mathbf{y}) \, \psi_m(\mathbf{y}) \, dm_n(y) = \int g(\mathbf{y}) \, \psi_m(\mathbf{x} - \mathbf{y}) \, dm_n(y)$$

where $\{\psi_m\}$ is a mollifier. It follows from Lemma 9.5.6 $g_m \in C_c^\infty(\mathbb{R}^n)$. It vanishes if $\mathbf{x} \notin \mathrm{spt}(g) + B(0, \frac{1}{m})$.

$$\begin{aligned}
\int |g - g_m| \, dm_n &= \int |g(\mathbf{x}) - \int g(\mathbf{x} - \mathbf{y}) \psi_m(\mathbf{y}) dm_n(\mathbf{y})| dm_n(\mathbf{x}) \\
&\leq \int (\int |g(\mathbf{x}) - g(\mathbf{x} - \mathbf{y})| \psi_m(\mathbf{y}) dm_n(\mathbf{y})) dm_n(\mathbf{x}) \\
&\leq \int \int |g(\mathbf{x}) - g(\mathbf{x} - \mathbf{y})| dm_n(\mathbf{x}) \psi_m(\mathbf{y}) dm_n(\mathbf{y}) \\
&= \int_{B(0, \frac{1}{m})} \int |g - g_{\mathbf{y}}| dm_n(\mathbf{x}) \psi_m(\mathbf{y}) dm_n(\mathbf{y}) < \frac{\varepsilon}{2}
\end{aligned}$$

whenever m is large enough. This follows because since g has compact support, it is uniformly continuous on $\mathbb{R}^n$ and so if $\eta > 0$ is given, then whenever $|\mathbf{y}|$ is sufficiently small,

$$|g(\mathbf{x}) - g(\mathbf{x} - \mathbf{y})| < \eta$$

for all $\mathbf{x}$. Thus, since $\mathbf{g}$ has compact support, if $\mathbf{y}$ is small enough, it follows

$$\int |g - g_{\mathbf{y}}| dm_n(\mathbf{x}) < \varepsilon/2.$$

There is no measurability problem in the use of Fubini's theorem because the function

$$(\mathbf{x}, \mathbf{y}) \to |g(\mathbf{x}) - g(\mathbf{x} - \mathbf{y})| \psi_m(\mathbf{y})$$

is continuous. Thus when m is large enough,

$$\int |f - g_m| \, dm_n \leq \int |f - g| \, dm_n + \int |g - g_m| \, dm_n < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

This proves the theorem.

9.6 The Vitali Covering Theorem

The Vitali covering theorem is a profound result about coverings of a set in $\mathbb{R}^n$ with open balls. The balls can be defined in terms of any norm for $\mathbb{R}^n$. For example, the norm could be

$$||\mathbf{x}|| \equiv \max\{|x_k| : k = 1, \cdots, n\}$$

or the usual norm

$$|\mathbf{x}| = \sqrt{\sum_k |x_k|^2}$$

or any other. The proof given here is from Basic Analysis [26]. It first considers the case of open balls and then generalizes to balls which may be neither open nor closed.

Lemma 9.6.1 *Let $\mathcal{F}$ be a countable collection of balls satisfying*

$$\infty > M \equiv \sup\{r : B(\mathbf{p}, r) \in \mathcal{F}\} > 0$$

and let $k \in (0, \infty)$. Then there exists $\mathcal{G} \subseteq \mathcal{F}$ such that

$$\text{If } B(\mathbf{p}, r) \in \mathcal{G} \text{ then } r > k, \tag{9.4}$$

$$\text{If } B_1, B_2 \in \mathcal{G} \text{ then } B_1 \cap B_2 = \emptyset, \tag{9.5}$$

$$\mathcal{G} \text{ is maximal with respect to 9.4 and 9.5.} \tag{9.6}$$

By this is meant that if $\mathcal{H}$ is a collection of balls satisfying 9.4 and 9.5, then $\mathcal{H}$ cannot properly contain $\mathcal{G}$.

Proof: If no ball of $\mathcal{F}$ has radius larger than k, let $\mathcal{G} = \emptyset$. Assume therefore, that some balls have radius larger than k. Let $\mathcal{F} \equiv \{B_i\}_{i=1}^{\infty}$. Now let B_{n_1} be the first ball in the list which has radius greater than k. If every ball having radius larger than k intersects this one, then stop. The maximal set is $\{B_{n_1}\}$. Otherwise, let B_{n_2} be the next ball having radius larger than k which is disjoint from B_{n_1}. Continue this way obtaining $\{B_{n_i}\}_{i=1}^{\infty}$, a finite or infinite sequence of disjoint balls having radius larger than k. Then let $\mathcal{G} \equiv \{B_{n_i}\}$. To see $\mathcal{G}$ is maximal with respect to 9.4 and 9.5, suppose $B \in \mathcal{F}$, B has radius larger than k, and $\mathcal{G} \cup \{B\}$ satisfies 9.4 and 9.5. Then at some point in the process, B would have been chosen because it would be the ball of radius larger than k which has the smallest index. Therefore, $B \in \mathcal{G}$ and this shows $\mathcal{G}$ is maximal with respect to 9.4 and 9.5.

For an open ball, $B = B(\mathbf{x}, r)$, denote by $\widetilde{B}$ the open ball, $B(\mathbf{x}, 4r)$.

Lemma 9.6.2 *Let $\mathcal{F}$ be a collection of open balls, and let*

$$A \equiv \cup\{B : B \in \mathcal{F}\}.$$

Suppose

$$\infty > M \equiv \sup\{r : B(\mathbf{p}, r) \in \mathcal{F}\} > 0.$$

Then there exists $\mathcal{G} \subseteq \mathcal{F}$ such that $\mathcal{G}$ consists of disjoint balls and

$$A \subseteq \cup\{\widetilde{B} : B \in \mathcal{G}\}.$$

Proof: Without loss of generality assume $\mathcal{F}$ is countable. This is because there is a countable subset of $\mathcal{F}$, $\mathcal{F}'$ such that $\cup\mathcal{F}' = A$. To see this, consider the set of balls having rational radii and centers having all components rational. This is a countable set of balls and you should verify that every open set is the union of balls of this form. Therefore, you can consider the subset of this set of balls consisting of those which are contained in some open set of $\mathcal{F}$, G so $\cup G = A$ and use the axiom of choice to define a subset of $\mathcal{F}$ consisting of a single set from $\mathcal{F}$ containing each set of G. Then this is $\mathcal{F}'$. The union of these sets equals A. Then consider $\mathcal{F}'$ instead of $\mathcal{F}$. Therefore, assume at the outset $\mathcal{F}$ is countable.

By Lemma 9.6.1, there exists $\mathcal{G}_1 \subseteq \mathcal{F}$ which satisfies 9.4, 9.5, and 9.6 with $k = \frac{2M}{3}$.

Suppose $\mathcal{G}_1, \cdots, \mathcal{G}_{m-1}$ have been chosen for $m \geq 2$. Let

$$\mathcal{F}_m = \{B \in \mathcal{F} : B \subseteq \mathbb{R}^n \setminus \overbrace{\cup\{\mathcal{G}_1 \cup \cdots \cup \mathcal{G}_{m-1}\}}^{\text{union of the balls in these } \mathcal{G}_j}\}$$

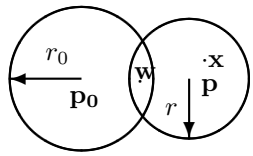
and using Lemma 9.6.1, let $\mathcal{G}_m$ be a maximal collection of disjoint balls from $\mathcal{F}_m$ with the property that each ball has radius larger than $\left(\frac{2}{3}\right)^m M$. Let $\mathcal{G} \equiv \cup_{k=1}^{\infty}\mathcal{G}_k$. Let $\mathbf{x} \in B(\mathbf{p}, r) \in \mathcal{F}$. Choose m such that

$$\left(\frac{2}{3}\right)^m M < r \leq \left(\frac{2}{3}\right)^{m-1} M$$

Then $B(\mathbf{p},r)$ must have nonempty intersection with some ball from $\mathcal{G}_1 \cup \cdots \cup \mathcal{G}_m$ because if it didn't, then $\mathcal{G}_m$ would fail to be maximal. Denote by $B(\mathbf{p}_0,r_0)$ a ball in $\mathcal{G}_1 \cup \cdots \cup \mathcal{G}_m$ which has nonempty intersection with $B(\mathbf{p},r)$. Thus

$$r_0 > \left(\frac{2}{3}\right)^m M.$$

Consider the picture, in which $\mathbf{w} \in B(\mathbf{p}_0,r_0) \cap B(\mathbf{p},r)$.



Then

$$\begin{aligned} |\mathbf{x}-\mathbf{p}_0| &\leq |\mathbf{x}-\mathbf{p}| + |\mathbf{p}-\mathbf{w}| + \overbrace{|\mathbf{w}-\mathbf{p}_0|}^{<r_0} \\ &< r + r + r_0 \leq 2\overbrace{\left(\frac{2}{3}\right)^{m-1}}^{<\frac{3}{2}r_0} M + r_0 \\ &< 2\left(\frac{3}{2}\right) r_0 + r_0 = 4r_0. \end{aligned}$$

This proves the lemma since it shows $B(\mathbf{p},r) \subseteq B(\mathbf{p}_0,4r_0)$.

With this Lemma consider a version of the Vitali covering theorem in which the balls do not have to be open. In this theorem, B will denote an open ball, $B(\mathbf{x},r)$ along with either part or all of the points where $||\mathbf{x}|| = r$ and $||\cdot||$ is any norm for $\mathbb{R}^n$.

Definition 9.6.3 *Let B be a ball centered at $\mathbf{x}$ having radius r. Denote by $\widehat{B}$ the open ball, $B(\mathbf{x},5r)$.*

Theorem 9.6.4 *(Vitali) Let $\mathcal{F}$ be a collection of balls, and let*

$$A \equiv \cup\{B : B \in \mathcal{F}\}.$$

Suppose

$$\infty > M \equiv \sup\{r : B(\mathbf{p},r) \in \mathcal{F}\} > 0.$$

Then there exists $\mathcal{G} \subseteq \mathcal{F}$ such that $\mathcal{G}$ consists of disjoint balls and

$$A \subseteq \cup\{\widehat{B} : B \in \mathcal{G}\}.$$

Proof: For B one of these balls, say $\overline{B(\mathbf{x},r)} \supseteq B \supseteq B(\mathbf{x},r)$, denote by B_1, the open ball $B\left(\mathbf{x},\frac{5r}{4}\right)$. Let $\mathcal{F}_1 \equiv \{B_1 : B \in \mathcal{F}\}$ and let A_1 denote the union of the balls in $\mathcal{F}_1$. Apply Lemma 9.6.2 to $\mathcal{F}_1$ to obtain

$$A_1 \subseteq \cup\{\widetilde{B_1} : B_1 \in \mathcal{G}_1\}$$

where $\mathcal{G}_1$ consists of disjoint balls from $\mathcal{F}_1$. Now let $\mathcal{G} \equiv \{B \in \mathcal{F} : B_1 \in \mathcal{G}_1\}$. Thus $\mathcal{G}$ consists of disjoint balls from $\mathcal{F}$ because they are contained in the disjoint open balls, $\mathcal{G}_1$. Then

$$A \subseteq A_1 \subseteq \cup\{\widetilde{B_1} : B_1 \in \mathcal{G}_1\} = \cup\{\widehat{B} : B \in \mathcal{G}\}$$

because for $B_1 = B\left(\mathbf{x},\frac{5r}{4}\right)$, it follows $\widetilde{B_1} = B(\mathbf{x},5r) = \widehat{B}$. This proves the theorem.

9.7 Vitali Coverings

There is another version of the Vitali covering theorem which is also of great importance. In this one, disjoint balls from the original set of balls almost cover the set, leaving out only a set of measure zero. It is like packing a truck with stuff. You keep trying to fill in the holes with smaller and smaller things so as to not waste space. It is remarkable that you can avoid wasting any space at all when you are dealing with balls of any sort provided you can use arbitrarily small balls.

Definition 9.7.1 *Let $\mathcal{F}$ be a collection of balls that cover a set, E, which have the property that if $\mathbf{x} \in E$ and $\varepsilon > 0$, then there exists $B \in \mathcal{F}$, diameter of $B < \varepsilon$ and $\mathbf{x} \in B$. Such a collection covers E in the sense of Vitali.*

In the following covering theorem, $\overline{m_n}$ denotes the outer measure determined by n dimensional Lebesgue measure. Thus, letting $\mathcal{F}$ denote the Lebesgue measurable sets,

$$\overline{m_n}(S) \equiv \inf\left\{\sum_{k=1}^{\infty} m_n(E_k) : S \subseteq \cup_k E_k, E_k \in \mathcal{F}\right\}$$

Recall that from this definition, if $S \subseteq \mathbb{R}^n$ there exists $E_1 \supseteq S$ such that $m_n(E_1) = \overline{m_n}(S)$. To see this, note that it suffices to assume in the above definition of $\overline{m_n}$ that the E_k are also disjoint. If not, replace with the sequence given by

$$F_1 = E_1, F_2 \equiv E_2 \setminus F_1, \cdots, F_m \equiv E_m \setminus F_{m-1},$$

etc. Then for each $l > \overline{m_n}(S)$, there exists $\{E_k\}$ such that

$$l > \sum_k m_n(E_k) \geq \sum_k m_n(F_k) = m_n(\cup_k E_k) \geq \overline{m_n}(S).$$

If $\overline{m_n}(S) = \infty$, let $E_1 = \mathbb{R}^n$. Otherwise, there exists $G_k \in \mathcal{F}$ such that

$$\overline{m_n}(S) \leq m_n(G_k) \leq \overline{m_n}(S) + 1/k.$$

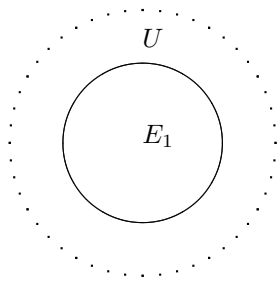
then let $E_1 = \cap_k G_k$.

Note this implies that if $\overline{m_n}(S) = 0$ then S must be in $\mathcal{F}$ because of completeness of Lebesgue measure.

Theorem 9.7.2 *Let $E \subseteq \mathbb{R}^n$ and suppose $0 < \overline{m_n}(E) < \infty$ where $\overline{m_n}$ is the outer measure determined by m_n, n dimensional Lebesgue measure, and let $\mathcal{F}$ be a collection of closed balls of bounded radii such that $\mathcal{F}$ covers E in the sense of Vitali. Then there exists a countable collection of disjoint balls from $\mathcal{F}$, $\{B_j\}_{j=1}^{\infty}$, such that $\overline{m_n}(E \setminus \cup_{j=1}^{\infty} B_j) = 0$.*

Proof: From the definition of outer measure there exists a Lebesgue measurable set, $E_1 \supseteq E$ such that $m_n(E_1) = \overline{m_n}(E)$. Now by outer regularity of Lebesgue measure, there exists U, an open set which satisfies

$$m_n(E_1) > (1 - 10^{-n}) m_n(U),\ U \supseteq E_1.$$



Each point of E is contained in balls of $\mathcal{F}$ of arbitrarily small radii and so there exists a covering of E with balls of $\mathcal{F}$ which are themselves contained in U. Therefore, by the Vitali covering theorem, there exist disjoint balls, $\{B_i\}_{i=1}^{\infty} \subseteq \mathcal{F}$ such that

$$E \subseteq \cup_{j=1}^{\infty} \widehat{B}_j,\ B_j \subseteq U.$$

Therefore,

$$\begin{aligned} m_n(E_1) &= \overline{m_n}(E) \leq m_n\left(\cup_{j=1}^{\infty} \widehat{B}_j\right) \leq \sum_j m_n\left(\widehat{B}_j\right) \\ &= 5^n \sum_j m_n(B_j) = 5^n m_n\left(\cup_{j=1}^{\infty} B_j\right) \end{aligned}$$

Then

$$\begin{aligned} m_n(E_1) &> (1-10^{-n}) m_n(U) \\ &\geq (1-10^{-n})[m_n(E_1 \setminus \cup_{j=1}^{\infty} B_j) + m_n(\cup_{j=1}^{\infty} B_j)] \\ &\geq (1-10^{-n})[m_n(E_1 \setminus \cup_{j=1}^{\infty} B_j) + 5^{-n} \overbrace{\overline{m_n}(E)}^{=m_n(E_1)}]. \end{aligned}$$

and so

$$\left(1 - \left(1-10^{-n}\right) 5^{-n}\right) m_n(E_1) \geq (1-10^{-n}) m_n(E_1 \setminus \cup_{j=1}^{\infty} B_j)$$

which implies

$$m_n(E_1 \setminus \cup_{j=1}^{\infty} B_j) \leq \frac{\left(1-\left(1-10^{-n}\right)5^{-n}\right)}{\left(1-10^{-n}\right)} m_n(E_1)$$

Now a short computation shows

$$0 < \frac{\left(1-\left(1-10^{-n}\right)5^{-n}\right)}{\left(1-10^{-n}\right)} < 1$$

Hence, denoting by θ_n a number such that

$$\frac{\left(1-\left(1-10^{-n}\right)5^{-n}\right)}{\left(1-10^{-n}\right)} < \theta_n < 1,$$

$$\overline{m_n}\left(E \setminus \cup_{j=1}^{\infty} B_j\right) \leq m_n(E_1 \setminus \cup_{j=1}^{\infty} B_j) < \theta_n m_n(E_1) = \theta_n \overline{m_n}(E)$$

Now using Theorem 7.3.2 on Page 148 there exists N_1 large enough that

$$\theta_n \overline{m_n}(E) \geq m_n(E_1 \setminus \cup_{j=1}^{N_1} B_j) \geq \overline{m_n}(E \setminus \cup_{j=1}^{N_1} B_j) \tag{9.7}$$

Let $\mathcal{F}_1 = \{B \in \mathcal{F} : B_j \cap B = \emptyset,\ j = 1, \cdots, N_1\}$. If $E \setminus \cup_{j=1}^{N_1} B_j = \emptyset$, then $\mathcal{F}_1 = \emptyset$ and

$$\overline{m_n}\left(E \setminus \cup_{j=1}^{N_1} B_j\right) = 0$$

Therefore, in this case let $B_k = \emptyset$ for all $k > N_1$. Consider the case where

$$E \setminus \cup_{j=1}^{N_1} B_j \neq \emptyset.$$

In this case, since the balls are closed and $\mathcal{F}$ is a Vitali cover, $\mathcal{F}_1 \neq \emptyset$ and covers $E \setminus \cup_{j=1}^{N_1} B_j$ in the sense of Vitali. Repeat the same argument, letting $E \setminus \cup_{j=1}^{N_1} B_j$ play the role of E. (You pick a different E_1 whose measure equals the outer measure of $E \setminus \cup_{j=1}^{N_1} B_j$ and proceed as before.) Then choosing B_j for $j = N_1 + 1, \cdots, N_2$ as in the above argument,

$$\theta_n \overline{m_n}(E \setminus \cup_{j=1}^{N_1} B_j) \geq \overline{m_n}(E \setminus \cup_{j=1}^{N_2} B_j)$$

and so from 9.7,

$$\theta_n^2 \overline{m_n}(E) \geq \overline{m_n}(E \setminus \cup_{j=1}^{N_2} B_j).$$

Continuing this way

$$\theta_n^k \overline{m_n}(E) \geq \overline{m_n}\left(E \setminus \cup_{j=1}^{N_k} B_j\right).$$

If it is ever the case that $E \setminus \cup_{j=1}^{N_k} B_j = \emptyset$, then as in the above argument,

$$\overline{m_n}\left(E \setminus \cup_{j=1}^{N_k} B_j\right) = 0.$$

Otherwise, the process continues and

$$\overline{m_n}\left(E \setminus \cup_{j=1}^{\infty} B_j\right) \leq \overline{m_n}\left(E \setminus \cup_{j=1}^{N_k} B_j\right) \leq \theta_n^k \overline{m_n}\left(E\right)$$

for every $k \in \mathbb{N}$. Therefore, the conclusion holds in this case also because $\theta_n < 1$. This proves the Theorem.

There is an obvious corollary which removes the assumption that $0 < \overline{m_n}(E)$.

Corollary 9.7.3 *Let $E \subseteq \mathbb{R}^n$ and suppose $\overline{m_n}(E) < \infty$ where $\overline{m_n}$ is the outer measure determined by m_n, n dimensional Lebesgue measure, and let $\mathcal{F}$, be a collection of closed balls of bounded radii such that$\mathcal{F}$ covers E in the sense of Vitali. Then there exists a countable collection of disjoint balls from $\mathcal{F}$, $\{B_j\}_{j=1}^{\infty}$, such that $\overline{m_n}(E \setminus \cup_{j=1}^{\infty} B_j) = 0$.*

Proof: If $0 = \overline{m_n}(E)$ you simply pick any ball from $\mathcal{F}$ for your collection of disjoint balls.

It is also not hard to remove the assumption that $\overline{m_n}\left(E\right) < \infty$.

Corollary 9.7.4 *Let $E \subseteq \mathbb{R}^n$ and let $\mathcal{F}$, be a collection of closed balls of bounded radii such that $\mathcal{F}$ covers E in the sense of Vitali. Then there exists a countable collection of disjoint balls from $\mathcal{F}$, $\{B_j\}_{j=1}^{\infty}$, such that $\overline{m_n}(E \setminus \cup_{j=1}^{\infty} B_j) = 0$.*

Proof: Let $R_m \equiv (-m, m)^n$ be the open rectangle having sides of length $2m$ which is centered at $\mathbf{0}$ and let $R_0 = \emptyset$. Let $H_m \equiv \overline{R_m} \setminus R_m$. Since both $\overline{R_m}$ and R_m have the same measure, $(2m)^n$, it follows $m_n\left(H_m\right) = 0$. Now for all $k \in \mathbb{N}$, $R_k \subseteq \overline{R_k} \subseteq R_{k+1}$. Consider the disjoint open sets, $U_k \equiv R_{k+1} \setminus \overline{R_k}$. Thus $\mathbb{R}^n = \cup_{k=0}^{\infty} U_k \cup N$ where N is a set of measure zero equal to the union of the H_k. Let $\mathcal{F}_k$ denote those balls of $\mathcal{F}$ which are contained in U_k and let $E_k \equiv U_k \cap E$. Then from Theorem 9.7.2, there exists a sequence of disjoint balls,

$D_k \equiv \left\{B_i^k\right\}_{i=1}^{\infty}$ of $\mathcal{F}_k$ such that $\overline{m_n}(E_k \setminus \cup_{j=1}^{\infty} B_j^k) = 0$. Letting $\{B_i\}_{i=1}^{\infty}$ be an enumeration of all the balls of $\cup_k D_k$, it follows that

$$\overline{m_n}(E \setminus \cup_{j=1}^{\infty} B_j) \leq m_n(N) + \sum_{k=1}^{\infty} \overline{m_n}(E_k \setminus \cup_{j=1}^{\infty} B_j^k) = 0.$$

Also, you don't have to assume the balls are closed.

Corollary 9.7.5 *Let $E \subseteq \mathbb{R}^n$ and let $\mathcal{F}$, be a collection of open balls of bounded radii such that $\mathcal{F}$ covers E in the sense of Vitali. Then there exists a countable collection of disjoint balls from $\mathcal{F}$, $\{B_j\}_{j=1}^{\infty}$, such that $\overline{m_n}(E \setminus \cup_{j=1}^{\infty} B_j) = 0$.*

Proof: Let $\overline{\mathcal{F}}$ be the collection of closures of balls in $\mathcal{F}$. Then $\overline{\mathcal{F}}$ covers E in the sense of Vitali and so from Corollary 9.7.4 there exists a sequence of disjoint closed balls from $\overline{\mathcal{F}}$ satisfying $\overline{m_n}\left(E \setminus \cup_{i=1}^{\infty} \overline{B_i}\right) = 0$. Now boundaries of the balls, B_i have measure zero and so $\{B_i\}$ is a sequence of disjoint open balls satisfying $\overline{m_n}\left(E \setminus \cup_{i=1}^{\infty} B_i\right) = 0$. The reason for this is that

$$\left(E \setminus \cup_{i=1}^{\infty} B_i\right) \setminus \left(E \setminus \cup_{i=1}^{\infty} \overline{B_i}\right) \subseteq \cup_{i=1}^{\infty} \overline{B_i} \setminus \cup_{i=1}^{\infty} B_i \subseteq \cup_{i=1}^{\infty} \overline{B_i} \setminus B_i,$$

a set of measure zero. Therefore,

$$E \setminus \cup_{i=1}^{\infty} B_i \subseteq \left(E \setminus \cup_{i=1}^{\infty} \overline{B_i}\right) \cup \left(\cup_{i=1}^{\infty} \overline{B_i} \setminus B_i\right)$$

and so

$$\begin{aligned} \overline{m_n}\left(E \setminus \cup_{i=1}^{\infty} B_i\right) &\leq \overline{m_n}\left(E \setminus \cup_{i=1}^{\infty} \overline{B_i}\right) + m_n\left(\cup_{i=1}^{\infty} \overline{B_i} \setminus B_i\right) \\ &= \overline{m_n}\left(E \setminus \cup_{i=1}^{\infty} \overline{B_i}\right) = 0. \end{aligned}$$

This implies you can fill up an open set with balls which cover the open set in the sense of Vitali.

Corollary 9.7.6 *Let $U \subseteq \mathbb{R}^n$ be an open set and let $\mathcal{F}$ be a collection of closed or even open balls of bounded radii contained in U such that $\mathcal{F}$ covers U in the sense of Vitali. Then there exists a countable collection of disjoint balls from $\mathcal{F}$, $\{B_j\}_{j=1}^{\infty}$, such that $\overline{m_n}(U \setminus \cup_{j=1}^{\infty} B_j) = 0$.*

9.8 Change Of Variables For Linear Maps

To begin with certain kinds of functions map measurable sets to measurable sets. It will be assumed that U is an open set in $\mathbb{R}^n$ and that $\mathbf{h} : U \to \mathbb{R}^n$ satisfies

$$D\mathbf{h}(\mathbf{x}) \text{ exists for all } \mathbf{x} \in U, \tag{9.8}$$

Note that if

$$\mathbf{h}(\mathbf{x}) = L\mathbf{x}$$

where $L \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^n)$, then L is included in 9.8 because

$$L(\mathbf{x} + \mathbf{v}) = L(\mathbf{x}) + L(\mathbf{v}) + \mathbf{o}(\mathbf{v})$$

In fact, $\mathbf{o}(\mathbf{v}) = \mathbf{0}$.

It is convenient in the following lemma to use the norm on $\mathbb{R}^n$ given by

$$||\mathbf{x}|| = \max\{|x_k| : k = 1, 2, \cdots, n\}.$$

Thus $B(\mathbf{x},r)$ is the open box,

$$\prod_{k=1}^{n}(x_k-r,x_k+r)$$

and so $m_n(B(\mathbf{x},r))=(2r)^n$.

Lemma 9.8.1 *Let* $\mathbf{h}$ *satisfy 9.8. If* $T\subseteq U$ *and* $m_n(T)=0$, *then* $m_n(\mathbf{h}(T))=0$.

Proof: Let

$$T_k\equiv\{\mathbf{x}\in T:||D\mathbf{h}(\mathbf{x})||<k\}$$

and let $\varepsilon>0$ be given. Now by outer regularity, there exists an open set, V, containing T_k which is contained in U such that $m_n(V)<\varepsilon$. Let $\mathbf{x}\in T_k$. Then by differentiability,

$$\mathbf{h}(\mathbf{x}+\mathbf{v})=\mathbf{h}(\mathbf{x})+D\mathbf{h}(\mathbf{x})\,\mathbf{v}+o(\mathbf{v})$$

and so there exist arbitrarily small $r_{\mathbf{x}}<1$ such that $B(\mathbf{x},5r_{\mathbf{x}})\subseteq V$ and whenever $||\mathbf{v}||\leq r_{\mathbf{x}}$, $||o(\mathbf{v})||<k||\mathbf{v}||$. Thus

$$\mathbf{h}(B(\mathbf{x},r_{\mathbf{x}}))\subseteq B(\mathbf{h}(\mathbf{x}),2kr_{\mathbf{x}}).$$

From the Vitali covering theorem there exists a countable disjoint sequence of these balls, $\{B(\mathbf{x}_i,r_i)\}_{i=1}^{\infty}$ such that $\{B(\mathbf{x}_i,5r_i)\}_{i=1}^{\infty}=\left\{\widehat{B}_i\right\}_{i=1}^{\infty}$ covers T_k Then letting $\overline{m_n}$ denote the outer measure determined by m_n,

$$\begin{aligned}\overline{m_n}(\mathbf{h}(T_k))&\leq\overline{m_n}\left(\mathbf{h}\left(\cup_{i=1}^{\infty}\widehat{B}_i\right)\right)\\&\leq\sum_{i=1}^{\infty}\overline{m_n}\left(\mathbf{h}\left(\widehat{B}_i\right)\right)\leq\sum_{i=1}^{\infty}m_n(B(\mathbf{h}(\mathbf{x}_i),2kr_{\mathbf{x}_i}))\\&=\sum_{i=1}^{\infty}m_n(B(\mathbf{x}_i,2kr_{\mathbf{x}_i}))=(2k)^n\sum_{i=1}^{\infty}m_n(B(\mathbf{x}_i,r_{\mathbf{x}_i}))\\&\leq(2k)^n m_n(V)\leq(2k)^n\varepsilon.\end{aligned}$$

Since $\varepsilon>0$ is arbitrary, this shows $m_n(\mathbf{h}(T_k))=0$. Now

$$m_n(\mathbf{h}(T))=\lim_{k\to\infty}m_n(\mathbf{h}(T_k))=0.$$

This proves the lemma.

Lemma 9.8.2 *Let* $\mathbf{h}$ *satisfy 9.8. If* S *is a Lebesgue measurable subset of* U, *then* $\mathbf{h}(S)$ *is Lebesgue measurable.*

Proof: By Theorem 9.4.2 there exists F which is a countable union of compact sets, $F=\cup_{k=1}^{\infty}K_k$ such that

$$F\subseteq S,\ m_n(S\setminus F)=0.$$

Then since $\mathbf{h}$ is continuous

$$\mathbf{h}(F)=\cup_k\mathbf{h}(K_k)\in\mathcal{B}(\mathbb{R}^n)$$

because the continuous image of a compact set is compact. Also, $\mathbf{h}(S\setminus F)$ is a set of measure zero by Lemma 9.8.1 and so

$$\mathbf{h}(S)=\mathbf{h}(F)\cup\mathbf{h}(S\setminus F)\in\mathcal{F}_n$$

because it is the union of two sets which are in $\mathcal{F}_n$. This proves the lemma.

In particular, this proves the following corollary.

Corollary 9.8.3 *Suppose $A \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^n)$. Then if S is a Lebesgue measurable set, it follows AS is also a Lebesgue measurable set.*

In the next lemma, the norm used for defining balls will be the usual norm,

$$|\mathbf{x}| = \left(\sum_{k=1}^{n} |x_k|^2 \right)^{1/2}.$$

Thus a unitary transformation preserves distances measured with respect to this norm. In particular, if R is unitary, ($R^*R = RR^* = I$) then

$$R(B(\mathbf{0}, r)) = B(\mathbf{0}, r).$$

Lemma 9.8.4 *Let R be unitary and let V be a an open set. Then $m_n(RV) = m_n(V)$.*

Proof: First assume V is a bounded open set. By Corollary 9.7.6 there is a disjoint sequence of closed balls, $\{B_i\}$ such that $U = \cup_{i=1}^{\infty} B_i \cup N$ where $m_n(N) = 0$. Denote by $\mathbf{x}_i$ the center of B_i and let r_i be the radius of B_i. Then by Lemma 9.8.1 $m_n(RV) = \sum_{i=1}^{\infty} m_n(RB_i)$. Now by invariance of translation of Lebesgue measure, this equals $\sum_{i=1}^{\infty} m_n(RB_i - R\mathbf{x}_i$ $\sum_{i=1}^{\infty} m_n(RB(\mathbf{0}, r_i))$. Since R is unitary, it preserves all distances and so $RB(\mathbf{0}, r_i) = B(\mathbf{0}, r_i)$ and therefore,

$$m_n(RV) = \sum_{i=1}^{\infty} m_n(B(\mathbf{0}, r_i)) = \sum_{i=1}^{\infty} m_n(B_i) = m_n(V).$$

This proves the lemma in the case that V is bounded. Suppose now that V is just an open set. Let $V_k = V \cap B(\mathbf{0}, k)$. Then $m_n(RV_k) = m_n(V_k)$. Letting $k \to \infty$, this yields the desired conclusion. This proves the lemma in the case that V is open.

Lemma 9.8.5 *Let E be Lebesgue measurable set in $\mathbb{R}^n$ and let R be unitary. Then $m_n(RE) = m_n(E)$.*

Proof: Let $\mathcal{K}$ be the open sets. Thus $\mathcal{K}$ is a π system. Let $\mathcal{G}$ denote those Borel sets F such that for each $p \in \mathbb{N}$,

$$m_n(R(F \cap (-p, p)^n)) = m_n(F \cap (-p, p)^n).$$

Thus $\mathcal{G}$ contains $\mathcal{K}$ from Lemma 9.8.4. It is also routine to verify $\mathcal{G}$ is closed with respect to complements and countable disjoint unions. Therefore from the π systems lemma,

$$\mathcal{G} \supseteq \sigma(\mathcal{K}) = \mathcal{B}(\mathbb{R}^n) \supseteq \mathcal{G}$$

and this proves the lemma whenever $E \in \mathcal{B}(\mathbb{R}^n)$. If E is only in $\mathcal{F}_n$, it follows from Theorem 9.4.2

$$E = F \cup N$$

where $m_n(N) = 0$ and F is a countable union of compact sets. Thus by Lemma 9.8.1

$$m_n(RE) = m_n(RF) + m_n(RN) = m_n(RF) = m_n(F) = m_n(E).$$

This proves the theorem.

Lemma 9.8.6 *Let $D \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^n)$ be of the form*

$$D = \sum_j d_j \mathbf{e}_j \mathbf{e}_j$$

where $d_j \geq 0$ and $\{\mathbf{e}_j\}$ is the usual orthonormal basis of $\mathbb{R}^n$. Then for all $E \in \mathcal{F}_n$

$$m_n(DE) = |\det(D)|\, m_n(E).$$

Proof: Let $\mathcal{K}$ consist of open sets of the form

$$\prod_{k=1}^n (a_k, b_k) \equiv \left\{ \sum_{k=1}^n x_k \mathbf{e}_k \text{ such that } x_k \in (a_k, b_k) \right\}$$

Hence

$$\begin{aligned} D\left(\prod_{k=1}^n (a_k, b_k)\right) &= \left\{ \sum_{k=1}^n d_k x_k \mathbf{e}_k \text{ such that } x_k \in (a_k, b_k) \right\} \\ &= \prod_{k=1}^n (d_k a_k, d_k b_k). \end{aligned}$$

It follows

$$\begin{aligned} m_n\left(D\left(\prod_{k=1}^n (a_k, b_k)\right)\right) &= \left(\prod_{k=1}^n d_k\right)\left(\prod_{k=1}^n (b_k - a_k)\right) \\ &= |\det(D)|\, m_n\left(\prod_{k=1}^n (a_k, b_k)\right). \end{aligned}$$

Now let $\mathcal{G}$ consist of Borel sets F with the property that

$$m_n(D(F \cap (-p,p)^n)) = |\det(D)|\, m_n(F \cap (-p,p)^n).$$

Thus $\mathcal{K} \subseteq \mathcal{G}$.

Suppose now that $F \in \mathcal{G}$ and first assume D is one to one. Then

$$m_n\left(D\left(F^C \cap (-p,p)^n\right)\right) + m_n(D(F \cap (-p,p)^n)) = m_n(D(-p,p)^n)$$

and so

$$m_n\left(D\left(F^C \cap (-p,p)^n\right)\right) + |\det(D)|\, m_n(F \cap (-p,p)^n) = |\det(D)|\, m_n((-p,p)^n)$$

which shows

$$\begin{aligned} m_n\left(D\left(F^C \cap (-p,p)^n\right)\right) &= |\det(D)|\left[m_n((-p,p)^n) - m_n(F \cap (-p,p)^n)\right] \\ &= |\det(D)|\, m_n\left(F^C \cap (-p,p)^n\right) \end{aligned}$$

In case D is not one to one, it follows some $d_j = 0$ and so $|\det(D)| = 0$ and

$$\begin{aligned} 0 &\leq m_n\left(D\left(F^C \cap (-p,p)^n\right)\right) \leq m_n(D(-p,p)^n) = \prod_{i=1}^n (d_i p + d_i p) = 0 \\ &= |\det(D)|\, m_n\left(F^C \cap (-p,p)^n\right) \end{aligned}$$

so $F^C \in \mathcal{G}$.

If $\{F_k\}$ is a sequence of disjoint sets of $\mathcal{G}$ and D is one to one

$$\begin{aligned} m_n\left(D\left(\cup_{k=1}^{\infty} F_k \cap (-p,p)^n\right)\right) &= \sum_{k=1}^{\infty} m_n\left(D\left(F_k \cap (-p,p)^n\right)\right) \\ &= |\det(D)| \sum_{k=1}^{\infty} m_n\left(F_k \cap (-p,p)^n\right) \\ &= |\det(D)|\, m_n\left(\cup_k F_k \cap (-p,p)^n\right). \end{aligned}$$

If D is not one to one, then $\det(D) = 0$ and so the right side of the above equals 0. The left side is also equal to zero because it is no larger than

$$m_n\left(D\left(-p,p\right)^n\right) = 0.$$

Thus $\mathcal{G}$ is closed with respect to complements and countable disjoint unions. Hence it contains $\sigma(\mathcal{K})$, the Borel sets. But also $\mathcal{G} \subseteq \mathcal{B}(\mathbb{R}^n)$ and so $\mathcal{G}$ equals $\mathcal{B}(\mathbb{R}^n)$. Letting $p \to \infty$ yields the conclusion of the lemma in case $E \in \mathcal{B}(\mathbb{R}^n)$.

Now for $E \in \mathcal{F}_n$ arbitrary, it follows from Theorem 9.4.2

$$E = F \cup N$$

where N is a set of measure zero and F is a countable union of compact sets. Hence as before,

$$\begin{aligned} m_n(D(E)) &= m_n(DF \cup DN) \le m_n(DF) + m_n(DN) \\ &= |\det(D)|\, m_n(F) = |\det(D)|\, m_n(E) \end{aligned}$$

Also from Theorem 9.4.2 there exists G Borel such that

$$G = E \cup S$$

where S is a set of measure zero. Therefore,

$$\begin{aligned} |\det(D)|\, m_n(E) &= |\det(D)|\, m_n(G) = m_n(DG) \\ &= m_n(DE \cup DS) \le m_n(DE) + m_n(DS) \\ &= m_n(DE) \end{aligned}$$

This proves the theorem.

The main result follows.

Theorem 9.8.7 *Let $E \in \mathcal{F}_n$ and let $A \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^n)$. Then $m_n(AV) = |\det(A)|\, m_n(V)$.*

Proof: Let RU be the right polar decomposition (Theorem 3.9.3 on Page 62) of A. Thus R is unitary and

$$U = \sum_k d_k \mathbf{w}_k \mathbf{w}_k$$

where each $d_k \ge 0$. It follows $|\det(A)| = |\det(U)|$ because

$$|\det(A)| = |\det(R)\det(U)| = |\det(R)|\,|\det(U)| = |\det(U)|.$$

Recall from Lemma 3.9.5 on Page 64 the determinant of a unitary transformation has absolute value equal to 1. Then from Lemma 9.8.5,

$$m_n(AE) = m_n(RUE) = m_n(UE).$$

Let

$$Q = \sum_j \mathbf{w}_j \mathbf{e}_j$$

and so by Lemma 3.8.17 on Page 59,

$$Q^* = \sum_k \mathbf{e}_k \mathbf{w}_k.$$

Thus Q and Q^* are both unitary and a simple computation shows

$$U = Q \sum_i d_i \mathbf{e}_i \mathbf{e}_i Q^* \equiv QDQ^*.$$

Do both sides to $\mathbf{w}_k$ and observe both sides give $d_k \mathbf{w}_k$. Since the two linear operators agree on a basis, they must be the same. Thus

$$|\det(D)| = |\det(U)| = |\det(A)|.$$

Therefore, from Lemma 9.8.5 and Lemma 9.8.6

$$\begin{aligned} m_n(AE) &= m_n(QDQ^*E) = m_n(DQ^*E) \\ &= |\det(D)|\, m_n(Q^*E) = |\det(A)|\, m_n(E). \end{aligned}$$

This proves the theorem.

9.9 Change Of Variables For C^1 Functions

In this section theorems are proved which yield change of variables formulas for C^1 functions. More general versions can be seen in Kuttler [26], Kuttler [27], and Rudin [34]. You can obtain more by exploiting the Radon Nikodym theorem and the Lebesgue fundamental theorem of calculus, two topics which are best studied in a real analysis course. Instead, I will present some good theorems using the Vitali covering theorem directly.

A basic version of the theorems to be presented is the following. If you like, let the balls be defined in terms of the norm

$$||\mathbf{x}|| \equiv \max\{|x_k| : k = 1, \cdots, n\}$$

Lemma 9.9.1 *Let U and V be bounded open sets in $\mathbb{R}^n$ and let $\mathbf{h}, \mathbf{h}^{-1}$ be C^1 functions such that $\mathbf{h}(U) = V$. Also let $f \in C_c(V)$. Then*

$$\int_V f(\mathbf{y})\, dm_n = \int_U f(\mathbf{h}(\mathbf{x}))\, |\det(D\mathbf{h}(\mathbf{x}))|\, dm_n$$

Proof: First note $\mathbf{h}^{-1}(\mathrm{spt}(f))$ is a closed subset of the bounded set, U and so it is compact. Thus $\mathbf{x} \to f(\mathbf{h}(\mathbf{x}))\, |\det(D\mathbf{h}(\mathbf{x}))|$ is bounded and continuous.

Let $\mathbf{x} \in U$. By the assumption that $\mathbf{h}$ and $\mathbf{h}^{-1}$ are C^1,

$$\begin{aligned} \mathbf{h}(\mathbf{x}+\mathbf{v}) - \mathbf{h}(\mathbf{x}) &= D\mathbf{h}(\mathbf{x})\,\mathbf{v} + \mathbf{o}(\mathbf{v}) \\ &= D\mathbf{h}(\mathbf{x})\left(\mathbf{v} + D\mathbf{h}^{-1}(\mathbf{h}(\mathbf{x}))\,\mathbf{o}(\mathbf{v})\right) \\ &= D\mathbf{h}(\mathbf{x})\,(\mathbf{v} + \mathbf{o}(\mathbf{v})) \end{aligned}$$

and so if $r > 0$ is small enough then $B(\mathbf{x}, r)$ is contained in U and

$$\mathbf{h}(B(\mathbf{x}, r)) - \mathbf{h}(\mathbf{x}) =$$

$$\mathbf{h}(\mathbf{x}+B(\mathbf{0},r)) - \mathbf{h}(\mathbf{x}) \subseteq D\mathbf{h}(\mathbf{x})(B(\mathbf{0},(1+\varepsilon)r)). \tag{9.9}$$

Making r still smaller if necessary, one can also obtain

$$|f(\mathbf{y}) - f(\mathbf{h}(\mathbf{x}))| < \varepsilon \tag{9.10}$$

for any $\mathbf{y} \in \mathbf{h}(B(\mathbf{x},r))$ and also

$$|f(\mathbf{h}(\mathbf{x}_1))|\det(D\mathbf{h}(\mathbf{x}_1))| - f(\mathbf{h}(\mathbf{x}))|\det(D\mathbf{h}(\mathbf{x}))|| < \varepsilon \tag{9.11}$$

whenever $\mathbf{x}_1 \in B(\mathbf{x},r)$. The collection of such balls is a Vitali cover of U. By Corollary 9.7.6 there is a sequence of disjoint closed balls $\{B_i\}$ such that $U = \cup_{i=1}^{\infty} B_i \cup N$ where $m_n(N) = 0$. Denote by $\mathbf{x}_i$ the center of B_i and r_i the radius. Then by Lemma 9.8.1, the monotone convergence theorem, and 9.9 - 9.11,

$$\begin{aligned}
&\int_V f(\mathbf{y})\,dm_n = \sum_{i=1}^{\infty} \int_{\mathbf{h}(B_i)} f(\mathbf{y})\,dm_n \\
&\leq \varepsilon m_n(V) + \sum_{i=1}^{\infty} \int_{\mathbf{h}(B_i)} f(\mathbf{h}(\mathbf{x}_i))\,dm_n \\
&\leq \varepsilon m_n(V) + \sum_{i=1}^{\infty} f(\mathbf{h}(\mathbf{x}_i))\, m_n(\mathbf{h}(B_i)) \\
&\leq \varepsilon m_n(V) + \sum_{i=1}^{\infty} f(\mathbf{h}(\mathbf{x}_i))\, m_n(D\mathbf{h}(\mathbf{x}_i)(B(\mathbf{0},(1+\varepsilon)r_i))) \\
&= \varepsilon m_n(V) + (1+\varepsilon)^n \sum_{i=1}^{\infty} \int_{B_i} f(\mathbf{h}(\mathbf{x}_i))\,|\det(D\mathbf{h}(\mathbf{x}_i))|\,dm_n \\
&\leq \varepsilon m_n(V) + (1+\varepsilon)^n \sum_{i=1}^{\infty} \left(\int_{B_i} f(\mathbf{h}(\mathbf{x}))\,|\det(D\mathbf{h}(\mathbf{x}))|\,dm_n + \varepsilon m_n(B_i) \right) \\
&\leq \varepsilon m_n(V) + (1+\varepsilon)^n \sum_{i=1}^{\infty} \int_{B_i} f(\mathbf{h}(\mathbf{x}))\,|\det(D\mathbf{h}(\mathbf{x}))|\,dm_n + (1+\varepsilon)^n \varepsilon m_n(U) \\
&= \varepsilon m_n(V) + (1+\varepsilon)^n \int_U f(\mathbf{h}(\mathbf{x}))\,|\det(D\mathbf{h}(\mathbf{x}))|\,dm_n + (1+\varepsilon)^n \varepsilon m_n(U)
\end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, this shows

$$\int_V f(\mathbf{y})\,dm_n \leq \int_U f(\mathbf{h}(\mathbf{x}))\,|\det(D\mathbf{h}(\mathbf{x}))|\,dm_n \tag{9.12}$$

whenever $f \in C_c(V)$. Now $\mathbf{x} \to f(\mathbf{h}(\mathbf{x}))\,|\det(D\mathbf{h}(\mathbf{x}))|$ is in $C_c(U)$ and so using the same argument with U and V switching roles and replacing $\mathbf{h}$ with $\mathbf{h}^{-1}$,

$$\begin{aligned}
&\int_U f(\mathbf{h}(\mathbf{x}))\,|\det(D\mathbf{h}(\mathbf{x}))|\,dm_n \\
\leq\ &\int_V f\left(\mathbf{h}\left(\mathbf{h}^{-1}(\mathbf{y})\right)\right) \left|\det\left(D\mathbf{h}\left(\mathbf{h}^{-1}(\mathbf{y})\right)\right)\right| \left|\det\left(D\mathbf{h}^{-1}(\mathbf{y})\right)\right| dm_n \\
=\ &\int_V f(\mathbf{y})\,dm_n
\end{aligned}$$

by the chain rule. This with 9.12 proves the lemma.

The next task is to relax the assumption that f is continuous.

Corollary 9.9.2 *Let U and V be bounded open sets in $\mathbb{R}^n$ and let $\mathbf{h}, \mathbf{h}^{-1}$ be C^1 functions such that $\mathbf{h}(U) = V$. Also let $E \subseteq V$ be measurable. Then*

$$\int_V \mathcal{X}_E(\mathbf{y})\,dm_n = \int_U \mathcal{X}_E(\mathbf{h}(\mathbf{x}))\,|\det(D\mathbf{h}(\mathbf{x}))|\,dm_n.$$

Proof: By regularity, there exist compact sets, K_k and a decreasing sequence of open sets G_k such that

$$K_k \subseteq E \subseteq G_k$$

and $m_n(G_k \setminus K_k) < 2^{-k}$. By Lemma 8.10.2, there exist f_k such that $K_k \prec f_k \prec G_k$. Then $f_k(\mathbf{y}) \to \mathcal{X}_E(\mathbf{y})$ a.e. because if $\mathbf{y}$ is such that convergence fails, it must be the case that $\mathbf{y}$ is in $G_k \setminus K_k$ for infinitely many k and $\sum_k m_n(G_k \setminus K_k) < \infty$. This set equals

$$N = \cap_{m=1}^{\infty} \cup_{k=m}^{\infty} G_k \setminus K_k$$

and so for each $m \in \mathbb{N}$

$$\begin{aligned} m_n(N) &\leq m_n(\cup_{k=m}^{\infty} G_k \setminus K_k) \\ &\leq \sum_{k=m}^{\infty} m_n(G_k \setminus K_k) < \sum_{k=m}^{\infty} 2^{-k} = 2^{-(m-1)} \end{aligned}$$

showing $m_n(N) = 0$.

Then $f_k(\mathbf{h}(\mathbf{x}))$ must converge to $\mathcal{X}_E(\mathbf{h}(\mathbf{x}))$ for all $\mathbf{x} \notin \mathbf{h}^{-1}(N)$, a set of measure zero by Lemma 9.8.1. Thus

$$\int_V f_k(\mathbf{y})\, dm_n = \int_U f_k(\mathbf{h}(\mathbf{x}))\, |\det(D\mathbf{h}(\mathbf{x}))|\, dm_n.$$

Since U is bounded $\overline{G_1}$ is compact. Therefore, $|\det(D\mathbf{h}(\mathbf{x}))|$ is bounded independent of k and so, by the dominated convergence theorem, using a dominating function, $\mathcal{X}_V$ in the integral on the left and $\mathcal{X}_{G_1} |\det(D\mathbf{h})|$ on the right, it follows

$$\int_V \mathcal{X}_E(\mathbf{y})\, dm_n = \int_U \mathcal{X}_E(\mathbf{h}(\mathbf{x}))\, |\det(D\mathbf{h}(\mathbf{x}))|\, dm_n.$$

This proves the corollary.

You don't need to assume the open sets are bounded.

Corollary 9.9.3 *Let U and V be open sets in $\mathbb{R}^n$ and let $\mathbf{h}, \mathbf{h}^{-1}$ be C^1 functions such that $\mathbf{h}(U) = V$. Also let $E \subseteq V$ be measurable. Then*

$$\int_V \mathcal{X}_E(\mathbf{y})\, dm_n = \int_U \mathcal{X}_E(\mathbf{h}(\mathbf{x}))\, |\det(D\mathbf{h}(\mathbf{x}))|\, dm_n.$$

Proof: For each $\mathbf{x} \in U$, there exists $r_{\mathbf{x}}$ such that $\overline{B(\mathbf{x}, r_{\mathbf{x}})} \subseteq U$ and $r_{\mathbf{x}} < 1$. Then by the mean value inequality Theorem 6.4.2, it follows $\mathbf{h}(B(\mathbf{x}, r_{\mathbf{x}}))$ is also bounded. These balls, $B(\mathbf{x}, r_{\mathbf{x}})$ give a Vitali cover of U and so by Corollary 9.7.6 there is a sequence of these balls, $\{B_i\}$ such that they are disjoint, $\mathbf{h}(B_i)$ is bounded and

$$m_n(U \setminus \cup_i B_i) = 0.$$

It follows from Lemma 9.8.1 that $\mathbf{h}(U \setminus \cup_i B_i)$ also has measure zero. Then from Corollary 9.9.2

$$\begin{aligned} \int_V \mathcal{X}_E(\mathbf{y})\, dm_n &= \sum_i \int_{\mathbf{h}(B_i)} \mathcal{X}_{E \cap \mathbf{h}(B_i)}(\mathbf{y})\, dm_n \\ &= \sum_i \int_{B_i} \mathcal{X}_E(\mathbf{h}(\mathbf{x}))\, |\det(D\mathbf{h}(\mathbf{x}))|\, dm_n \\ &= \int_U \mathcal{X}_E(\mathbf{h}(\mathbf{x}))\, |\det(D\mathbf{h}(\mathbf{x}))|\, dm_n. \end{aligned}$$

This proves the corollary.

With this corollary, the main theorem follows.

Theorem 9.9.4 *Let U and V be open sets in $\mathbb{R}^n$ and let $\mathbf{h}, \mathbf{h}^{-1}$ be C^1 functions such that $\mathbf{h}(U) = V$. Then if g is a nonnegative Lebesgue measurable function,*

$$\int_V g(\mathbf{y})\, dm_n = \int_U g(\mathbf{h}(\mathbf{x}))\, |\det(D\mathbf{h}(\mathbf{x}))|\, dm_n. \tag{9.13}$$

Proof: From Corollary 9.9.3, 9.13 holds for any nonnegative simple function in place of g. In general, let $\{s_k\}$ be an increasing sequence of simple functions which converges to g pointwise. Then from the monotone convergence theorem

$$\begin{aligned}\int_V g(\mathbf{y})\, dm_n &= \lim_{k\to\infty} \int_V s_k dm_n = \lim_{k\to\infty} \int_U s_k(\mathbf{h}(\mathbf{x}))\, |\det(D\mathbf{h}(\mathbf{x}))|\, dm_n \\ &= \int_U g(\mathbf{h}(\mathbf{x}))\, |\det(D\mathbf{h}(\mathbf{x}))|\, dm_n.\end{aligned}$$

This proves the theorem.

Of course this theorem implies the following corollary by splitting up the function into the positive and negative parts of the real and imaginary parts.

Corollary 9.9.5 *Let U and V be open sets in $\mathbb{R}^n$ and let $\mathbf{h}, \mathbf{h}^{-1}$ be C^1 functions such that $\mathbf{h}(U) = V$. Let $g \in L^1(V)$. Then*

$$\int_V g(\mathbf{y})\, dm_n = \int_U g(\mathbf{h}(\mathbf{x}))\, |\det(D\mathbf{h}(\mathbf{x}))|\, dm_n.$$

This is a pretty good theorem but it isn't too hard to generalize it. In particular, it is not necessary to assume $\mathbf{h}^{-1}$ is C^1.

Lemma 9.9.6 *Suppose V is an $n-1$ dimensional subspace of $\mathbb{R}^n$ and K is a compact subset of V. Then letting*

$$K_\varepsilon \equiv \cup_{\mathbf{x}\in K} B(\mathbf{x},\varepsilon) = K + B(\mathbf{0},\varepsilon),$$

it follows that

$$m_n(K_\varepsilon) \le 2^n \varepsilon (\operatorname{diam}(K) + \varepsilon)^{n-1}.$$

Proof: Using the Gram Schmidt procedure, there exists an orthonormal basis for V, $\{\mathbf{v}_1, \cdots, \mathbf{v}_{n-1}\}$ and let

$$\{\mathbf{v}_1, \cdots, \mathbf{v}_{n-1}, \mathbf{v}_n\}$$

be an orthonormal basis for $\mathbb{R}^n$. Now define a linear transformation, Q by $Q\mathbf{v}_i = \mathbf{e}_i$. Thus $QQ^* = Q^*Q = I$ and Q preserves all distances and is a unitary transformation because

$$\left|Q\sum_i a_i \mathbf{e}_i\right|^2 = \left|\sum_i a_i \mathbf{v}_i\right|^2 = \sum_i |a_i|^2 = \left|\sum_i a_i \mathbf{e}_i\right|^2.$$

Thus $m_n(K_\varepsilon) = m_n(QK_\varepsilon)$. Letting $\mathbf{k}_0 \in K$, it follows $K \subseteq B(\mathbf{k}_0, \operatorname{diam}(K))$ and so,

$$QK \subseteq B^{n-1}(Q\mathbf{k}_0, \operatorname{diam}(QK)) = B^{n-1}(Q\mathbf{k}_0, \operatorname{diam}(K))$$

where B^{n-1} refers to the ball taken with respect to the usual norm in $\mathbb{R}^{n-1}$. Every point of K_ε is within ε of some point of K and so it follows that every point of QK_ε is within ε of some point of QK. Therefore,

$$QK_\varepsilon \subseteq B^{n-1}(Q\mathbf{k}_0, \operatorname{diam}(QK) + \varepsilon) \times (-\varepsilon, \varepsilon),$$

To see this, let $\mathbf{x} \in QK_\varepsilon$. Then there exists $\mathbf{k} \in QK$ such that $|\mathbf{k}-\mathbf{x}| < \varepsilon$. Therefore, $|(x_1,\cdots,x_{n-1})-(k_1,\cdots,k_{n-1})| < \varepsilon$ and $|x_n - k_n| < \varepsilon$ and so $\mathbf{x}$ is contained in the set on the right in the above inclusion because $k_n = 0$. However, the measure of the set on the right is smaller than

$$[2(\operatorname{diam}(QK)+\varepsilon)]^{n-1}(2\varepsilon) = 2^n[(\operatorname{diam}(K)+\varepsilon)]^{n-1}\varepsilon.$$

This proves the lemma.

Note this is a very sloppy estimate. You can certainly do much better but this estimate is sufficient to prove Sard's lemma which follows.

Definition 9.9.7 *If T, S are two nonempty sets in a normed vector space,*

$$\operatorname{dist}(S,T) \equiv \inf\{||\mathbf{s}-\mathbf{t}|| : \mathbf{s} \in S, \mathbf{t} \in T\}.$$

Lemma 9.9.8 *Let $\mathbf{h}$ be a C^1 function defined on an open set, $U \subseteq \mathbb{R}^n$ and let K be a compact subset of U. Then if $\varepsilon > 0$ is given, there exists $r_1 > 0$ such that if $|\mathbf{v}| \le r_1$, then for all $\mathbf{x} \in K$,*

$$|\mathbf{h}(\mathbf{x}+\mathbf{v}) - \mathbf{h}(\mathbf{x}) - D\mathbf{h}(\mathbf{x})\mathbf{v}| < \varepsilon|\mathbf{v}|.$$

Proof: Let $0 < \delta < \operatorname{dist}\left(K, U^C\right)$. Such a positive number exists because if there exists a sequence of points in $K, \{\mathbf{k}_k\}$ and points in $U^C, \{\mathbf{s}_k\}$ such that $|\mathbf{k}_k - \mathbf{s}_k| \to 0$, then you could take a subsequence, still denoted by k such that $\mathbf{k}_k \to \mathbf{k} \in K$ and then $\mathbf{s}_k \to \mathbf{k}$ also. But U^C is closed so $\mathbf{k} \in K \cap U^C$, a contradiction. Then for $|\mathbf{v}| < \delta$ it follows that for every $\mathbf{x} \in K$,

$$\mathbf{x}+t\mathbf{v} \in U$$

and

$$\begin{aligned}\frac{|\mathbf{h}(\mathbf{x}+\mathbf{v}) - \mathbf{h}(\mathbf{x}) - D\mathbf{h}(\mathbf{x})\mathbf{v}|}{|\mathbf{v}|} &\le \frac{\left|\int_0^1 D\mathbf{h}(\mathbf{x}+t\mathbf{v})\mathbf{v}dt - D\mathbf{h}(\mathbf{x})\mathbf{v}\right|}{|\mathbf{v}|} \\ &\le \frac{\int_0^1 |D\mathbf{h}(\mathbf{x}+t\mathbf{v})\mathbf{v} - D\mathbf{h}(\mathbf{x})\mathbf{v}|\,dt}{|\mathbf{v}|}.\end{aligned}$$

The integral in the above involves integrating componentwise. Thus $t \to D\mathbf{h}(\mathbf{x}+t\mathbf{v})\mathbf{v}$ is a function having values in $\mathbb{R}^n$

$$\begin{pmatrix} Dh_1(\mathbf{x}+t\mathbf{v})\mathbf{v} \\ \vdots \\ Dh_n(\mathbf{x}+t\mathbf{v})\mathbf{v} \end{pmatrix}$$

and the integral is defined by

$$\begin{pmatrix} \int_0^1 Dh_1(\mathbf{x}+t\mathbf{v})\mathbf{v}dt \\ \vdots \\ \int_0^1 Dh_n(\mathbf{x}+t\mathbf{v})\mathbf{v}dt \end{pmatrix}$$

Now from uniform continuity of $D\mathbf{h}$ on the compact set, $\{\mathbf{x} : \operatorname{dist}(\mathbf{x},K) \le \delta\}$ it follows there exists $r_1 < \delta$ such that if $|\mathbf{v}| \le r_1$, then $||D\mathbf{h}(\mathbf{x}+t\mathbf{v}) - D\mathbf{h}(\mathbf{x})|| < \varepsilon$ for every $\mathbf{x} \in K$. From the above formula, it follows that if $|\mathbf{v}| \le r_1$,

$$\begin{aligned}\frac{|\mathbf{h}(\mathbf{x}+\mathbf{v}) - \mathbf{h}(\mathbf{x}) - D\mathbf{h}(\mathbf{x})\mathbf{v}|}{|\mathbf{v}|} &\le \frac{\int_0^1 |D\mathbf{h}(\mathbf{x}+t\mathbf{v})\mathbf{v} - D\mathbf{h}(\mathbf{x})\mathbf{v}|\,dt}{|\mathbf{v}|} \\ &< \frac{\int_0^1 \varepsilon|\mathbf{v}|\,dt}{|\mathbf{v}|} = \varepsilon.\end{aligned}$$

This proves the lemma.

The following is Sard's lemma. In the proof, it does not matter which norm you use in defining balls.

Lemma 9.9.9 *(Sard) Let U be an open set in $\mathbb{R}^n$ and let $\mathbf{h}: U \to \mathbb{R}^n$ be C^1. Let*

$$Z \equiv \{\mathbf{x} \in U : \det D\mathbf{h}(\mathbf{x}) = 0\}.$$

Then $m_n(\mathbf{h}(Z)) = 0$.

Proof: Let $\{U_k\}_{k=1}^{\infty}$ be an increasing sequence of open sets whose closures are compact and whose union equals U and let $Z_k \equiv Z \cap \overline{U_k}$. To obtain such a sequence, let

$$U_k = \left\{\mathbf{x} \in U : \operatorname{dist}\left(\mathbf{x}, U^C\right) < \frac{1}{k}\right\} \cap B(\mathbf{0}, k).$$

First it is shown that $\mathbf{h}(Z_k)$ has measure zero. Let W be an open set contained in U_{k+1} which contains Z_k and satisfies

$$m_n(Z_k) + \varepsilon > m_n(W)$$

where here and elsewhere, $\varepsilon < 1$. Let

$$r = \operatorname{dist}\left(\overline{U_k}, U_{k+1}^C\right)$$

and let $r_1 > 0$ be a constant as in Lemma 9.9.8 such that whenever $\mathbf{x} \in \overline{U_k}$ and $0 < |\mathbf{v}| \leq r_1$,

$$|\mathbf{h}(\mathbf{x}+\mathbf{v}) - \mathbf{h}(\mathbf{x}) - D\mathbf{h}(\mathbf{x})\mathbf{v}| < \varepsilon |\mathbf{v}|. \tag{9.14}$$

Now the closures of balls which are contained in W and which have the property that their diameters are less than r_1 yield a Vitali covering of W. Therefore, by Corollary 9.7.6 there is a disjoint sequence of these closed balls, $\left\{\widetilde{B}_i\right\}$ such that

$$W = \cup_{i=1}^{\infty} \widetilde{B}_i \cup N$$

where N is a set of measure zero. Denote by $\{B_i\}$ those closed balls in this sequence which have nonempty intersection with Z_k, let d_i be the diameter of B_i, and let $\mathbf{z}_i$ be a point in $B_i \cap Z_k$. Since $\mathbf{z}_i \in Z_k$, it follows $D\mathbf{h}(\mathbf{z}_i) B(\mathbf{0}, d_i) = D_i$ where D_i is contained in a subspace, V which has dimension $n-1$ and the diameter of D_i is no larger than $2C_k d_i$ where

$$C_k \geq \max\{||D\mathbf{h}(\mathbf{x})|| : \mathbf{x} \in Z_k\}$$

Then by 9.14, if $\mathbf{z} \in B_i$,

$$\mathbf{h}(\mathbf{z}) - \mathbf{h}(\mathbf{z}_i) \in D_i + B(\mathbf{0}, \varepsilon d_i) \subseteq \overline{D_i} + B(\mathbf{0}, \varepsilon d_i).$$

Thus

$$\mathbf{h}(B_i) \subseteq \mathbf{h}(\mathbf{z}_i) + \overline{D_i} + B(\mathbf{0}, \varepsilon d_i)$$

By Lemma 9.9.6

$$\begin{aligned} m_n(\mathbf{h}(B_i)) &\leq 2^n (2C_k d_i + \varepsilon d_i)^{n-1} \varepsilon d_i \\ &\leq d_i^n \left(2^n [2C_k + \varepsilon]^{n-1}\right) \varepsilon \\ &\leq C_{n,k} m_n(B_i) \varepsilon. \end{aligned}$$

Therefore, by Lemma 9.8.1

$$\begin{aligned} m_n(\mathbf{h}(Z_k)) &\leq m_n(W) = \sum_i m_n(\mathbf{h}(B_i)) \leq C_{n,k}\varepsilon \sum_i m_n(B_i) \\ &\leq \varepsilon C_{n,k} m_n(W) \leq \varepsilon C_{n,k}(m_n(Z_k) + \varepsilon) \end{aligned}$$

Since ε is arbitrary, this shows $m_n(\mathbf{h}(Z_k)) = 0$ and so $0 = \lim_{k\to\infty} m_n(\mathbf{h}(Z_k)) = m_n(\mathbf{h}(Z))$.

With this important lemma, here is a generalization of Theorem 9.9.4.

Theorem 9.9.10 *Let U be an open set and let $\mathbf{h}$ be a $1-1$, $C^1(U)$ function with values in $\mathbb{R}^n$. Then if g is a nonnegative Lebesgue measurable function,*

$$\int_{\mathbf{h}(U)} g(\mathbf{y})\, dm_n = \int_U g(\mathbf{h}(\mathbf{x}))\, |\det(D\mathbf{h}(\mathbf{x}))|\, dm_n. \tag{9.15}$$

Proof: Let $Z = \{\mathbf{x} : \det(D\mathbf{h}(\mathbf{x})) = 0\}$, a closed set. Then by the inverse function theorem, $\mathbf{h}^{-1}$ is C^1 on $\mathbf{h}(U \setminus Z)$ and $\mathbf{h}(U \setminus Z)$ is an open set. Therefore, from Lemma 9.9.9, $\mathbf{h}(Z)$ has measure zero and so by Theorem 9.9.4,

$$\begin{aligned} \int_{\mathbf{h}(U)} g(\mathbf{y})\, dm_n &= \int_{\mathbf{h}(U\setminus Z)} g(\mathbf{y})\, dm_n = \int_{U\setminus Z} g(\mathbf{h}(\mathbf{x}))\, |\det(D\mathbf{h}(\mathbf{x}))|\, dm_n \\ &= \int_U g(\mathbf{h}(\mathbf{x}))\, |\det(D\mathbf{h}(\mathbf{x}))|\, dm_n. \end{aligned}$$

This proves the theorem.

Of course the next generalization considers the case when $\mathbf{h}$ is not even one to one.

9.10 Change Of Variables For Mappings Which Are Not One To One

Now suppose $\mathbf{h}$ is only C^1, not necessarily one to one. For

$$U_+ \equiv \{\mathbf{x} \in U : |\det D\mathbf{h}(x)| > 0\}$$

and Z the set where $|\det D\mathbf{h}(\mathbf{x})| = 0$, Lemma 9.9.9 implies $m_n(\mathbf{h}(Z)) = 0$. For $\mathbf{x} \in U_+$, the inverse function theorem implies there exists an open set $B_{\mathbf{x}} \subseteq U_+$, such that $\mathbf{h}$ is one to one on $B_{\mathbf{x}}$.

Let $\{B_i\}$ be a countable subset of $\{B_{\mathbf{x}}\}_{\mathbf{x}\in U_+}$ such that $U_+ = \cup_{i=1}^\infty B_i$. Let $E_1 = B_1$. If $E_1, \cdots, E_k$ have been chosen, $E_{k+1} = B_{k+1} \setminus \cup_{i=1}^k E_i$. Thus

$$\cup_{i=1}^\infty E_i = U_+, \ \ \mathbf{h} \text{ is one to one on } E_i, \ \ E_i \cap E_j = \emptyset,$$

and each E_i is a Borel set contained in the open set B_i. Now define

$$n(\mathbf{y}) \equiv \sum_{i=1}^\infty \mathcal{X}_{\mathbf{h}(E_i)}(\mathbf{y}) + \mathcal{X}_{\mathbf{h}(Z)}(\mathbf{y}).$$

The set, $\mathbf{h}(E_i), \mathbf{h}(Z)$ are measurable by Lemma 9.8.2. Thus $n(\cdot)$ is measurable.

Lemma 9.10.1 *Let $F \subseteq \mathbf{h}(U)$ be measurable. Then*

$$\int_{\mathbf{h}(U)} n(\mathbf{y}) \mathcal{X}_F(\mathbf{y}) dm_n = \int_U \mathcal{X}_F(\mathbf{h}(\mathbf{x}))|\det D\mathbf{h}(\mathbf{x})| dm_n.$$

Proof: Using Lemma 9.9.9 and the Monotone Convergence Theorem

$$\begin{aligned}
\int_{\mathbf{h}(U)} n(\mathbf{y})\mathcal{X}_F(\mathbf{y})dm_n &= \int_{\mathbf{h}(U)} \left(\sum_{i=1}^{\infty} \mathcal{X}_{\mathbf{h}(E_i)}(\mathbf{y}) + \overbrace{\mathcal{X}_{\mathbf{h}(Z)}(\mathbf{y})}^{m_n(\mathbf{h}(Z))=0} \right) \mathcal{X}_F(\mathbf{y})dm_n \\
&= \sum_{i=1}^{\infty} \int_{\mathbf{h}(U)} \mathcal{X}_{\mathbf{h}(E_i)}(\mathbf{y})\mathcal{X}_F(\mathbf{y})dm_n \\
&= \sum_{i=1}^{\infty} \int_{\mathbf{h}(B_i)} \mathcal{X}_{\mathbf{h}(E_i)}(\mathbf{y})\mathcal{X}_F(\mathbf{y})dm_n \\
&= \sum_{i=1}^{\infty} \int_{B_i} \mathcal{X}_{E_i}(\mathbf{x})\mathcal{X}_F(\mathbf{h}(\mathbf{x}))|\det D\mathbf{h}(\mathbf{x})|dm_n \\
&= \sum_{i=1}^{\infty} \int_{U} \mathcal{X}_{E_i}(\mathbf{x})\mathcal{X}_F(\mathbf{h}(\mathbf{x}))|\det D\mathbf{h}(\mathbf{x})|dm_n \\
&= \int_{U} \sum_{i=1}^{\infty} \mathcal{X}_{E_i}(\mathbf{x})\mathcal{X}_F(\mathbf{h}(\mathbf{x}))|\det D\mathbf{h}(\mathbf{x})|dm_n
\end{aligned}$$

$$= \int_{U_+} \mathcal{X}_F(\mathbf{h}(\mathbf{x}))|\det D\mathbf{h}(\mathbf{x})|dm_n = \int_U \mathcal{X}_F(\mathbf{h}(\mathbf{x}))|\det D\mathbf{h}(\mathbf{x})|dm_n.$$

This proves the lemma.

Definition 9.10.2 *For $\mathbf{y} \in \mathbf{h}(U)$, define a function, $\#$, according to the formula*

$$\#(\mathbf{y}) \equiv \textit{number of elements in } \mathbf{h}^{-1}(\mathbf{y}).$$

Observe that

$$\#(\mathbf{y}) = n(\mathbf{y}) \quad \text{a.e.} \tag{9.16}$$

because $n(\mathbf{y}) = \#(\mathbf{y})$ if $\mathbf{y} \notin \mathbf{h}(Z)$, a set of measure 0. Therefore, $\#$ is a measurable function because of completeness of Lebesgue measure.

Theorem 9.10.3 *Let $g \geq 0$, g measurable, and let $\mathbf{h}$ be $C^1(U)$. Then*

$$\int_{\mathbf{h}(U)} \#(\mathbf{y})g(\mathbf{y})dm_n = \int_U g(\mathbf{h}(\mathbf{x}))|\det D\mathbf{h}(\mathbf{x})|dm_n. \tag{9.17}$$

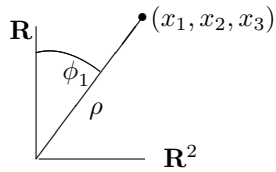
Proof: From 9.16 and Lemma 9.10.1, 9.17 holds for all g, a nonnegative simple function. Approximating an arbitrary measurable nonnegative function, g, with an increasing pointwise convergent sequence of simple functions and using the monotone convergence theorem, yields 9.17 for an arbitrary nonnegative measurable function, g. This proves the theorem.

9.11 Spherical Coordinates In n Dimensions

Sometimes there is a need to deal with spherical coordinates in more than three dimensions. In this section, this concept is defined and formulas are derived for these coordinate systems. Recall polar coordinates are of the form

$$\begin{aligned}
y_1 &= \rho\cos\theta \\
y_2 &= \rho\sin\theta
\end{aligned}$$

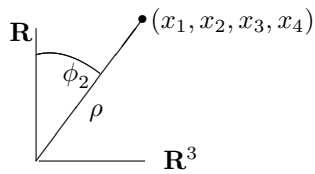
where $\rho > 0$ and $\theta \in \mathbb{R}$. Thus these transformation equations are not one to one but they are one to one on $(0,\infty) \times [0,2\pi)$. Here I am writing ρ in place of r to emphasize a pattern which is about to emerge. I will consider polar coordinates as spherical coordinates in two dimensions. I will also simply refer to such coordinate systems as polar coordinates regardless of the dimension. This is also the reason I am writing y_1 and y_2 instead of the more usual x and y. Now consider what happens when you go to three dimensions. The situation is depicted in the following picture.



From this picture, you see that $y_3 = \rho\cos\phi_1$. Also the distance between (y_1, y_2) and $(0,0)$ is $\rho\sin(\phi_1)$. Therefore, using polar coordinates to write (y_1, y_2) in terms of θ and this distance,

$$\begin{aligned} y_1 &= \rho\sin\phi_1\cos\theta, \\ y_2 &= \rho\sin\phi_1\sin\theta, \\ y_3 &= \rho\cos\phi_1. \end{aligned}$$

where $\phi_1 \in \mathbb{R}$ and the transformations are one to one if ϕ_1 is restricted to be in $[0,\pi]$. What was done is to replace ρ with $\rho\sin\phi_1$ and then to add in $y_3 = \rho\cos\phi_1$. Having done this, there is no reason to stop with three dimensions. Consider the following picture:



From this picture, you see that $y_4 = \rho\cos\phi_2$. Also the distance between (y_1, y_2, y_3) and $(0,0,0)$ is $\rho\sin(\phi_2)$. Therefore, using polar coordinates to write (y_1, y_2, y_3) in terms of θ, ϕ_1, and this distance,

$$\begin{aligned} y_1 &= \rho\sin\phi_2\sin\phi_1\cos\theta, \\ y_2 &= \rho\sin\phi_2\sin\phi_1\sin\theta, \\ y_3 &= \rho\sin\phi_2\cos\phi_1, \\ y_4 &= \rho\cos\phi_2 \end{aligned}$$

where $\phi_2 \in \mathbb{R}$ and the transformations will be one to one if

$$\phi_2, \phi_1 \in [0,\pi], \theta \in [0,2\pi), \rho \in (0,\infty).$$

Continuing this way, given spherical coordinates in $\mathbb{R}^n$, to get the spherical coordinates in $\mathbb{R}^{n+1}$, you let $y_{n+1} = \rho\cos\phi_{n-1}$ and then replace every occurance of ρ with $\rho\sin\phi_{n-1}$ to obtain $y_1 \cdots y_n$ in terms of $\phi_1, \phi_2, \cdots, \phi_{n-1}$,$\theta$, and ρ.

It is always the case that ρ measures the distance from the point in $\mathbb{R}^n$ to the origin in $\mathbb{R}^n$, $\mathbf{0}$. Each $\phi_i \in \mathbb{R}$ and the transformations will be one to one if each $\phi_i \in [0,\pi]$, and $\theta \in [0,2\pi)$. It can be shown using math induction that these coordinates map $\prod_{i=1}^{n-2}[0,\pi] \times [0,2\pi) \times (0,\infty)$ one to one onto $\mathbb{R}^n \setminus \{\mathbf{0}\}$.

Theorem 9.11.1 *Let $\mathbf{y} = \mathbf{h}(\boldsymbol{\phi}, \theta, \rho)$ be the spherical coordinate transformations in $\mathbb{R}^n$. Then letting $A = \prod_{i=1}^{n-2} [0, \pi] \times [0, 2\pi)$, it follows $\mathbf{h}$ maps $A \times (0, \infty)$ one to one onto $\mathbb{R}^n \setminus \{\mathbf{0}\}$. Also $|\det D\mathbf{h}(\boldsymbol{\phi}, \theta, \rho)|$ will always be of the form*

$$|\det D\mathbf{h}(\boldsymbol{\phi}, \theta, \rho)| = \rho^{n-1} \Phi(\boldsymbol{\phi}, \theta). \tag{9.18}$$

where Φ is a continuous function of $\boldsymbol{\phi}$ and θ.[1] *Furthermore whenever f is Lebesgue measurable and nonnegative,*

$$\int_{\mathbb{R}^n} f(\mathbf{y})\, dy = \int_0^\infty \rho^{n-1} \int_A f(\mathbf{h}(\boldsymbol{\phi}, \theta, \rho)) \Phi(\boldsymbol{\phi}, \theta)\, d\boldsymbol{\phi}\, d\theta d\rho \tag{9.19}$$

where here $d\boldsymbol{\phi}\, d\theta$ denotes dm_{n-1} on A. The same formula holds if $f \in L^1(\mathbb{R}^n)$.

Proof: Formula 9.18 is obvious from the definition of the spherical coordinates. The first claim is also clear from the definition and math induction. It remains to verify 9.19. Let $A_0 \equiv \prod_{i=1}^{n-2} (0, \pi) \times (0, 2\pi)$. Then it is clear that $(A \setminus A_0) \times (0, \infty) \equiv N$ is a set of measure zero in $\mathbb{R}^n$. Therefore, from Lemma 9.8.1 it follows $\mathbf{h}(N)$ is also a set of measure zero. Therefore, using the change of variables theorem, Fubini's theorem, and Sard's lemma,

$$\begin{aligned}
\int_{\mathbb{R}^n} f(\mathbf{y})\, dy &= \int_{\mathbb{R}^n \setminus \{\mathbf{0}\}} f(\mathbf{y})\, dy = \int_{\mathbb{R}^n \setminus (\{\mathbf{0}\} \cup \mathbf{h}(N))} f(\mathbf{y})\, dy \\
&= \int_{A_0 \times (0,\infty)} f(\mathbf{h}(\boldsymbol{\phi}, \theta, \rho)) \rho^{n-1} \Phi(\boldsymbol{\phi}, \theta)\, dm_n \\
&= \int \mathcal{X}_{A \times (0,\infty)}(\boldsymbol{\phi}, \theta, \rho) f(\mathbf{h}(\boldsymbol{\phi}, \theta, \rho)) \rho^{n-1} \Phi(\boldsymbol{\phi}, \theta)\, dm_n \\
&= \int_0^\infty \rho^{n-1} \left(\int_A f(\mathbf{h}(\boldsymbol{\phi}, \theta, \rho)) \Phi(\boldsymbol{\phi}, \theta)\, d\boldsymbol{\phi}\, d\theta \right) d\rho.
\end{aligned}$$

Now the claim about $f \in L^1$ follows routinely from considering the positive and negative parts of the real and imaginary parts of f in the usual way. This proves the theorem.

Notation 9.11.2 *Often this is written differently. Note that from the spherical coordinate formulas, $f(\mathbf{h}(\boldsymbol{\phi}, \theta, \rho)) = f(\rho\boldsymbol{\omega})$ where $|\boldsymbol{\omega}| = 1$. Letting S^{n-1} denote the unit sphere, $\{\boldsymbol{\omega} \in \mathbb{R}^n : |\boldsymbol{\omega}| = 1\}$, the inside integral in the above formula is sometimes written as*

$$\int_{S^{n-1}} f(\rho\boldsymbol{\omega})\, d\sigma$$

where σ is a measure on S^{n-1}. See [26] for another description of this measure. It isn't an important issue here. Either 9.19 or the formula

$$\int_0^\infty \rho^{n-1} \left(\int_{S^{n-1}} f(\rho\boldsymbol{\omega})\, d\sigma \right) d\rho$$

will be referred to as polar coordinates and is very useful in establishing estimates. Here $\sigma\left(S^{n-1}\right) \equiv \int_A \Phi(\boldsymbol{\phi}, \theta)\, d\boldsymbol{\phi}\, d\theta$.

Example 9.11.3 *For what values of s is the integral $\int_{B(\mathbf{0},R)} \left(1 + |\mathbf{x}|^2\right)^s dy$ bounded independent of R? Here $B(\mathbf{0}, R)$ is the ball, $\{\mathbf{x} \in \mathbb{R}^n : |\mathbf{x}| \leq R\}$.*

[1] Actually it is only a function of the first but this is not important in what follows.

I think you can see immediately that s must be negative but exactly how negative? It turns out it depends on n and using polar coordinates, you can find just exactly what is needed. From the polar coordinats formula above,

$$\begin{aligned}\int_{B(\mathbf{0},R)} \left(1+|\mathbf{x}|^2\right)^s dy &= \int_0^R \int_{S^{n-1}} \left(1+\rho^2\right)^s \rho^{n-1} d\sigma d\rho \\ &= C_n \int_0^R \left(1+\rho^2\right)^s \rho^{n-1} d\rho\end{aligned}$$

Now the very hard problem has been reduced to considering an easy one variable problem of finding when

$$\int_0^R \rho^{n-1} \left(1+\rho^2\right)^s d\rho$$

is bounded independent of R. You need $2s+(n-1)<-1$ so you need $s<-n/2$.

9.12 Brouwer Fixed Point Theorem

The Brouwer fixed point theorem is one of the most significant theorems in mathematics. There exist relatively easy proofs of this important theorem. The proof I am giving here is the one given in Evans [14]. I think it is one of the shortest and easiest proofs of this important theorem. It is based on the following lemma which is an interesting result about cofactors of a matrix.

Recall that for A an $n\times n$ matrix, $\operatorname{cof}(A)_{ij}$ is the determinant of the matrix which results from deleting the i^{th} row and the j^{th} column and multiplying by $(-1)^{i+j}$. In the proof and in what follows, I am using $D\mathbf{g}$ to equal the matrix of the linear transformation $D\mathbf{g}$ taken with respect to the usual basis on $\mathbb{R}^n$. Thus

$$D\mathbf{g}(\mathbf{x}) = \sum_{ij} (D\mathbf{g})_{ij} \mathbf{e}_i \mathbf{e}_j$$

and recall that $(D\mathbf{g})_{ij} = \partial g_i/\partial x_j$ where $\mathbf{g} = \sum_i g_i \mathbf{e}_i$.

Lemma 9.12.1 *Let $\mathbf{g}: U \to \mathbb{R}^n$ be C^2 where U is an open subset of $\mathbb{R}^n$. Then*

$$\sum_{j=1}^n \operatorname{cof}(D\mathbf{g})_{ij,j} = 0,$$

where here $(D\mathbf{g})_{ij} \equiv g_{i,j} \equiv \frac{\partial g_i}{\partial x_j}$. Also, $\operatorname{cof}(D\mathbf{g})_{ij} = \frac{\partial \det(D\mathbf{g})}{\partial g_{i,j}}$.

Proof: From the cofactor expansion theorem,

$$\det(D\mathbf{g}) = \sum_{i=1}^n g_{i,j} \operatorname{cof}(D\mathbf{g})_{ij}$$

and so

$$\frac{\partial \det(D\mathbf{g})}{\partial g_{i,j}} = \operatorname{cof}(D\mathbf{g})_{ij} \tag{9.20}$$

which shows the last claim of the lemma. Also

$$\delta_{kj} \det(D\mathbf{g}) = \sum_i g_{i,k} \left(\operatorname{cof}(D\mathbf{g})\right)_{ij} \tag{9.21}$$

because if $k \neq j$ this is just the cofactor expansion of the determinant of a matrix in which the k^{th} and j^{th} columns are equal. Differentiate 9.21 with respect to x_j and sum on j. This yields

$$\sum_{r,s,j} \delta_{kj} \frac{\partial \left(\det D\mathbf{g}\right)}{\partial g_{r,s}} g_{r,sj} = \sum_{ij} g_{i,kj} \left(\operatorname{cof}\left(D\mathbf{g}\right)\right)_{ij} + \sum_{ij} g_{i,k} \operatorname{cof}\left(D\mathbf{g}\right)_{ij,j}.$$

Hence, using $\delta_{kj} = 0$ if $j \neq k$ and 9.20,

$$\sum_{rs} \left(\operatorname{cof}\left(D\mathbf{g}\right)\right)_{rs} g_{r,sk} = \sum_{rs} g_{r,ks} \left(\operatorname{cof}\left(D\mathbf{g}\right)\right)_{rs} + \sum_{ij} g_{i,k} \operatorname{cof}\left(D\mathbf{g}\right)_{ij,j}.$$

Subtracting the first sum on the right from both sides and using the equality of mixed partials,

$$\sum_{i} g_{i,k} \left(\sum_{j} \left(\operatorname{cof}\left(D\mathbf{g}\right)\right)_{ij,j} \right) = 0.$$

If $\det\left(g_{i,k}\right) \neq 0$ so that $\left(g_{i,k}\right)$ is invertible, this shows $\sum_j \left(\operatorname{cof}\left(D\mathbf{g}\right)\right)_{ij,j} = 0$. If $\det\left(D\mathbf{g}\right) = 0$, let

$$g_k = g + \varepsilon_k I$$

where $\varepsilon_k \to 0$ and $\det\left(D\mathbf{g} + \varepsilon_k I\right) \equiv \det\left(D\mathbf{g}_k\right) \neq 0$. Then

$$\sum_{j} \left(\operatorname{cof}\left(D\mathbf{g}\right)\right)_{ij,j} = \lim_{k\to\infty} \sum_{j} \left(\operatorname{cof}\left(D\mathbf{g}_k\right)\right)_{ij,j} = 0$$

and this proves the lemma.

Definition 9.12.2 *Let* $\mathbf{h}$ *be a function defined on an open set,* $U \subseteq \mathbb{R}^n$. *Then* $\mathbf{h} \in C^k\left(\overline{U}\right)$ *if there exists a function* $\mathbf{g}$ *defined on an open set,* W *containng* $\overline{U}$ *such that* $\mathbf{g} = \mathbf{h}$ *on* U *and* $\mathbf{g}$ *is* $C^k\left(W\right)$.

In the following lemma, you could use any norm in defining the balls and everything would work the same but I have in mind the usual norm.

Lemma 9.12.3 *There does not exist* $\mathbf{h} \in C^2\left(\overline{B\left(\mathbf{0},R\right)}\right)$ *such that* $\mathbf{h} : \overline{B\left(\mathbf{0},R\right)} \to \partial B\left(\mathbf{0},R\right)$ *which also has the property that* $\mathbf{h}\left(\mathbf{x}\right) = \mathbf{x}$ *for all* $\mathbf{x} \in \partial B\left(\mathbf{0},R\right)$. *Such a function is called a retraction.*

Proof: Suppose such an $\mathbf{h}$ exists. Let $\lambda \in [0,1]$ and let $\mathbf{p}_\lambda\left(\mathbf{x}\right) \equiv \mathbf{x} + \lambda\left(\mathbf{h}\left(\mathbf{x}\right) - \mathbf{x}\right)$. This function, $\mathbf{p}_\lambda$ is called a homotopy of the identity map and the retraction, $\mathbf{h}$. Let

$$I\left(\lambda\right) \equiv \int_{B(\mathbf{0},R)} \det\left(D\mathbf{p}_\lambda\left(\mathbf{x}\right)\right) dx.$$

Then using the dominated convergence theorem,

$$\begin{aligned} I'\left(\lambda\right) &= \int_{B(\mathbf{0},R)} \sum_{i.j} \frac{\partial \det\left(D\mathbf{p}_\lambda\left(\mathbf{x}\right)\right)}{\partial p_{\lambda i,j}} \frac{\partial p_{\lambda ij}\left(\mathbf{x}\right)}{\partial \lambda} dx \\ &= \int_{B(\mathbf{0},R)} \sum_{i} \sum_{j} \frac{\partial \det\left(D\mathbf{p}_\lambda\left(\mathbf{x}\right)\right)}{\partial p_{\lambda i,j}} \left(h_i\left(\mathbf{x}\right) - x_i\right)_{,j} dx \\ &= \int_{B(\mathbf{0},R)} \sum_{i} \sum_{j} \operatorname{cof}\left(D\mathbf{p}_\lambda\left(\mathbf{x}\right)\right)_{ij} \left(h_i\left(\mathbf{x}\right) - x_i\right)_{,j} dx \end{aligned}$$

Now by assumption, $h_i(\mathbf{x}) = x_i$ on $\partial B(\mathbf{0}, R)$ and so one can form iterated integrals and integrate by parts in each of the one dimensional integrals to obtain

$$I'(\lambda) = -\sum_i \int_{B(\mathbf{0},R)} \sum_j \operatorname{cof}(D\mathbf{p}_\lambda(\mathbf{x}))_{ij,j} (h_i(\mathbf{x}) - x_i)\, dx = 0.$$

Therefore, $I(\lambda)$ equals a constant. However,

$$I(0) = m_n(B(\mathbf{0}, R)) > 0$$

but

$$I(1) = \int_{B(\mathbf{0},1)} \det(D\mathbf{h}(\mathbf{x}))\, dm_n = \int_{\partial B(\mathbf{0},1)} \#(\mathbf{y})\, dm_n = 0$$

because from polar coordinates or other elementary reasoning, $m_n(\partial B(\mathbf{0}, 1)) = 0$. This proves the lemma.

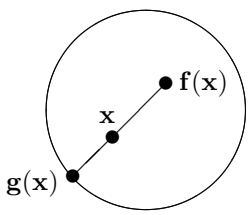
The following is the Brouwer fixed point theorem for C^2 maps.

Lemma 9.12.4 *If* $\mathbf{h} \in C^2\left(\overline{B(\mathbf{0}, R)}\right)$ *and* $\mathbf{h} : \overline{B(\mathbf{0}, R)} \to \overline{B(\mathbf{0}, R)}$, *then* $\mathbf{h}$ *has a fixed point,* $\mathbf{x}$ *such that* $\mathbf{h}(\mathbf{x}) = \mathbf{x}$.

Proof: Suppose the lemma is not true. Then for all $\mathbf{x}, |\mathbf{x} - \mathbf{h}(\mathbf{x})| \neq 0$. Then define

$$\mathbf{g}(\mathbf{x}) = \mathbf{h}(\mathbf{x}) + \frac{\mathbf{x} - \mathbf{h}(\mathbf{x})}{|\mathbf{x} - \mathbf{h}(\mathbf{x})|} t(\mathbf{x})$$

where $t(\mathbf{x})$ is nonnegative and is chosen such that $\mathbf{g}(\mathbf{x}) \in \partial B(\mathbf{0}, R)$. This mapping is illustrated in the following picture.



If $\mathbf{x} \to t(\mathbf{x})$ is C^2 near $\overline{B(\mathbf{0}, R)}$, it will follow $\mathbf{g}$ is a C^2 retraction onto $\partial B(\mathbf{0}, R)$ contrary to Lemma 9.12.3. Now $t(\mathbf{x})$ is the nonnegative solution, t to

$$H(\mathbf{x}, \mathbf{t}) = |\mathbf{h}(\mathbf{x})|^2 + 2\left(\mathbf{h}(\mathbf{x}), \frac{\mathbf{x} - \mathbf{h}(\mathbf{x})}{|\mathbf{x} - \mathbf{h}(\mathbf{x})|}\right) t + t^2 = R^2 \tag{9.22}$$

Then

$$H_t(\mathbf{x}, t) = 2\left(\mathbf{h}(\mathbf{x}), \frac{\mathbf{x} - \mathbf{h}(\mathbf{x})}{|\mathbf{x} - \mathbf{h}(\mathbf{x})|}\right) + 2t.$$

If this is nonzero for all $\mathbf{x}$ near $\overline{B(\mathbf{0}, R)}$, it follows from the implicit function theorem that t is a C^2 function of $\mathbf{x}$. From 9.22

$$\begin{aligned} 2t &= -2\left(\mathbf{h}(\mathbf{x}), \frac{\mathbf{x} - \mathbf{h}(\mathbf{x})}{|\mathbf{x} - \mathbf{h}(\mathbf{x})|}\right) \\ &\quad \pm\sqrt{4\left(\mathbf{h}(\mathbf{x}), \frac{\mathbf{x} - \mathbf{h}(\mathbf{x})}{|\mathbf{x} - \mathbf{h}(\mathbf{x})|}\right)^2 - 4\left(|\mathbf{h}(\mathbf{x})|^2 - R^2\right)} \end{aligned}$$

and so

$$\begin{aligned} H_t(\mathbf{x},t) &= 2t + 2\left(\mathbf{h}(\mathbf{x}), \frac{\mathbf{x}-\mathbf{h}(\mathbf{x})}{|\mathbf{x}-\mathbf{h}(\mathbf{x})|}\right) \\ &= \pm\sqrt{4\left(R^2 - |\mathbf{h}(\mathbf{x})|^2\right) + 4\left(\mathbf{h}(\mathbf{x}), \frac{\mathbf{x}-\mathbf{h}(\mathbf{x})}{|\mathbf{x}-\mathbf{h}(\mathbf{x})|}\right)^2} \end{aligned}$$

If $|\mathbf{h}(\mathbf{x})| < R$, this is nonzero. If $|\mathbf{h}(\mathbf{x})| = R$, then it is still nonzero unless

$$(\mathbf{h}(\mathbf{x}), \mathbf{x}-\mathbf{h}(\mathbf{x})) = 0.$$

But this cannot happen because the angle between $\mathbf{h}(\mathbf{x})$ and $\mathbf{x}-\mathbf{h}(\mathbf{x})$ cannot be $\pi/2$. Alternatively, if the above equals zero, you would need

$$(\mathbf{h}(\mathbf{x}), \mathbf{x}) = |\mathbf{h}(\mathbf{x})|^2 = R^2$$

which cannot happen unless $\mathbf{x} = \mathbf{h}(\mathbf{x})$ which is assumed not to happen. Therefore, $\mathbf{x} \to t(\mathbf{x})$ is C^2 near $\overline{B(\mathbf{0},R)}$ and so $\mathbf{g}(\mathbf{x})$ given above contradicts Lemma 9.12.3. This proves the lemma.

Now it is easy to prove the Brouwer fixed point theorem.

Theorem 9.12.5 *Let* $\mathbf{f} : \overline{B(\mathbf{0},R)} \to \overline{B(\mathbf{0},R)}$ *be continuous. Then* $\mathbf{f}$ *has a fixed point.*

Proof: If this is not so, there exists $\varepsilon > 0$ such that for all $\mathbf{x} \in \overline{B(\mathbf{0},R)}$,

$$|\mathbf{x} - \mathbf{f}(\mathbf{x})| > \varepsilon.$$

By the Weierstrass approximation theorem, there exists $\mathbf{h}$, a polynomial such that

$$\max\left\{|\mathbf{h}(\mathbf{x}) - \mathbf{f}(\mathbf{x})| : \mathbf{x} \in \overline{B(\mathbf{0},R)}\right\} < \frac{\varepsilon}{2}.$$

Then for all $\mathbf{x} \in \overline{B(\mathbf{0},R)}$,

$$|\mathbf{x} - \mathbf{h}(\mathbf{x})| \geq |\mathbf{x} - \mathbf{f}(\mathbf{x})| - |\mathbf{h}(\mathbf{x}) - \mathbf{f}(\mathbf{x})| > \varepsilon - \frac{\varepsilon}{2} = \frac{\varepsilon}{2}$$

contradicting Lemma 9.12.4. This proves the theorem.

9.13 Exercises

1. Recall the definition of $f_{\mathbf{y}}$. Prove that if $f \in L^1(\mathbb{R}^n)$, then

$$\lim_{\mathbf{y}\to\mathbf{0}} \int_{\mathbb{R}^n} |f - f_{\mathbf{y}}| \, dm_n = 0$$

   This is known as continuity of translation. **Hint:** Use the theorem about being able to approximate an arbitrary function in $L^1(\mathbb{R}^n)$ with a function in $C_c(\mathbb{R}^n)$.

2. Let E be a Lebesgue measurable set in $\mathbb{R}$. Suppose $m(E) > 0$. Consider the set

$$E - E = \{x - y : x \in E, y \in E\}.$$

   Show that $E - E$ contains an interval. **Hint:** Let

$$f(x) = \int \mathcal{X}_E(t)\mathcal{X}_E(x+t)dt.$$

   Explain why f is continuous at 0 and $f(0) > 0$ and use continuity of translation in L^1.

3. If $f \in L^1(\mathbb{R}^n)$, show there exists $g \in L^1(\mathbb{R}^n)$ such that g is also Borel measurable such that $g(\mathbf{x}) = f(\mathbf{x})$ for a.e. $\mathbf{x}$.

4. Suppose $f, g \in L^1(\mathbb{R}^n)$. Define $f * g(\mathbf{x})$ by

$$\int f(\mathbf{x}-\mathbf{y})\, g(\mathbf{y})\, dm_n(y).$$

Show this makes sense for a.e. $\mathbf{x}$ and that in fact for a.e. $\mathbf{x}$

$$\int |f(\mathbf{x}-\mathbf{y})|\, |g(\mathbf{y})|\, dm_n(y)$$

Next show

$$\int |f * g(\mathbf{x})|\, dm_n(\mathbf{x}) \leq \int |f|\, dm_n \int |g|\, dm_n.$$

Hint: Use Problem 3. Show first there is no problem if f, g are Borel measurable. The reason for this is that you can use Fubini's theorem to write

$$\begin{aligned} & \int\int |f(\mathbf{x}-\mathbf{y})|\, |g(\mathbf{y})|\, dm_n(y)\, dm_n(x) \\ = & \int\int |f(\mathbf{x}-\mathbf{y})|\, |g(\mathbf{y})|\, dm_n(x)\, dm_n(y) \\ = & \int |f(\mathbf{z})|\, dm_n \int |g(\mathbf{y})|\, dm_n. \end{aligned}$$

Explain. Then explain why if f and g are replaced by functions which are equal to f and g a.e. but are Borel measurable, the convolution is unchanged.

5. In the situation of Problem 4 Show $\mathbf{x} \to f * g(\mathbf{x})$ is continuous whenever g is also bounded. **Hint:** Use Problem 1.

6. Let $f : [0,\infty) \to \mathbb{R}$ be in $L^1(\mathbb{R}, m)$. The Laplace transform is given by $\widehat{f}(x) = \int_0^\infty e^{-xt} f(t)dt$. Let f, g be in $L^1(\mathbb{R}, m)$, and let $h(x) = \int_0^x f(x-t)g(t)dt$. Show $h \in L^1$, and $\widehat{h} = \widehat{f}\widehat{g}$.

7. Suppose A is covered by a finite collection of Balls, $\mathcal{F}$. Show that then there exists a disjoint collection of these balls, $\{B_i\}_{i=1}^p$, such that $A \subseteq \cup_{i=1}^p \widehat{B}_i$ where $\widehat{B}_i$ has the same center as B_i but 3 times the radius. **Hint:** Since the collection of balls is finite, they can be arranged in order of decreasing radius.

8. Let f be a function defined on an interval, (a, b). The Dini derivates are defined as

$$\begin{aligned} D_+ f(x) &\equiv \lim\inf_{h\to 0+} \frac{f(x+h)-f(x)}{h}, \\ D^+ f(x) &\equiv \lim\sup_{h\to 0+} \frac{f(x+h)-f(x)}{h} \end{aligned}$$

$$\begin{aligned} D_- f(x) &\equiv \lim\inf_{h\to 0+} \frac{f(x)-f(x-h)}{h}, \\ D^- f(x) &\equiv \lim\sup_{h\to 0+} \frac{f(x)-f(x-h)}{h}. \end{aligned}$$

Suppose f is continuous on (a,b) and for all $x \in (a,b)$, $D_+f(x) \geq 0$. Show that then f is increasing on (a,b). **Hint:** Consider the function, $H(x) \equiv f(x)(d-c) - x(f(d)-f(c))$ where $a<c<d<b$. Thus $H(c)=H(d)$. Also it is easy to see that H cannot be constant if $f(d)<f(c)$ due to the assumption that $D_+f(x)\geq 0$. If there exists $x_1 \in (a,b)$ where $H(x_1)>H(c)$, then let $x_0 \in (c,d)$ be the point where the maximum of f occurs. Consider $D_+f(x_0)$. If, on the other hand, $H(x)<H(c)$ for all $x \in (c,d)$, then consider $D_+H(c)$.

9. ↑ Suppose in the situation of the above problem we only know

$$D_+f(x) \geq 0 \text{ a.e.}$$

Does the conclusion still follow? What if we only know $D_+f(x)\geq 0$ for every x outside a countable set? **Hint:** In the case of $D_+f(x)\geq 0$,consider the bad function in the exercises for the chapter on the construction of measures which was based on the Cantor set. In the case where $D_+f(x)\geq 0$ for all but countably many x, by replacing $f(x)$ with $\widetilde{f}(x) \equiv f(x)+\varepsilon x$, consider the situation where $D_+\widetilde{f}(x)>0$ for all but countably many x. If in this situation, $\widetilde{f}(c)>\widetilde{f}(d)$ for some $c<d$, and $y \in \left(\widetilde{f}(d), \widetilde{f}(c)\right)$,let

$$z \equiv \sup\left\{x \in [c,d] : \widetilde{f}(x) > y_0\right\}.$$

Show that $\widetilde{f}(z)=y_0$ and $D_+\widetilde{f}(z)\leq 0$. Conclude that if $\widetilde{f}$ fails to be increasing, then $D_+\widetilde{f}(z)\leq 0$ for uncountably many points, z. Now draw a conclusion about f.

10. ↑ Let $f:[a,b]\to\mathbb{R}$ be increasing. Show

$$m\left(\overbrace{\left[D^+f(x)>q>p>D_+f(x)\right]}^{N_{pq}}\right)=0 \tag{9.23}$$

and conclude that aside from a set of measure zero, $D^+f(x)=D_+f(x)$. Similar reasoning will show $D^-f(x)=D_-f(x)$ a.e. and $D^+f(x)=D_-f(x)$ a.e. and so off some set of measure zero, we have

$$D_-f(x)=D^-f(x)=D^+f(x)=D_+f(x)$$

which implies the derivative exists and equals this common value. **Hint**: To show 9.23, let U be an open set containing N_{pq} such that $\overline{m}(N_{pq})+\varepsilon > m(U)$. For each $x \in N_{pq}$ there exist $y>x$ arbitrarily close to x such that

$$f(y)-f(x)<p(y-x).$$

Thus the set of such intervals, $\{[x,y]\}$ which are contained in U constitutes a Vitali cover of N_{pq}. Let $\{[x_i,y_i]\}$ be disjoint and

$$\overline{m}(N_{pq}\setminus \cup_i [x_i,y_i])=0.$$

Now let $V \equiv \cup_i (x_i,y_i)$. Then also we have

$$\overline{m}\left(N_{pq}\setminus \overbrace{\cup_i (x_i,y_i)}^{=V}\right)=0.$$

and so $\overline{m}(N_{pq} \cap V) = \overline{m}(N_{pq})$. For each $x \in N_{pq} \cap V$, there exist $y > x$ arbitrarily close to x such that

$$f(y) - f(x) > q(y - x).$$

Thus the set of such intervals, $\{[x', y']\}$ which are contained in V is a Vitali cover of $N_{pq} \cap V$. Let $\{[x_i', y_i']\}$ be disjoint and

$$\overline{m}(N_{pq} \cap V \setminus \cup_i [x_i', y_i']) = 0.$$

Then verify the following:

$$\begin{aligned} \sum_i f(y_i') - f(x_i') &> q \sum_i (y_i' - x_i') \geq q\overline{m}(N_{pq} \cap V) = q\overline{m}(N_{pq}) \\ &\geq p\overline{m}(N_{pq}) > p(m(U) - \varepsilon) \geq p \sum_i (y_i - x_i) - p\varepsilon \\ &\geq \sum_i (f(y_i) - f(x_i)) - p\varepsilon \geq \sum_i f(y_i') - f(x_i') - p\varepsilon \end{aligned}$$

and therefore, $(q - p)\overline{m}(N_{pq}) \leq p\varepsilon$. Since $\varepsilon > 0$ is arbitrary, this proves that there is a right derivative a.e. A similar argument does the other cases.

11. Suppose f is a function in $L^1(\mathbb{R})$ and f is infinitely differentiable. Does it follow that $f' \in L^1(\mathbb{R})$? **Hint:** What if $\phi \in C_c^\infty(0,1)$ and $f(x) = \phi(2^n(x - n))$ for $x \in (n, n+1)$, $f(x) = 0$ if $x < 0$?

12. For a function $f \in L^1(\mathbb{R}^n)$, the Fourier transform, Ff is given by

$$Ff(\mathbf{t}) \equiv \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}^n} e^{-i\mathbf{t}\cdot\mathbf{x}} f(\mathbf{x})\, dx$$

and the so called inverse Fourier transform, $F^{-1}f$ is defined by

$$Ff(\mathbf{t}) \equiv \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}^n} e^{i\mathbf{t}\cdot\mathbf{x}} f(\mathbf{x})\, dx$$

Show that if $f \in L^1(\mathbb{R}^n)$, then $\lim_{|\mathbf{x}|\to\infty} Ff(\mathbf{x}) = 0$. **Hint:** You might try to show this first for $f \in C_c^\infty(\mathbb{R}^n)$.

13. For this problem define $\int_a^\infty f(t)\, dt \equiv \lim_{r\to\infty} \int_a^r f(t)\, dt$. Note this coincides with the Lebesgue integral when $f \in L^1(a, \infty)$. Show

    (a) $\int_0^\infty \frac{\sin(u)}{u} du = \frac{\pi}{2}$

    (b) $\lim_{r\to\infty} \int_\delta^\infty \frac{\sin(ru)}{u} du = 0$ whenever $\delta > 0$.

    (c) If $f \in L^1(\mathbb{R})$, then $\lim_{r\to\infty} \int_{\mathbb{R}} \sin(ru) f(u)\, du = 0$.

    Hint: For the first two, use $\frac{1}{u} = \int_0^\infty e^{-ut} dt$ and apply Fubini's theorem to $\int_0^R \sin u \int_{\mathbb{R}} e^{-ut} dt du$. For the last part, first establish it for $f \in C_c^\infty(\mathbb{R})$ and then use the density of this set in $L^1(\mathbb{R})$ to obtain the result. This is called the Riemann Lebesgue lemma.

14. ↑Suppose that $g \in L^1(\mathbb{R})$ and that at some $x > 0$, g is locally Holder continuous from the right and from the left. This means

$$\lim_{r\to 0+} g(x + r) \equiv g(x+)$$

exists,

$$\lim_{r\to 0+} g(x-r) \equiv g(x-)$$

exists and there exist constants $K, \delta > 0$ and $r \in (0,1]$ such that for $|x-y| < \delta$,

$$|g(x+) - g(y)| < K|x-y|^r$$

for $y > x$ and

$$|g(x-) - g(y)| < K|x-y|^r$$

for $y < x$. Show that under these conditions,

$$\lim_{r\to\infty} \frac{2}{\pi}\int_0^\infty \frac{\sin(ur)}{u}\left(\frac{g(x-u)+g(x+u)}{2}\right)du$$

$$= \frac{g(x+)+g(x-)}{2}.$$

15. ↑ Let $g \in L^1(\mathbb{R})$ and suppose g is locally Holder continuous from the right and from the left at x. Show that then

$$\lim_{R\to\infty} \frac{1}{2\pi}\int_{-R}^R e^{ixt}\int_{-\infty}^\infty e^{-ity} g(y)\,dydt = \frac{g(x+)+g(x-)}{2}.$$

This is very interesting. This shows $F^{-1}(Fg)(x) = \frac{g(x+)+g(x-)}{2}$, the midpoint of the jump in g at the point, x provided Fg is in L^1. **Hint:** Show the left side of the above equation reduces to

$$\frac{2}{\pi}\int_0^\infty \frac{\sin(ur)}{u}\left(\frac{g(x-u)+g(x+u)}{2}\right)du$$

and then use Problem 14 to obtain the result.

16. ↑ A measurable function g defined on $(0,\infty)$ has exponential growth if $|g(t)| \le Ce^{\eta t}$ for some η. For $\mathrm{Re}(s) > \eta$, define the Laplace Transform by

$$Lg(s) \equiv \int_0^\infty e^{-su} g(u)\,du.$$

Assume that g has exponential growth as above and is Holder continuous from the right and from the left at t. Pick $\gamma > \eta$. Show that

$$\lim_{R\to\infty} \frac{1}{2\pi}\int_{-R}^R e^{\gamma t}e^{iyt} Lg(\gamma+iy)\,dy = \frac{g(t+)+g(t-)}{2}.$$

This formula is sometimes written in the form

$$\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} e^{st} Lg(s)\,ds$$

and is called the complex inversion integral for Laplace transforms. It can be used to find inverse Laplace transforms. **Hint:**

$$\frac{1}{2\pi}\int_{-R}^R e^{\gamma t}e^{iyt} Lg(\gamma+iy)\,dy =$$

$$\frac{1}{2\pi}\int_{-R}^{R} e^{\gamma t}e^{iyt}\int_0^\infty e^{-(\gamma+iy)u}g(u)\,dudy.$$

Now use Fubini's theorem and do the integral from $-R$ to R to get this equal to

$$\frac{e^{\gamma t}}{\pi}\int_{-\infty}^{\infty} e^{-\gamma u}\overline{g}(u)\frac{\sin(R(t-u))}{t-u}du$$

where $\overline{g}$ is the zero extension of g off $[0,\infty)$. Then this equals

$$\frac{e^{\gamma t}}{\pi}\int_{-\infty}^{\infty} e^{-\gamma(t-u)}\overline{g}(t-u)\frac{\sin(Ru)}{u}du$$

which equals

$$\frac{2e^{\gamma t}}{\pi}\int_0^\infty \frac{\overline{g}(t-u)\,e^{-\gamma(t-u)}+\overline{g}(t+u)\,e^{-\gamma(t+u)}}{2}\frac{\sin(Ru)}{u}du$$

and then apply the result of Problem 14.

17. Let K be a nonempty closed and convex subset of $\mathbb{R}^n$. Recall K is convex means that if $\mathbf{x},\mathbf{y}\in K$, then for all $t\in[0,1]$, $t\mathbf{x}+(1-t)\mathbf{y}\in K$. Show that if $\mathbf{x}\in\mathbb{R}^n$ there exists a unique $\mathbf{z}\in K$ such that

$$|\mathbf{x}-\mathbf{z}| = \min\{|\mathbf{x}-\mathbf{y}| : \mathbf{y}\in K\}.$$

This $\mathbf{z}$ will be denoted as $P\mathbf{x}$. **Hint:** First note you do not know K is compact. Establish the parallelogram identity if you have not already done so,

$$|\mathbf{u}-\mathbf{v}|^2+|\mathbf{u}+\mathbf{v}|^2 = 2|\mathbf{u}|^2+2|\mathbf{v}|^2.$$

Then let $\{\mathbf{z}_k\}$ be a minimizing sequence,

$$\lim_{k\to\infty}|\mathbf{z}_k-\mathbf{x}|^2 = \inf\{|\mathbf{x}-\mathbf{y}| : \mathbf{y}\in K\}\equiv\lambda.$$

Now using convexity, explain why

$$\left|\frac{\mathbf{z}_k-\mathbf{z}_m}{2}\right|^2+\left|\mathbf{x}-\frac{\mathbf{z}_k+\mathbf{z}_m}{2}\right|^2 = 2\left|\frac{\mathbf{x}-\mathbf{z}_k}{2}\right|^2+2\left|\frac{\mathbf{x}-\mathbf{z}_m}{2}\right|^2$$

and then use this to argue $\{\mathbf{z}_k\}$ is a Cauchy sequence. Then if $\mathbf{z}_i$ works for $i=1,2$, consider $(\mathbf{z}_1+\mathbf{z}_2)/2$ to get a contradiction.

18. In Problem 17 show that $P\mathbf{x}$ satisfies the following variational inequality.

$$(\mathbf{x}-P\mathbf{x})\cdot(\mathbf{y}-P\mathbf{x})\le 0$$

for all $\mathbf{y}\in K$. Then show that $|P\mathbf{x}_1-P\mathbf{x}_2|\le|\mathbf{x}_1-\mathbf{x}_2|$. **Hint:** For the first part note that if $\mathbf{y}\in K$, the function $t\to|\mathbf{x}-(P\mathbf{x}+t(\mathbf{y}-P\mathbf{x}))|^2$ achieves its minimum on $[0,1]$ at $t=0$. For the second part,

$$(\mathbf{x}_1-P\mathbf{x}_1)\cdot(P\mathbf{x}_2-P\mathbf{x}_1)\le 0,\ (\mathbf{x}_2-P\mathbf{x}_2)\cdot(P\mathbf{x}_1-P\mathbf{x}_2)\le 0.$$

Explain why

$$(\mathbf{x}_2-P\mathbf{x}_2-(\mathbf{x}_1-P\mathbf{x}_1))\cdot(P\mathbf{x}_2-P\mathbf{x}_1)\ge 0$$

and then use a some manipulations and the Cauchy Schwarz inequality to get the desired inequality.

19. Establish the Brouwer fixed point theorem for any convex compact set in $\mathbb{R}^n$. **Hint:** If K is a compact and convex set, let R be large enough that the closed ball, $D(\mathbf{0},R) \supseteq K$. Let P be the projection onto K as in Problem 18 above. If $\mathbf{f}$ is a continuous map from K to K, consider $\mathbf{f}\circ P$. You want to show $\mathbf{f}$ has a fixed point in K.

20. In the situation of the implicit function theorem, suppose $\mathbf{f}(\mathbf{x}_0,\mathbf{y}_0)=\mathbf{0}$ and assume $\mathbf{f}$ is C^1. Show that for $(\mathbf{x},\mathbf{y}) \in B(\mathbf{x}_0,\delta)\times B(\mathbf{y}_0,r)$ where δ, r are small enough, the mapping

$$\mathbf{x} \to T_{\mathbf{y}}(\mathbf{x}) \equiv \mathbf{x} - D_1\mathbf{f}(\mathbf{x}_0,\mathbf{y}_0)^{-1}\mathbf{f}(\mathbf{x},\mathbf{y})$$

is continuous and maps $\overline{B(\mathbf{x}_0,\delta)}$ to $\overline{B(\mathbf{x}_0,\delta/2)} \subseteq \overline{B(\mathbf{x}_0,\delta)}$. Apply the Brouwer fixed point theorem to obtain a shorter proof of the implicit function theorem.

21. Here is a really interesting little theorem which depends on the Brouwer fixed point theorem. It plays a prominent role in the treatment of the change of variables formula in Rudin's book, [34] and is useful in other contexts as well. The idea is that if a continuous function mapping a ball in $\mathbb{R}^k$ to $\mathbb{R}^k$ doesn't move any point very much, then the image of the ball must contain a slightly smaller ball.

Lemma: Let $B = B(\mathbf{0},r)$, a ball in $\mathbb{R}^k$ and let $\mathbf{F}: \overline{B} \to \mathbb{R}^k$ be continuous and suppose for some $\varepsilon < 1$,

$$|\mathbf{F}(\mathbf{v}) - \mathbf{v}| < \varepsilon r \tag{9.24}$$

for all $\mathbf{v} \in \overline{B}$. Then

$$\mathbf{F}(B) \supseteq B(\mathbf{0}, r(1-\varepsilon)).$$

Hint: Suppose $\mathbf{a} \in B(\mathbf{0},r(1-\varepsilon)) \setminus \mathbf{F}(B)$ so it didn't work. First explain why $\mathbf{a} \neq \mathbf{F}(\mathbf{v})$ for all $\mathbf{v} \in \overline{B}$. Now letting $\mathbf{G}: \overline{B} \to \overline{B}$, be defined by $\mathbf{G}(\mathbf{v}) \equiv \frac{r(\mathbf{a}-\mathbf{F}(\mathbf{v}))}{|\mathbf{a}-\mathbf{F}(\mathbf{v})|}$, it follows $\mathbf{G}$ is continuous. Then by the Brouwer fixed point theorem, $\mathbf{G}(\mathbf{v}) = \mathbf{v}$ for some $\mathbf{v} \in \overline{B}$. Explain why $|\mathbf{v}| = r$. Then take the inner product with $\mathbf{v}$ and explain the following steps.

$$\begin{aligned}
(\mathbf{G}(\mathbf{v}),\mathbf{v}) &= |\mathbf{v}|^2 = r^2 = \frac{r}{|\mathbf{a}-\mathbf{F}(\mathbf{v})|}(\mathbf{a}-\mathbf{F}(\mathbf{v}),\mathbf{v}) \\
&= \frac{r}{|\mathbf{a}-\mathbf{F}(\mathbf{v})|}(\mathbf{a}-\mathbf{v}+\mathbf{v}-\mathbf{F}(\mathbf{v}),\mathbf{v}) \\
&= \frac{r}{|\mathbf{a}-\mathbf{F}(\mathbf{v})|}[(\mathbf{a}-\mathbf{v},\mathbf{v})+(\mathbf{v}-\mathbf{F}(\mathbf{v}),\mathbf{v})] \\
&= \frac{r}{|\mathbf{a}-\mathbf{F}(\mathbf{v})|}\left[(\mathbf{a},\mathbf{v})-|\mathbf{v}|^2+(\mathbf{v}-\mathbf{F}(\mathbf{v}),\mathbf{v})\right] \\
&\leq \frac{r}{|\mathbf{a}-\mathbf{F}(\mathbf{v})|}\left[r^2(1-\varepsilon)-r^2+r^2\varepsilon\right] = 0.
\end{aligned}$$

22. Using Problem 21 establish the following interesting result. Suppose $\mathbf{f}: U \to \mathbb{R}^n$ is differentiable. Let

$$S = \overline{\{\mathbf{x} \in U : \det D\mathbf{f}(\mathbf{x}) = 0\}}.$$

Show $\mathbf{f}(U \setminus S)$ is an open set.

23. Let K be a closed, bounded and convex set in $\mathbb{R}^n$ and let $\mathbf{f}: K \to \mathbb{R}^n$ be continuous and let $\mathbf{y} \in \mathbb{R}^n$. Show using the Brouwer fixed point theorem there exists a point $\mathbf{x} \in K$ such that $P(\mathbf{y}-\mathbf{f}(\mathbf{x})+\mathbf{x}) = \mathbf{x}$. Next show that $(\mathbf{y}-\mathbf{f}(\mathbf{x}),\mathbf{z}-\mathbf{x}) \leq 0$ for all $\mathbf{z} \in K$. The existence of this $\mathbf{x}$ is known as Browder's lemma and it has great significance

in the study of certain types of nolinear operators. Now suppose $\mathbf{f} : \mathbb{R}^n \to \mathbb{R}^n$ is continuous and satisfies

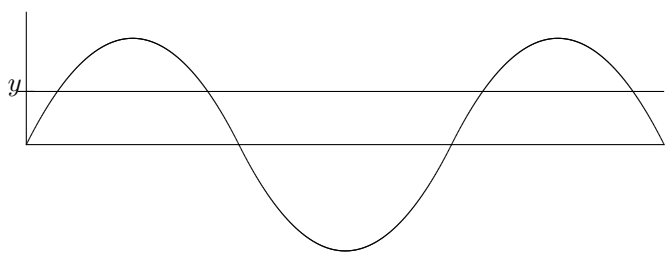
$$\lim_{|\mathbf{x}|\to\infty} \frac{(\mathbf{f}(\mathbf{x}), \mathbf{x})}{|\mathbf{x}|} = \infty.$$

Show using Browder's lemma that $\mathbf{f}$ is onto.

Brouwer Degree

This chapter is on the Brouwer degree, a very useful concept with numerous and important applications. The degree can be used to prove some difficult theorems in topology such as the Brouwer fixed point theorem, the Jordan separation theorem, and the invariance of domain theorem. It also is used in bifurcation theory and many other areas in which it is an essential tool. This is an advanced calculus course so the degree will be developed for $\mathbb{R}^n$. When this is understood, it is not too difficult to extend to versions of the degree which hold in Banach space. There is more on degree theory in the book by Deimling [9] and much of the presentation here follows this reference.

To give you an idea what the degree is about, consider a real valued C^1 function defined on an interval, I, and let $y \in f(I)$ be such that $f'(x) \neq 0$ for all $x \in f^{-1}(y)$. In this case the degree is the sum of the signs of $f'(x)$ for $x \in f^{-1}(y)$, written as $d(f, I, y)$.



In the above picture, $d(f, I, y)$ is 0 because there are two places where the sign is 1 and two where it is -1.

The amazing thing about this is the number you obtain in this simple manner is a specialization of something which is defined for continuous functions and which has nothing to do with differentiability.

There are many ways to obtain the Brouwer degree. The method I will use here is due to Heinz [22] and appeared in 1959. It involves first studying the degree for functions in C^2 and establishing all its most important topological properties with the aid of an integral. Then when this is done, it is very easy to extend to general continuous functions.

When you have the topological degree, you can get all sorts of amazing theorems like the invariance of domain theorem and others.

10.1 Preliminary Results

In this chapter Ω will refer to a bounded open set.

Definition 10.1.1 *For Ω a bounded open set, denote by $C\left(\overline{\Omega}\right)$ the set of functions which are continuous on $\overline{\Omega}$ and by $C^m\left(\overline{\Omega}\right), m \leq \infty$ the space of restrictions of functions in $C_c^m\left(\mathbb{R}^n\right)$ to $\overline{\Omega}$. The norm in $C\left(\overline{\Omega}\right)$ is defined as follows.*

$$||f||_\infty = ||f||_{C\left(\overline{\Omega}\right)} \equiv \sup\left\{|f(\mathbf{x})| : \mathbf{x} \in \overline{\Omega}\right\}.$$

If the functions take values in $\mathbb{R}^n$ write $C^m\left(\overline{\Omega};\mathbb{R}^n\right)$ or $C\left(\overline{\Omega};\mathbb{R}^n\right)$ for these functions if there is no differentiability assumed. The norm on $C\left(\overline{\Omega};\mathbb{R}^n\right)$ is defined in the same way as above,

$$||\mathbf{f}||_\infty = ||\mathbf{f}||_{C\left(\overline{\Omega};\mathbb{R}^n\right)} \equiv \sup\left\{|\mathbf{f}(\mathbf{x})| : \mathbf{x} \in \overline{\Omega}\right\}.$$

Also, $C\left(\Omega;\mathbb{R}^n\right)$ consists of functions which are continuous on Ω that have values in $\mathbb{R}^n$ and $C^m\left(\Omega;\mathbb{R}^n\right)$ denotes the functions which have m continuous derivatives defined on Ω.

Theorem 10.1.2 *Let Ω be a bounded open set in $\mathbb{R}^n$ and let $f \in C\left(\overline{\Omega}\right)$. Then there exists $g \in C^\infty\left(\overline{\Omega}\right)$ with $||g-f||_{C\left(\overline{\Omega}\right)} < \varepsilon$. In fact, g can be assumed to equal a polynomial for all $\mathbf{x} \in \Omega$.*

Proof: This follows immediately from the Weierstrass approximation theorem, Theorem 5.7.10. Pick a polynomial, p such that $||p-f||_{C\left(\overline{\Omega}\right)} < \varepsilon$. Now $p \notin C^\infty\left(\overline{\Omega}\right)$ because it does not vanish outside some compact subset of $\mathbb{R}^n$ so let g equal p multiplied by some function $\psi \in C_c^\infty\left(\mathbb{R}^n\right)$ where $\psi = 1$ on $\overline{\Omega}$. See Theorem 9.5.7.

Applying this result to the components of a vector valued function yields the following corollary.

Corollary 10.1.3 *If $\mathbf{f} \in C\left(\overline{\Omega};\mathbb{R}^n\right)$ for Ω a bounded subset of $\mathbb{R}^n$, then for all $\varepsilon > 0$, there exists $\mathbf{g} \in C^\infty\left(\overline{\Omega};\mathbb{R}^n\right)$ such that*

$$||\mathbf{g}-\mathbf{f}||_\infty < \varepsilon.$$

Lemma 9.12.1 on Page 235 will also play an important role in the definition of the Brouwer degree. Earlier it made possible an easy proof of the Brouwer fixed point theorem. Later in this chapter, it is used to show the definition of the degree is well defined. For convenience, here it is stated again.

Lemma 10.1.4 *Let $\mathbf{g} : U \to \mathbb{R}^n$ be C^2 where U is an open subset of $\mathbb{R}^n$. Then*

$$\sum_{j=1}^n \operatorname{cof}(D\mathbf{g})_{ij,j} = 0,$$

where here $(D\mathbf{g})_{ij} \equiv g_{i,j} \equiv \frac{\partial g_i}{\partial x_j}$. Also, $\operatorname{cof}(D\mathbf{g})_{ij} = \frac{\partial \det(D\mathbf{g})}{\partial g_{i,j}}$.

Another simple result which will be used whenever convenient is the following lemma, stated in somewhat more generality than needed.

Lemma 10.1.5 *Let K be a compact set and C a closed set in a complete normed vector space such that $K \cap C = \emptyset$. Then*

$$\operatorname{dist}(K,C) > 0.$$

Proof: Let

$$d \equiv \inf \{||k-c|| : k \in K, c \in C\}$$

Let $\{k_n\}, \{c_n\}$ be such that

$$d + \frac{1}{n} > ||k_n - c_n||.$$

Since K is compact, there is a subsequence still denoted by $\{k_n\}$ such that $k_n \to k \in K$. Then also

$$||c_n - c_m|| \leq ||c_n - k_n|| + ||k_n - k_m|| + ||c_m - k_m||$$

If $d = 0$, then as $m, n \to \infty$ it follows $||c_n - c_m|| \to 0$ and so $\{c_n\}$ is a Cauchy sequence which must converge to some $c \in C$. But then $||c-k|| = \lim_{n\to\infty} ||c_n - k_n|| = 0$ and so $c = k \in C \cap K$, a contradiction to these sets being disjoint. This proves the lemma.

In particular the distance between a point and a closed set is always posibive if the point is not in the closed set. Of course this is obvious even without the above lemma.

10.2 Definitions And Elementary Properties

In this section, $\mathbf{f} : \overline{\Omega} \to \mathbb{R}^n$ will be a continuous map. It is always assumed that $\mathbf{f}(\partial\Omega)$ misses the point $\mathbf{y}$ where $d(\mathbf{g}, \Omega, \mathbf{y})$ is the topological degree which is being defined. Also, it is assumed Ω is a bounded open set.

Definition 10.2.1 $\mathcal{U}_{\mathbf{y}} \equiv \left\{\mathbf{f} \in C\left(\overline{\Omega}; \mathbb{R}^n\right) : \mathbf{y} \notin \mathbf{f}(\partial\Omega)\right\}$. *(Recall that $\partial\Omega = \overline{\Omega} \setminus \Omega$) For two functions,*

$$\mathbf{f}, \mathbf{g} \in \mathcal{U}_{\mathbf{y}},$$

$\mathbf{f} \sim \mathbf{g}$ *if there exists a continuous function,*

$$\mathbf{h} : \overline{\Omega} \times [0,1] \to \mathbb{R}^n$$

such that $\mathbf{h}(\mathbf{x}, 1) = \mathbf{g}(\mathbf{x})$ *and* $\mathbf{h}(\mathbf{x}, 0) = \mathbf{f}(\mathbf{x})$ *and* $\mathbf{x} \to \mathbf{h}(\mathbf{x},t) \in \mathcal{U}_{\mathbf{y}}$ *for all* $t \in [0,1]$ $(\mathbf{y} \notin \mathbf{h}(\partial\Omega, \mathbf{t}))$. *This function,* $\mathbf{h}$*, is called a homotopy and* $\mathbf{f}$ *and* $\mathbf{g}$ *are homotopic.*

Definition 10.2.2 *For W an open set in $\mathbb{R}^n$ and $\mathbf{g} \in C^1(W; \mathbb{R}^n)$ $\mathbf{y}$ is called a regular value of $\mathbf{g}$ if whenever $\mathbf{x} \in \mathbf{g}^{-1}(\mathbf{y})$, $\det(D\mathbf{g}(\mathbf{x})) \neq 0$. Note that if $\mathbf{g}^{-1}(\mathbf{y}) = \emptyset$, it follows that $\mathbf{y}$ is a regular value from this definition. Denote by $S_{\mathbf{g}}$ the set of singular values of $\mathbf{g}$.*

Lemma 10.2.3 *The relation, $\sim$, is an equivalence relation and, denoting by $[\mathbf{f}]$ the equivalence class determined by $\mathbf{f}$, it follows that $[\mathbf{f}]$ is an open subset of*

$$\mathcal{U}_{\mathbf{y}} \equiv \left\{\mathbf{f} \in C\left(\overline{\Omega}; \mathbb{R}^n\right) : \mathbf{y} \notin \mathbf{f}(\partial\Omega)\right\}.$$

Furthermore, $\mathcal{U}_{\mathbf{y}}$ is an open set in $C\left(\overline{\Omega}; \mathbb{R}^n\right)$ and if $\mathbf{f} \in \mathcal{U}_{\mathbf{y}}$ and $\varepsilon > 0$, there exists $\mathbf{g} \in [\mathbf{f}] \cap C^2\left(\overline{\Omega}; \mathbb{R}^n\right)$ for which $\mathbf{y}$ is a regular value of $\mathbf{g}$ and $||\mathbf{f} - \mathbf{g}|| < \varepsilon$.

Proof: In showing that $\sim$ is an equivalence relation, it is easy to verify that $\mathbf{f} \sim \mathbf{f}$ and that if $\mathbf{f} \sim \mathbf{g}$, then $\mathbf{g} \sim \mathbf{f}$. To verify the transitive property for an equivalence relation, suppose $\mathbf{f} \sim \mathbf{g}$ and $\mathbf{g} \sim \mathbf{k}$, with the homotopy for $\mathbf{f}$ and $\mathbf{g}$, the function, $\mathbf{h}_1$ and the homotopy for $\mathbf{g}$ and $\mathbf{k}$, the function $\mathbf{h}_2$. Thus $\mathbf{h}_1(\mathbf{x},0) = \mathbf{f}(\mathbf{x})$, $\mathbf{h}_1(\mathbf{x},1) = \mathbf{g}(\mathbf{x})$ and $\mathbf{h}_2(\mathbf{x},0) = \mathbf{g}(\mathbf{x})$, $\mathbf{h}_2(\mathbf{x},1) = \mathbf{k}(\mathbf{x})$. Then define a homotopy of $\mathbf{f}$ and $\mathbf{k}$ as follows.

$$\mathbf{h}(\mathbf{x},t) \equiv \begin{cases} \mathbf{h}_1(\mathbf{x},2t) & \text{if } t \in \left[0, \frac{1}{2}\right] \\ \mathbf{h}_2(\mathbf{x},2t-1) & \text{if } t \in \left[\frac{1}{2}, 1\right] \end{cases}.$$

It is obvious that $\mathcal{U}_{\mathbf{y}}$ is an open subset of $C\left(\overline{\Omega};\mathbb{R}^n\right)$. Next consider the claim that $[\mathbf{f}]$ is also an open set. If $\mathbf{f}\in\mathcal{U}_{\mathbf{y}}$, There exists $\delta>0$ such that $B\left(\mathbf{y},2\delta\right)\cap\mathbf{f}\left(\partial\Omega\right)=\emptyset$. Let $\mathbf{f}_1\in C\left(\overline{\Omega};\mathbb{R}^n\right)$ with $||\mathbf{f}_1-\mathbf{f}||_\infty<\delta$. Then if $t\in[0,1]$, and $\mathbf{x}\in\partial\Omega$

$$|\mathbf{f}\left(\mathbf{x}\right)+t\left(\mathbf{f}_1\left(\mathbf{x}\right)-\mathbf{f}\left(\mathbf{x}\right)\right)-\mathbf{y}|\geq|\mathbf{f}\left(\mathbf{x}\right)-\mathbf{y}|-t\left|\left|\mathbf{f}-\mathbf{f}_1\right|\right|_\infty>2\delta-t\delta>0.$$

Therefore, $B\left(\mathbf{f},\delta\right)\subseteq[\mathbf{f}]$ because if $\mathbf{f}_1\in B\left(\mathbf{f},\delta\right)$, this shows that, letting $\mathbf{h}\left(\mathbf{x},t\right)\equiv\mathbf{f}\left(\mathbf{x}\right)+t\left(\mathbf{f}_1\left(\mathbf{x}\right)-\mathbf{f}\left(\mathbf{x}\right)\right)$, $\mathbf{f}_1\sim\mathbf{f}$.

It remains to verify the last assertion of the lemma. Since $[\mathbf{f}]$ is an open set, it follows from Theorem 10.1.2 there exists $\mathbf{g}\in[\mathbf{f}]\cap C^2\left(\overline{\Omega};\mathbb{R}^n\right)$ and $||\mathbf{g}-\mathbf{f}||<\varepsilon/2$. If $\mathbf{y}$ is a regular value of $\mathbf{g}$, leave $\mathbf{g}$ unchanged. The desired function has been found. In the other case, let δ be small enough that $B\left(\mathbf{y},2\delta\right)\cap\mathbf{g}\left(\partial\Omega\right)=\emptyset$. Next let

$$S\equiv\left\{\mathbf{x}\in\overline{\Omega}:\det D\mathbf{g}\left(\mathbf{x}\right)=0\right\}$$

By Sard's lemma, Lemma 9.9.9 on Page 230, $\mathbf{g}\left(S\right)$ is a set of measure zero and so in particular contains no open ball and so there exists a regular values of $\mathbf{g}$ arbitrarily close to $\mathbf{y}$. Let $\widetilde{\mathbf{y}}$ be one of these regular values and consider

$$\mathbf{g}_1\left(\mathbf{x}\right)\equiv\mathbf{g}\left(\mathbf{x}\right)+\mathbf{y}-\widetilde{\mathbf{y}}.$$

It follows $\mathbf{g}_1\left(\mathbf{x}\right)=\mathbf{y}$ if and only if $\mathbf{g}\left(\mathbf{x}\right)=\widetilde{\mathbf{y}}$ and so, since $D\mathbf{g}\left(\mathbf{x}\right)=D\mathbf{g}_1\left(\mathbf{x}\right)$, $\mathbf{y}$ is a regular value of $\mathbf{g}_1$. Then for $t\in[0,1]$ and $\mathbf{x}\in\partial\Omega$,

$$|\mathbf{g}\left(\mathbf{x}\right)+t\left(\mathbf{g}_1\left(\mathbf{x}\right)-\mathbf{g}\left(\mathbf{x}\right)\right)-\mathbf{y}|\geq|\mathbf{g}\left(\mathbf{x}\right)-\mathbf{y}|-t\left|\mathbf{y}-\widetilde{\mathbf{y}}\right|>2\delta-t\delta\geq\delta>0.$$

provided $|\mathbf{y}-\widetilde{\mathbf{y}}|$ is small enough. It follows $\mathbf{g}_1\sim\mathbf{g}$ and so $\mathbf{g}_1\sim\mathbf{f}$. Also provided $|\mathbf{y}-\widetilde{\mathbf{y}}|$ is small enough,

$$\begin{aligned}||\mathbf{f}-\mathbf{g}_1||&\leq||\mathbf{f}-\mathbf{g}||+||\mathbf{g}-\mathbf{g}_1||\\&<\varepsilon/2+\varepsilon/2=\varepsilon.\end{aligned}$$

This proves the lemma.

The main conclusion of this lemma is that for $\mathbf{f}\in\mathcal{U}_{\mathbf{y}}$, there always exists a function $\mathbf{g}$ of $C^2\left(\overline{\Omega};\mathbb{R}^n\right)$ which is uniformly close to $\mathbf{f}$, homotopic to $\mathbf{f}$ and also such that $\mathbf{y}$ is a regular value of $\mathbf{g}$.

10.2.1 The Degree For $C^2\left(\overline{\Omega};\mathbb{R}^n\right)$

Here I will give a definition of the degree which works for all functions in $C^2\left(\overline{\Omega};\mathbb{R}^n\right)$. Part of this is to show the integral definition about to be given reduces to the sort of thing described earlier where you add signs of the determinant of the derivative.

Definition 10.2.4 *Let $\mathbf{g}\in C^2\left(\overline{\Omega};\mathbb{R}^n\right)\cap\mathcal{U}_{\mathbf{y}}$ where Ω is a bounded open set. Also let ϕ_ε be a mollifier.*

$$\phi_\varepsilon\in C_c^\infty\left(B\left(\mathbf{0},\varepsilon\right)\right),\phi_\varepsilon\geq0,\ \int\phi_\varepsilon dx=1.$$

Then

$$d\left(\mathbf{g},\Omega,\mathbf{y}\right)\equiv\lim_{\varepsilon\to0}\int_\Omega\phi_\varepsilon\left(\mathbf{g}\left(\mathbf{x}\right)-\mathbf{y}\right)\det D\mathbf{g}\left(\mathbf{x}\right)dx$$

Lemma 10.2.5 *The above definition is well defined. In particular the limit exists. In fact*

$$\int_\Omega \phi_\varepsilon (\mathbf{g}(\mathbf{x}) - \mathbf{y}) \det D\mathbf{g}(\mathbf{x})\, dx$$

does not depend on ε whenenver ε is small enough. If $\mathbf{y}$ us a regular value for $\mathbf{g}$ then for all ε small enough,

$$\int_\Omega \phi_\varepsilon (\mathbf{g}(\mathbf{x}) - \mathbf{y}) \det D\mathbf{g}(\mathbf{x})\, dx \equiv$$

$$\sum \left\{ \operatorname{sgn}(\det D\mathbf{g}(\mathbf{x})) : \mathbf{x} \in \mathbf{g}^{-1}(\mathbf{y}) \right\} \tag{10.1}$$

If $\mathbf{f}, \mathbf{g}$ are two functions in $C^2\left(\overline{\Omega}; \mathbb{R}^n\right)$ such that for all $\mathbf{x} \in \partial\Omega$

$$\mathbf{y} \notin t\mathbf{f}(\mathbf{x}) + (1-t)\mathbf{g}(\mathbf{x}) \tag{10.2}$$

for all $t \in [0,1]$, then for each $\varepsilon > 0$,

$$\begin{aligned} & \int_\Omega \phi_\varepsilon (\mathbf{f}(\mathbf{x}) - \mathbf{y}) \det D\mathbf{f}(\mathbf{x})\, dx \\ = & \int_\Omega \phi_\varepsilon (\mathbf{g}(\mathbf{x}) - \mathbf{y}) \det D\mathbf{g}(\mathbf{x})\, dx \end{aligned} \tag{10.3}$$

If $\mathbf{g}, \mathbf{f} \in \mathcal{U}_\mathbf{y} \cap C^2\left(\overline{\Omega}; \mathbb{R}^n\right)$, and 10.2 holds, then

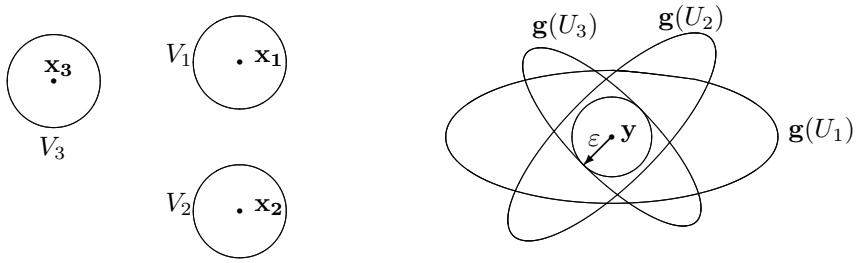
$$d(\mathbf{f}, \Omega, \mathbf{y}) = d(\mathbf{g}, \Omega, \mathbf{y})$$

Proof:

The case where y is a regular value

First consider the case where $\mathbf{y}$ is a regular value of $\mathbf{g}$. I will show that in this case, the integral expression is eventually constant for small $\varepsilon > 0$ and equals the right side of 10.1. I claim the right side of this equation is actually a finite sum. This follows from the inverse function theorem because $\mathbf{g}^{-1}(\mathbf{y})$ is a closed, hence compact subset of Ω due to the assumption that $\mathbf{y} \notin \mathbf{g}(\partial\Omega)$. If $\mathbf{g}^{-1}(\mathbf{y})$ had infinitely many points in it, there would exist a sequence of distinct points $\{\mathbf{x}_k\} \subseteq \mathbf{g}^{-1}(\mathbf{y})$. Since Ω is bounded, some subsequence $\{\mathbf{x}_{k_l}\}$ would converge to a limit point $\mathbf{x}_\infty$. By continuity of $\mathbf{g}$, it follows $\mathbf{x}_\infty \in \mathbf{g}^{-1}(\mathbf{y})$ also and so $\mathbf{x}_\infty \in \Omega$. Therefore, since $\mathbf{y}$ is a regular value, there is an open set, $U_{\mathbf{x}_\infty}$, containing $\mathbf{x}_\infty$ such that $\mathbf{g}$ is one to one on this open set contradicting the assertion that $\lim_{l\to\infty} \mathbf{x}_{k_l} = \mathbf{x}_\infty$. Therefore, this set is finite and so the sum is well defined.

Thus the right side of 10.1 is finite when $\mathbf{y}$ is a regular value. Next I need to show the left side of this equation is eventually constant. By what was just shown, there are finitely many points, $\{\mathbf{x}_i\}_{i=1}^m = \mathbf{g}^{-1}(\mathbf{y})$. By the inverse function theorem, there exist disjoint open sets, U_i with $\mathbf{x}_i \in U_i$, such that $\mathbf{g}$ is one to one on U_i with $\det(D\mathbf{g}(x))$ having constant sign on U_i and $\mathbf{g}(U_i)$ is an open set containing $\mathbf{y}$. Then let ε be small enough that $B(\mathbf{y}, \varepsilon) \subseteq \cap_{i=1}^m \mathbf{g}(U_i)$ and let $V_i \equiv \mathbf{g}^{-1}(B(\mathbf{y}, \varepsilon)) \cap U_i$.



Therefore, for any ε this small,

$$\int_\Omega \phi_\varepsilon(\mathbf{g}(\mathbf{x})-\mathbf{y})\det D\mathbf{g}(\mathbf{x})\,dx = \sum_{i=1}^m \int_{V_i} \phi_\varepsilon(\mathbf{g}(\mathbf{x})-\mathbf{y})\det D\mathbf{g}(\mathbf{x})\,dx$$

The reason for this is as follows. The integrand on the left is nonzero only if $\mathbf{g}(\mathbf{x})-\mathbf{y} \in B(\mathbf{0},\varepsilon)$ which occurs only if $\mathbf{g}(\mathbf{x}) \in B(\mathbf{y},\varepsilon)$ which is the same as $\mathbf{x} \in \mathbf{g}^{-1}(B(\mathbf{y},\varepsilon))$. Therefore, the integrand is nonzero only if $\mathbf{x}$ is contained in exactly one of the disjoint sets, V_i. Now using the change of variables theorem,

$$= \sum_{i=1}^m \int_{\mathbf{g}(V_i)-\mathbf{y}} \phi_\varepsilon(\mathbf{z})\det D\mathbf{g}\left(\mathbf{g}^{-1}(\mathbf{y}+\mathbf{z})\right)\left|\det D\mathbf{g}^{-1}(\mathbf{y}+\mathbf{z})\right| dz$$

By the chain rule, $I = D\mathbf{g}\left(\mathbf{g}^{-1}(\mathbf{y}+\mathbf{z})\right) D\mathbf{g}^{-1}(\mathbf{y}+\mathbf{z})$ and so

$$\begin{aligned} & \det D\mathbf{g}\left(\mathbf{g}^{-1}(\mathbf{y}+\mathbf{z})\right)\left|\det D\mathbf{g}^{-1}(\mathbf{y}+\mathbf{z})\right| \\ = \;& \operatorname{sgn}\left(\det D\mathbf{g}\left(\mathbf{g}^{-1}(\mathbf{y}+\mathbf{z})\right)\right)\cdot \end{aligned}$$

$$\begin{aligned} & \left|\det D\mathbf{g}\left(\mathbf{g}^{-1}(\mathbf{y}+\mathbf{z})\right)\right|\left|\det D\mathbf{g}^{-1}(\mathbf{y}+\mathbf{z})\right| \\ = \;& \operatorname{sgn}\left(\det D\mathbf{g}\left(\mathbf{g}^{-1}(\mathbf{y}+\mathbf{z})\right)\right) \end{aligned}$$

$$= \operatorname{sgn}(\det D\mathbf{g}(\mathbf{x})) = \operatorname{sgn}(\det D\mathbf{g}(\mathbf{x}_i)).$$

Therefore, this reduces to

$$\sum_{i=1}^m \operatorname{sgn}(\det D\mathbf{g}(\mathbf{x}_i)) \int_{\mathbf{g}(V_i)-\mathbf{y}} \phi_\varepsilon(\mathbf{z})\,dz =$$

$$\sum_{i=1}^m \operatorname{sgn}(\det D\mathbf{g}(\mathbf{x}_i)) \int_{B(\mathbf{0},\varepsilon)} \phi_\varepsilon(\mathbf{z})\,dz = \sum_{i=1}^m \operatorname{sgn}(\det D\mathbf{g}(\mathbf{x}_i)).$$

In case $\mathbf{g}^{-1}(\mathbf{y}) = \emptyset$, there exists $\varepsilon > 0$ such that $\mathbf{g}(\overline{\Omega}) \cap B(\mathbf{y},\varepsilon) = \emptyset$ and so for ε this small,

$$\int_\Omega \phi_\varepsilon(\mathbf{g}(\mathbf{x})-\mathbf{y})\det D\mathbf{g}(\mathbf{x})\,dx = 0.$$

Showing the integral is constant for small ε

With this done it is necessary to show that the integral in the definition of the degree is constant for small enough ε even if $\mathbf{y}$ is not a regular value. To do this, I will first show that if 10.2 holds, then 10.3 holds. This particular part of the argument is the trick which makes surprising things happen. This is where the fact the functions are twice continuously differentiable is used. Suppose then that $\mathbf{f}, \mathbf{g}$ satisfy 10.2. Also let $\varepsilon > 0$ be such that for all $t \in [0,1]$,

$$B(\mathbf{y}, \varepsilon) \cap (\mathbf{f} + t(\mathbf{g} - \mathbf{f}))(\partial\Omega) = \emptyset \tag{10.4}$$

Define for $t \in [0,1]$,

$$H(t) \equiv \int_\Omega \phi_\varepsilon(\mathbf{f} - \mathbf{y} + t(\mathbf{g} - \mathbf{f})) \det(D(\mathbf{f} + t(\mathbf{g} - \mathbf{f}))) \, dx.$$

Then if $t \in (0,1)$,

$$H'(t) = \int_\Omega \sum_\alpha \phi_{\varepsilon,\alpha}(\mathbf{f}(\mathbf{x}) - \mathbf{y} + t(\mathbf{g}(\mathbf{x}) - \mathbf{f}(\mathbf{x}))) \cdot$$

$$(g_\alpha(\mathbf{x}) - f_\alpha(\mathbf{x})) \det D(\mathbf{f} + t(\mathbf{g} - \mathbf{f})) \, dx$$

$$+ \int_\Omega \phi_\varepsilon(\mathbf{f} - \mathbf{y} + t(\mathbf{g} - \mathbf{f})) \cdot$$

$$\sum_{\alpha,j} \det D(\mathbf{f} + t(\mathbf{g} - \mathbf{f}))_{,\alpha j} (g_\alpha - f_\alpha)_{,j} \, dx \equiv \mathbf{A} + \mathbf{B}.$$

In this formula, the function det is considered as a function of the n^2 entries in the $n \times n$ matrix and the $,\alpha j$ represents the derivative with respect to the αj^{th} entry. Now as in the proof of Lemma 9.12.1 on Page 235,

$$\det D(\mathbf{f} + t(\mathbf{g} - \mathbf{f}))_{,\alpha j} = (\operatorname{cof} D(\mathbf{f} + t(\mathbf{g} - \mathbf{f})))_{\alpha j}$$

and so

$$\mathbf{B} = \int_\Omega \sum_\alpha \sum_j \phi_\varepsilon(\mathbf{f} - \mathbf{y} + t(\mathbf{g} - \mathbf{f})) \cdot$$

$$(\operatorname{cof} \ D(\mathbf{f} + t(\mathbf{g} - \mathbf{f})))_{\alpha j} (g_\alpha - f_\alpha)_{,j} \, dx.$$

By hypothesis

$$\begin{aligned} \mathbf{x} \quad \to \quad & \phi_\varepsilon(\mathbf{f}(\mathbf{x}) - \mathbf{y} + t(\mathbf{g}(\mathbf{x}) - \mathbf{f}(\mathbf{x}))) \cdot \\ & (\operatorname{cof} \ D(\mathbf{f}(\mathbf{x}) + t(\mathbf{g}(\mathbf{x}) - \mathbf{f}(\mathbf{x}))))_{\alpha j} \end{aligned}$$

is in $C_c^1(\Omega)$ because if $\mathbf{x} \in \partial\Omega$, it follows by 10.4 that for all $t \in [0,1]$

$$\mathbf{f}(\mathbf{x}) - \mathbf{y} + t(\mathbf{g}(\mathbf{x}) - \mathbf{f}(\mathbf{x})) \notin B(\mathbf{0}, \varepsilon)$$

and so $\phi_\varepsilon(\mathbf{f}(\mathbf{x}) - \mathbf{y} + t(\mathbf{g}(\mathbf{x}) - \mathbf{f}(\mathbf{x}))) = 0$. Furthermore this situation persists for $\mathbf{x}$ near $\partial\Omega$. Therefore, integrate by parts and write

$$\mathbf{B} = -\int_\Omega \sum_\alpha \sum_j \frac{\partial}{\partial x_j} (\phi_\varepsilon(\mathbf{f} - \mathbf{y} + t(\mathbf{g} - \mathbf{f}))) \cdot$$

$$(\operatorname{cof} \ D(\mathbf{f} + t(\mathbf{g} - \mathbf{f})))_{\alpha j} (g_\alpha - f_\alpha) \, dx +$$

$$-\int_\Omega \sum_\alpha \sum_j \phi_\varepsilon\left(\mathbf{f}-\mathbf{y}+t\left(\mathbf{g}-\mathbf{f}\right)\right)$$
$$\cdot\left(\operatorname{cof} D\left(\mathbf{f}+t\left(\mathbf{g}-\mathbf{f}\right)\right)\right)_{\alpha j,j}\left(g_\alpha-f_\alpha\right)dx.$$

The second term equals zero by Lemma 10.1.4. Simplifying the first term yields

$$\mathbf{B}=-\int_\Omega \sum_\alpha \sum_j \sum_\beta \phi_{\varepsilon,\beta}\left(\mathbf{f}-\mathbf{y}+t\left(\mathbf{g}-\mathbf{f}\right)\right)\cdot$$
$$\left(f_{\beta,j}+t\left(g_{\beta,j}-f_{\beta,j}\right)\right)\left(\operatorname{cof} D\left(\mathbf{f}+t\left(\mathbf{g}-\mathbf{f}\right)\right)\right)_{\alpha j}\left(g_\alpha-f_\alpha\right)dx$$

$$\begin{aligned}=&-\int_\Omega \sum_\alpha \sum_\beta \phi_{\varepsilon,\beta}\left(\mathbf{f}-\mathbf{y}+t\left(\mathbf{g}-\mathbf{f}\right)\right)\delta_{\beta\alpha}\cdot\\&\det\left(D\left(\mathbf{f}+t\left(\mathbf{g}-\mathbf{f}\right)\right)\right)\left(g_\alpha-f_\alpha\right)dx\end{aligned}$$

$$\begin{aligned}=&-\int_\Omega \sum_\alpha \phi_{\varepsilon,\alpha}\left(\mathbf{f}-\mathbf{y}+t\left(\mathbf{g}-\mathbf{f}\right)\right)\\&\cdot\det\left(D\left(\mathbf{f}+t\left(\mathbf{g}-\mathbf{f}\right)\right)\right)\left(g_\alpha-f_\alpha\right)dx\\=&-\mathbf{A}.\end{aligned}$$

Therefore, $H'(t)=0$ and so H is a constant.

Now let $\mathbf{g}\in\mathcal{U}_{\mathbf{y}}\cap C^2\left(\overline{\Omega};\mathbb{R}^n\right)$. By Sard's lemma, Lemma 9.9.9 there exists a regular value $\mathbf{y}_1$ of $\mathbf{g}$ which is very close to $\mathbf{y}$. This is because, by this lemma, the set of points which are not regular values has measure zero so this set of points must have empty interior. Let

$$\mathbf{g}_1(\mathbf{x})\equiv\mathbf{g}(\mathbf{x})+\mathbf{y}-\mathbf{y}_1$$

and let $\mathbf{y}_1-\mathbf{y}$ be so small that

$$\mathbf{y}\notin(1-t)\,\mathbf{g}_1+\mathbf{g}t\left(\partial\Omega\right)\equiv\mathbf{g}_1+t\left(\mathbf{g}-\mathbf{g}_1\right)\left(\partial\Omega\right)\ \text{ for all } t\in[0,1].$$

Then $\mathbf{g}_1(\mathbf{x})=\mathbf{y}$ if and only if $\mathbf{g}(\mathbf{x})=\mathbf{y}_1$ which is a regular value. Note also $D\left(\mathbf{g}(\mathbf{x})\right)=D\left(\mathbf{g}_1(\mathbf{x})\right)$. Then from what was just shown, letting $\mathbf{f}=\mathbf{g}$ and $\mathbf{g}=\mathbf{g}_1$ in the above and using $\mathbf{g}-\mathbf{y}_1=\mathbf{g}_1-\mathbf{y}$,

$$\begin{aligned}&\int_\Omega \phi_\varepsilon\left(\mathbf{g}(\mathbf{x})-\mathbf{y}_1\right)\det\left(D\left(\mathbf{g}(\mathbf{x})\right)\right)dx\\=&\int_\Omega \phi_\varepsilon\left(\mathbf{g}_1(\mathbf{x})-\mathbf{y}\right)\det\left(D\left(\mathbf{g}(\mathbf{x})\right)\right)dx\\=&\int_\Omega \phi_\varepsilon\left(\mathbf{g}(\mathbf{x})-\mathbf{y}\right)\det\left(D\left(\mathbf{g}(\mathbf{x})\right)\right)dx\end{aligned}$$

Since $\mathbf{y}_1$ is a regular value of $\mathbf{g}$ it follows from the first part of the argument that the first integral in the above is eventually constant for small enough ε. It follows the last integral is also eventually constant for small enough ε. This proves the claim about the limit existing and in fact being constant for small ε. The last claim follows right away from the above. Suppose 10.2 holds. Then choosing ε small enough, it follows $d\left(\mathbf{f},\Omega,\mathbf{y}\right)=d\left(\mathbf{g},\Omega,\mathbf{y}\right)$ because the two integrals defining the degree for small ε are equal. This proves the lemma.

Next I will show that if $\mathbf{f}\sim\mathbf{g}$ where $\mathbf{f},\mathbf{g}\in\mathcal{U}_{\mathbf{y}}\cap C^2\left(\overline{\Omega};\mathbb{R}^n\right)$ then $d\left(\mathbf{f},\Omega,\mathbf{y}\right)=d\left(\mathbf{g},\Omega,\mathbf{y}\right)$. In the special case where

$$\mathbf{h}(\mathbf{x},t)=t\mathbf{f}(\mathbf{x})+(1-t)\,\mathbf{g}(\mathbf{x})$$

this has already been done in the above lemma. In the following lemma the two functions $\mathbf{k},\mathbf{l}$ are only assumed to be continuous.

Lemma 10.2.6 *Suppose* $\mathbf{k} \sim \mathbf{l}$. *Then there exists a sequence of functions of* $\mathcal{U}_{\mathbf{y}}$,

$$\{\mathbf{g}_i\}_{i=1}^m,$$

such that $\mathbf{g}_i \in C^2\left(\overline{\Omega}; \mathbb{R}^n\right)$, *and defining* $\mathbf{g}_0 \equiv \mathbf{k}$ *and* $\mathbf{g}_{m+1} \equiv \mathbf{l}$, *there exists* $\delta > 0$ *such that for* $i = 1, \cdots, m+1$,

$$B(\mathbf{y}, \delta) \cap (t\mathbf{g}_i + (1-t)\mathbf{g}_{i-1})(\partial\Omega) = \emptyset, \text{ for all } t \in [0,1]. \tag{10.5}$$

Proof: This lemma is not really very surprising. By Lemma 10.2.3, $[\mathbf{k}]$ is an open set and since everything in $[\mathbf{k}]$ is homotopic to $\mathbf{k}$, it is also connected because it is path connected. The Lemma merely asserts there exists a piecewise linear curve joining $\mathbf{k}$ and $\mathbf{l}$ which stays within the open connected set, $[\mathbf{k}]$ in such a way that the vertices of this curve are in the dense set $C^2\left(\overline{\Omega}; \mathbb{R}^n\right)$. This is the abstract idea. Now here is a more down to earth treatment.

Let $\mathbf{h} : \overline{\Omega} \times [0,1] \to \mathbb{R}^n$ be a homotopy of $\mathbf{k}$ and $\mathbf{l}$ with the property that $\mathbf{y} \notin \mathbf{h}(\partial\Omega, t)$ for all $t \in [0,1]$. Such a homotopy exists because $\mathbf{k} \sim \mathbf{l}$. Now let $0 = t_0 < t_1 < \cdots < t_{m+1} = 1$ be such that for $i = 1, \cdots, m+1$,

$$||\mathbf{h}(\cdot, t_i) - \mathbf{h}(\cdot, t_{i-1})||_\infty < \delta \tag{10.6}$$

where $\delta > 0$ is small. By Lemma 10.2.3, for each $i \in \{1, \cdots, m\}$, there exists $\mathbf{g}_i \in \mathcal{U}_{\mathbf{y}} \cap C^2\left(\overline{\Omega}; \mathbb{R}^n\right)$ such that

$$||\mathbf{g}_i - \mathbf{h}(\cdot, t_i)||_\infty < \delta \tag{10.7}$$

Thus

$$\begin{aligned} ||\mathbf{g}_i - \mathbf{g}_{i-1}||_\infty &\le ||\mathbf{g}_i - \mathbf{h}(\cdot, t_i)||_\infty \\ + ||\mathbf{h}(\cdot, t_i) - \mathbf{h}(\cdot, t_{i-1})||_\infty &+ ||\mathbf{g}_{i-1} - \mathbf{h}(\cdot, t_{i-1})||_\infty < 3\delta. \end{aligned}$$

(Recall $\mathbf{g}_0 \equiv \mathbf{k}$ in case $i = 1$.)

It was just shown that for each $\mathbf{x} \in \partial\Omega$,

$$\mathbf{g}_i(\mathbf{x}) \in B(\mathbf{g}_{i-1}(\mathbf{x}), 3\delta)$$

Also for each $\mathbf{x} \in \partial\Omega$

$$\begin{aligned} |\mathbf{g}_j(\mathbf{x}) - \mathbf{y}| &\ge |\mathbf{h}(\mathbf{x}, t_j) - \mathbf{y}| - |\mathbf{g}_j(\mathbf{x}) - \mathbf{h}(\mathbf{x}, t_j)| \\ &\ge |\mathbf{h}(\mathbf{x}, t_j) - \mathbf{y}| - \delta \end{aligned}$$

For $\mathbf{x} \in \partial\Omega$

$$\begin{aligned} & |t\mathbf{g}_i(\mathbf{x}) + (1-t)\mathbf{g}_{i-1}(\mathbf{x}) - \mathbf{y}| \\ = \; & |\mathbf{g}_{i-1}(\mathbf{x}) + t(\mathbf{g}_i(\mathbf{x}) - \mathbf{g}_{i-1}(\mathbf{x})) - \mathbf{y}| \\ \ge \; & |\mathbf{g}_{i-1}(\mathbf{x}) - \mathbf{y}| - t|\mathbf{g}_i(\mathbf{x}) - \mathbf{g}_{i-1}(\mathbf{x})| \\ \ge \; & (|\mathbf{h}(\mathbf{x}, t_{i-1}) - \mathbf{y}| - \delta) - t3\delta \\ & \ge |\mathbf{h}(\mathbf{x}, t_{i-1}) - \mathbf{y}| - 4\delta > \delta \end{aligned}$$

provided δ was small enough that

$$B(\mathbf{y}, 5\delta) \cap \mathbf{h}(\partial\Omega \times [0,1]) = \emptyset \tag{10.8}$$

so that for all $t \in [0,1]$, $|\mathbf{h}(\mathbf{x}, t) - \mathbf{y}| > 5\delta$. This proves the lemma.

With this lemma the homotopy invariance of the degree on $C^2\left(\overline{\Omega}; \mathbb{R}^n\right)$ is easy to obtain. The following theorem gives this homotopy invariance and summarizes one of the results of Lemma 10.2.5.

Theorem 10.2.7 *Let* $\mathbf{f},\mathbf{g}\in\mathcal{U}_{\mathbf{y}}\cap C^2\left(\overline{\Omega};\mathbb{R}^n\right)$ *and suppose* $\mathbf{f}\sim\mathbf{g}$. *Then*

$$d\left(\mathbf{f},\Omega,\mathbf{y}\right)=d\left(\mathbf{g},\Omega,\mathbf{y}\right).$$

When $\mathbf{f}\in\mathcal{U}_{\mathbf{y}}\cap C^2\left(\overline{\Omega};\mathbb{R}^n\right)$,

$$d\left(\mathbf{f},\Omega,\mathbf{y}\right)=\sum\left\{\operatorname{sgn}\left(\det D\mathbf{g}\left(\mathbf{x}\right)\right):\mathbf{x}\in\mathbf{g}^{-1}\left(\mathbf{y}\right)\right\}.$$

The degree is an integer. Also

$$\mathbf{y}\to d\left(\mathbf{f},\Omega,\mathbf{y}\right)$$

is continuous on $\mathbb{R}^n\setminus\mathbf{f}\left(\partial\Omega\right)$ *and* $\mathbf{y}\to d\left(\mathbf{f},\Omega,\mathbf{y}\right)$ *is constant on every connected component of* $\mathbb{R}^n\setminus\mathbf{f}\left(\partial\Omega\right)$.

Proof: From Lemma 10.2.6 there exists a sequence of functions in $C^2\left(\overline{\Omega};\mathbb{R}^n\right)$ having the properties listed there. Then from Lemma 10.2.5

$$d\left(\mathbf{f},\Omega,\mathbf{y}\right)=d\left(\mathbf{g}_1,\Omega,\mathbf{y}\right)=d\left(\mathbf{g}_2,\Omega,\mathbf{y}\right)=\cdots=d\left(\mathbf{g},\Omega,\mathbf{y}\right).$$

The second assertion follows from Lemma 10.2.5. Finally consider the claim the degree is an integer. This is obvious if $\mathbf{y}$ is a regular point. If $\mathbf{y}$ is not a regular point, let

$$\mathbf{g}_1\left(\mathbf{x}\right)\equiv\mathbf{g}\left(\mathbf{x}\right)+\mathbf{y}-\mathbf{y}_1$$

where $\mathbf{y}_1$ is a regular point of $\mathbf{g}$ and $|\mathbf{y}-\mathbf{y}_1|$ is so small that

$$\mathbf{y}\notin\left(t\mathbf{g}_1+\left(1-t\right)\mathbf{g}\right)\left(\partial\Omega\right).$$

From Lemma 10.2.5

$$d\left(\mathbf{g}_1,\Omega,\mathbf{y}\right)=d\left(\mathbf{g},\Omega,\mathbf{y}\right).$$

But since $\mathbf{g}_1-\mathbf{y}=\mathbf{g}-\mathbf{y}_1$ and $\det D\mathbf{g}\left(\mathbf{x}\right)=\det D\mathbf{g}_1\left(\mathbf{x}\right)$,

$$\begin{aligned}d\left(\mathbf{g}_1,\Omega,\mathbf{y}\right)&=\lim_{\varepsilon\to0}\int_\Omega\phi_\varepsilon\left(\mathbf{g}_1\left(\mathbf{x}\right)-\mathbf{y}\right)\det D\mathbf{g}\left(\mathbf{x}\right)dx\\&=\lim_{\varepsilon\to0}\int_\Omega\phi_\varepsilon\left(\mathbf{g}\left(\mathbf{x}\right)-\mathbf{y}_1\right)\det D\mathbf{g}\left(\mathbf{x}\right)dx\end{aligned}$$

which by Lemma 10.2.5 equals $\sum\left\{\operatorname{sgn}\left(\det D\mathbf{g}\left(\mathbf{x}\right)\right):\mathbf{x}\in\mathbf{g}^{-1}\left(\mathbf{y}_1\right)\right\}$, an integer.

What about the continuity assertion and being constant on connected components? Let U be a connected component of $\mathbb{R}^n\setminus\mathbf{f}\left(\partial\Omega\right)$ and let $\mathbf{y},\mathbf{y}_1$ be two points of U. The argument is similar to some of the above. Let $\mathbf{y}\in\mathbb{R}^n\setminus\mathbf{f}\left(\partial\Omega\right)$ and suppose $\mathbf{y}_1$ is very close to $\mathbf{y}$, close enough that for

$$\mathbf{f}_1\left(\mathbf{x}\right)\equiv\mathbf{f}\left(\mathbf{x}\right)+\mathbf{y}-\mathbf{y}_1$$

it follows

$$\mathbf{y}\notin t\mathbf{f}+\left(1-t\right)\mathbf{f}_1\left(\partial\Omega\right)$$

Then from Lemma 10.2.5

$$\begin{aligned}d\left(\mathbf{f},\Omega,\mathbf{y}\right)&=d\left(\mathbf{f}_1,\Omega,\mathbf{y}\right)=\lim_{\varepsilon\to0}\int_\Omega\phi_\varepsilon\left(\mathbf{f}_1\left(\mathbf{x}\right)-\mathbf{y}\right)D\mathbf{f}\left(\mathbf{x}\right)dx\\&=\lim_{\varepsilon\to0}\int_\Omega\phi_\varepsilon\left(\mathbf{f}\left(\mathbf{x}\right)-\mathbf{y}_1\right)D\mathbf{f}\left(\mathbf{x}\right)dx\equiv d\left(\mathbf{f},\Omega,\mathbf{y}_1\right)\end{aligned}$$

which shows that $\mathbf{y}\to d\left(\mathbf{f},\Omega,\mathbf{y}\right)$ is continuous. Since it has integer values, it follows from Corollary 5.3.15 on Page 89 that this function must be constant on every connected component. This proves the theorem.

10.2.2 Definition Of The Degree For Continuous Functions

With the above results, it is now possible to extend the definition of the degree to continuous functions which have no differentiability. It is desired to preserve the homotopy invariance. This requires the following definition.

Definition 10.2.8 *Let* $\mathbf{f} \in \mathcal{U}_{\mathbf{y}}$ *where* $\mathbf{y} \in \mathbb{R}^n \setminus \mathbf{f}(\partial\Omega)$. *Then*

$$d(\mathbf{f}, \Omega, \mathbf{y}) \equiv f(\mathbf{g}, \Omega, \mathbf{y})$$

where $\mathbf{g} \in \mathcal{U}_{\mathbf{y}} \cap C^2\left(\overline{\Omega}; \mathbb{R}^n\right)$ *and* $\mathbf{f} \sim \mathbf{g}$.

Theorem 10.2.9 *The definition of the degree given in Definition 10.2.8 is well defined, equals an integer, and satisfies the following properties. In what follows,* $\mathrm{id}(\mathbf{x}) = \mathbf{x}$.

1. $d(\mathrm{id}, \Omega, \mathbf{y}) = 1$ *if* $\mathbf{y} \in \Omega$.
2. *If* $\Omega_i \subseteq \Omega, \Omega_i$ *open, and* $\Omega_1 \cap \Omega_2 = \emptyset$ *and if* $\mathbf{y} \notin \mathbf{f}\left(\overline{\Omega} \setminus (\Omega_1 \cup \Omega_2)\right)$, *then* $d(\mathbf{f}, \Omega_1, \mathbf{y}) + d(\mathbf{f}, \Omega_2, \mathbf{y}) = d(\mathbf{f}, \Omega, \mathbf{y})$.
3. *If* $\mathbf{y} \notin \mathbf{f}\left(\overline{\Omega} \setminus \Omega_1\right)$ *and* Ω_1 *is an open subset of* Ω, *then*
$$d(\mathbf{f}, \Omega, \mathbf{y}) = d(\mathbf{f}, \Omega_1, \mathbf{y}).$$
4. *For* $\mathbf{y} \in \mathbb{R}^n \setminus \mathbf{f}(\partial\Omega)$, *if* $d(\mathbf{f}, \Omega, \mathbf{y}) \neq 0$ *then* $\mathbf{f}^{-1}(\mathbf{y}) \cap \Omega \neq \emptyset$.
5. *If* $\mathbf{f}, \mathbf{g}$ *are homotopic with a homotopy,* $\mathbf{h} : \overline{\Omega} \times [0,1]$ *for which* $\mathbf{h}(\partial\Omega, t)$ *does not contain* $\mathbf{y}$, *then* $d(\mathbf{g},\Omega,\mathbf{y}) = d(\mathbf{f},\Omega,\mathbf{y})$.
6. $d(\cdot, \Omega, \mathbf{y})$ *is defined and constant on*
$$\left\{\mathbf{g} \in C\left(\overline{\Omega}; \mathbb{R}^n\right) : ||\mathbf{g} - \mathbf{f}||_\infty < r\right\}$$
where $r = \mathrm{dist}(\mathbf{y}, \mathbf{f}(\partial\Omega))$.
7. $d(\mathbf{f}, \Omega, \cdot)$ *is constant on every connected component of* $\mathbb{R}^n \setminus \mathbf{f}(\partial\Omega)$.
8. $d(\mathbf{g},\Omega,\mathbf{y}) = d(\mathbf{f},\Omega,\mathbf{y})$ *if* $\mathbf{g}|_{\partial\Omega} = \mathbf{f}|_{\partial\Omega}$.

Proof: First it is necessary to show the definition is well defined. There are two parts to this. First I need to show there exists $\mathbf{g}$ with the desired properties and then I need to show that it doesn't matter which $\mathbf{g}$ I happen to pick. The first part is easy. Let δ be small enough that

$$B(\mathbf{y}, \delta) \cap \mathbf{f}(\partial\Omega) = \emptyset.$$

Then by Lemma 10.2.3 there exists $\mathbf{g} \in C^2\left(\overline{\Omega}; \mathbb{R}^n\right)$ such that $||\mathbf{g} - \mathbf{f}||_\infty < \delta$. It follows that for $t \in [0,1]$,

$$\mathbf{y} \notin (t\mathbf{g} + (1-t)\mathbf{f})(\partial\Omega)$$

and so $\mathbf{g} \sim \mathbf{f}$. This does the first part. Now consider the second part. Suppose $\mathbf{g} \sim \mathbf{f}$ and $\mathbf{g}_1 \sim \mathbf{f}$. Then by Lemma 10.2.3 again

$$\mathbf{g} \sim \mathbf{g}_1$$

and by Theorem 10.2.7 it follows $d(\mathbf{g},\Omega,\mathbf{y}) = d(\mathbf{g}_1,\Omega,\mathbf{y})$ which shows the definition is well defined. Also $d(\mathbf{f},\Omega,\mathbf{y})$ must be an integer because it equals $d(\mathbf{g},\Omega,\mathbf{y})$ which is an integer.

Now consider the properties. The first one is obvious from Theorem 10.2.7 since $\mathbf{y}$ is a regular point of id.

Consider the second property. The assumption implies

$$\mathbf{y} \notin \mathbf{f}(\partial\Omega) \cup \mathbf{f}(\partial\Omega_1) \cup \mathbf{f}(\partial\Omega_2)$$

Let $\mathbf{g} \in C^2\left(\overline{\Omega}; \mathbb{R}^n\right)$ such that $||\mathbf{f}-\mathbf{g}||_\infty$ is small enough that

$$\mathbf{y} \notin \mathbf{g}\left(\overline{\Omega} \setminus (\Omega_1 \cup \Omega_2)\right) \tag{10.9}$$

and also small enough that

$$\begin{array}{rcl} \mathbf{y} & \notin & (t\mathbf{g} + (1-t)\mathbf{f})(\partial\Omega),\ \mathbf{y} \notin (t\mathbf{g} + (1-t)\mathbf{f})(\partial\Omega_1) \\ \mathbf{y} & \notin & (t\mathbf{g} + (1-t)\mathbf{f})(\partial\Omega_2) \end{array} \tag{10.10}$$

for all $t \in [0,1]$. Then it follows from Lemma 10.2.5, for all ε small enough,

$$d(\mathbf{g}, \Omega, \mathbf{y}) = \int_\Omega \phi_\varepsilon(\mathbf{g}(\mathbf{x}) - \mathbf{y})\, D\mathbf{g}(\mathbf{x})\, dx$$

From 10.9 there is a positive distance between the compact set

$$\mathbf{g}\left(\overline{\Omega} \setminus (\Omega_1 \cup \Omega_2)\right)$$

and $\mathbf{y}$. Therefore, making ε still smaller if necessary,

$$\phi_\varepsilon(\mathbf{g}(\mathbf{x}) - \mathbf{y}) = 0 \text{ if } \mathbf{x} \notin \Omega_1 \cup \Omega_2$$

Therefore, using the definition of the degree and 10.10,

$$d(\mathbf{f}, \Omega, \mathbf{y}) = d(\mathbf{g}, \Omega, \mathbf{y}) = \lim_{\varepsilon \to 0} \int_\Omega \phi_\varepsilon(\mathbf{g}(\mathbf{x}) - \mathbf{y})\, D\mathbf{g}(\mathbf{x})\, dx$$

$$\begin{array}{rcl} & = & \lim_{\varepsilon \to 0} \left(\int_{\Omega_1} \phi_\varepsilon(\mathbf{g}(\mathbf{x}) - \mathbf{y})\, D\mathbf{g}(\mathbf{x})\, dx + \right. \\ & & \left. \int_{\Omega_2} \phi_\varepsilon(\mathbf{g}(\mathbf{x}) - \mathbf{y})\, D\mathbf{g}(\mathbf{x})\, dx \right) \\ & = & d(\mathbf{g}, \Omega_1, \mathbf{y}) + d(\mathbf{g}, \Omega_2, \mathbf{y}) \\ & = & d(\mathbf{f}, \Omega_1, \mathbf{y}) + d(\mathbf{f}, \Omega_2, \mathbf{y}) \end{array}$$

This proves the second property.

Consider the third. This really follows from the second property. You can take $\Omega_2 = \emptyset$. I leave the details to you. To be more careful, you can modify the proof of property 2 slightly.

The fourth property is very important because it can be used to deduce the existence of solutions to a nonlinear equation. Suppose $\mathbf{f}^{-1}(\mathbf{y}) \cap \Omega = \emptyset$. I will show this requires $d(\mathbf{f}, \Omega, \mathbf{y}) = 0$. It is assumed $\mathbf{y} \notin \mathbf{f}(\partial\Omega)$ and so if $\mathbf{f}^{-1}(\mathbf{y}) \cap \Omega = \emptyset$, then $\mathbf{y} \notin \mathbf{f}\left(\overline{\Omega}\right)$. Choosing $\mathbf{g} \in C^2\left(\overline{\Omega}; \mathbb{R}^n\right)$ such that $||\mathbf{f}-\mathbf{g}||_\infty$ is sufficiently small, it can be assumed

$$\mathbf{y} \notin \mathbf{g}\left(\overline{\Omega}\right),\ \mathbf{y} \notin (t\mathbf{f} + (1-t)\mathbf{g})(\partial\Omega) \text{ for all } t \in [0,1].$$

Then it follows from the definition of the degree

$$d(\mathbf{f}, \Omega, \mathbf{y}) = d(\mathbf{g}, \Omega, \mathbf{y}) \equiv \lim_{\varepsilon \to 0} \int_\Omega \phi_\varepsilon(\mathbf{g}(\mathbf{x}) - \mathbf{y})\, D\mathbf{g}(\mathbf{x})\, dx = 0$$

because eventually ε is smaller than the distance from $\mathbf{y}$ to $\mathbf{g}\left(\overline{\Omega}\right)$ and so $\phi_\varepsilon\left(\mathbf{g}\left(\mathbf{x}\right)-\mathbf{y}\right)=0$ for all $\mathbf{x}\in\Omega$.

Property 5 follows from the definition of the degree.

Consider the sixth property. If $\mathbf{g}$ is in the specified set, and $t\in[0,1]$ with $\mathbf{x}\in\partial\Omega$

$$t\mathbf{g}\left(\mathbf{x}\right)+\left(1-t\right)\mathbf{f}\left(\mathbf{x}\right)=\mathbf{f}\left(\mathbf{x}\right)+t\left(\mathbf{g}\left(\mathbf{x}\right)-\mathbf{f}\left(\mathbf{x}\right)\right)$$

and so

$$\begin{aligned}&\left|t\mathbf{g}\left(\mathbf{x}\right)+\left(1-t\right)\mathbf{f}\left(\mathbf{x}\right)-\mathbf{y}\right|\\ \geq\ &\left|\mathbf{f}\left(\mathbf{x}\right)-\mathbf{y}\right|-t\left|\mathbf{g}\left(\mathbf{x}\right)-\mathbf{f}\left(\mathbf{x}\right)\right|>r-tr\geq 0\end{aligned}$$

and from the definition of the degree

$$d\left(\mathbf{f},\Omega,\mathbf{y}\right)=d\left(\mathbf{g},\Omega,\mathbf{y}\right)$$

The seventh claim is done already for the case where $\mathbf{f}\in C^2\left(\overline{\Omega};\mathbb{R}^n\right)$ in Theorem 10.2.7. It remains to verify this for the case where $\mathbf{f}$ is only continuous. This will be done by showing $\mathbf{y}\to d\left(\mathbf{f},\Omega,\mathbf{y}\right)$ is continuous. Let $\mathbf{y}_0\in\mathbb{R}^n\setminus\mathbf{f}\left(\partial\Omega\right)$ and let δ be small enough that

$$B\left(\mathbf{y}_0,4\delta\right)\cap\mathbf{f}\left(\partial\Omega\right)=\emptyset.$$

Now let $\mathbf{g}\in C^2\left(\overline{\Omega};\mathbb{R}^n\right)$ such that $\left|\left|\mathbf{g}-\mathbf{f}\right|\right|_\infty<\delta$. Then for $\mathbf{x}\in\partial\Omega, t\in[0,1]$, and $\mathbf{y}\in B\left(\mathbf{y}_0,\delta\right)$,

$$\left|\left(t\mathbf{g}+\left(1-t\right)\mathbf{f}\right)\left(\mathbf{x}\right)-\mathbf{y}\right|\geq\left|\mathbf{f}\left(\mathbf{x}\right)-\mathbf{y}\right|-t\left|\mathbf{g}\left(\mathbf{x}\right)-\mathbf{f}\left(\mathbf{x}\right)\right|$$

$$\begin{aligned}&\geq\ \left|\mathbf{f}\left(\mathbf{x}\right)-\mathbf{y}_0\right|-\left|\mathbf{y}_0-\mathbf{y}\right|-\left|\left|\mathbf{g}-\mathbf{f}\right|\right|_\infty\\ &\geq\ 4\delta-\delta-\delta>0.\end{aligned}$$

Therefore, for all such $\mathbf{y}\in B\left(\mathbf{y}_0,\delta\right)$

$$d\left(\mathbf{f},\Omega,\mathbf{y}\right)=d\left(\mathbf{g},\Omega,\mathbf{y}\right)$$

and it was shown in Theorem 10.2.7 that $\mathbf{y}\to d\left(\mathbf{g},\Omega,\mathbf{y}\right)$ is continuous. In particular $d\left(\mathbf{f},\Omega,\cdot\right)$ is continuous at $\mathbf{y}_0$. Since $\mathbf{y}_0$ was arbitrary, this shows $\mathbf{y}\to d\left(\mathbf{f},\Omega,\mathbf{y}\right)$ is continuous. Therefore, since it has integer values, this function is constant on every connected component of $\mathbb{R}^n\setminus\mathbf{f}\left(\partial\Omega\right)$ by Corollary 5.3.15.

Consider the eighth claim about the degree in which $\mathbf{f}=\mathbf{g}$ on $\partial\Omega$. This one is easy because for

$$\mathbf{y}\in\mathbb{R}^n\setminus\mathbf{f}\left(\partial\Omega\right)=\mathbb{R}^n\setminus\mathbf{g}\left(\partial\Omega\right),$$

and $\mathbf{x}\in\partial\Omega$,

$$t\mathbf{f}\left(\mathbf{x}\right)+\left(1-t\right)\mathbf{g}\left(\mathbf{x}\right)-\mathbf{y}=\mathbf{f}\left(\mathbf{x}\right)-\mathbf{y}\neq\mathbf{0}$$

for all $t\in[0,1]$ and so by the fifth claim, $d\left(\mathbf{f},\Omega,\mathbf{y}\right)=d\left(\mathbf{g},\Omega,\mathbf{y}\right)$ and this proves the theorem.

10.3 Borsuk's Theorem

In this section is an important theorem which can be used to verify that $d\left(\mathbf{f},\Omega,\mathbf{y}\right)\neq 0$. This is significant because when this is known, it follows from Theorem 10.2.9 that $\mathbf{f}^{-1}\left(\mathbf{y}\right)\neq\emptyset$. In other words there exists $\mathbf{x}\in\Omega$ such that $\mathbf{f}\left(\mathbf{x}\right)=\mathbf{y}$.

Definition 10.3.1 *A bounded open set, Ω is symmetric if $-\Omega=\Omega$. A continuous function, $\mathbf{f}:\overline{\Omega}\to\mathbb{R}^n$ is odd if $\mathbf{f}\left(-\mathbf{x}\right)=-\mathbf{f}\left(\mathbf{x}\right)$.*

Suppose Ω is symmetric and $\mathbf{g} \in C^2\left(\overline{\Omega}; \mathbb{R}^n\right)$ is an odd map for which $\mathbf{0}$ is a regular value. Then the chain rule implies $D\mathbf{g}(-\mathbf{x}) = D\mathbf{g}(\mathbf{x})$ and so $d(\mathbf{g}, \Omega, \mathbf{0})$ must equal an odd integer because if $\mathbf{x} \in \mathbf{g}^{-1}(\mathbf{0})$, it follows that $-\mathbf{x} \in \mathbf{g}^{-1}(\mathbf{0})$ also and since $D\mathbf{g}(-\mathbf{x}) = D\mathbf{g}(\mathbf{x})$, it follows the overall contribution to the degree from $\mathbf{x}$ and $-\mathbf{x}$ must be an even integer. Also $\mathbf{0} \in \mathbf{g}^{-1}(\mathbf{0})$ and so the degree equals an even integer added to $\operatorname{sgn}(\det D\mathbf{g}(\mathbf{0}))$, an odd integer, either -1 or 1. It seems reasonable to expect that something like this would hold for an arbitrary continuous odd function defined on symmetric Ω. In fact this is the case and this is next. The following lemma is the key result used. This approach is due to Gromes [19]. See also Deimling [9] which is where I found this argument.

The idea is to start with a smooth odd map and approximate it with a smooth odd map which also has $\mathbf{0}$ a regular value.

Lemma 10.3.2 *Let $\mathbf{g} \in C^2\left(\overline{\Omega}; \mathbb{R}^n\right)$ be an odd map. Then for every $\varepsilon > 0$, there exists $\mathbf{h} \in C^2\left(\overline{\Omega}; \mathbb{R}^n\right)$ such that $\mathbf{h}$ is also an odd map, $||\mathbf{h} - \mathbf{g}||_\infty < \varepsilon$, and $\mathbf{0}$ is a regular value of $\mathbf{h}$.*

Proof: In this argument $\eta > 0$ will be a small positive number and C will be a constant which depends only on the diameter of Ω. Let $\mathbf{h}_0(\mathbf{x}) = \mathbf{g}(\mathbf{x}) + \eta\mathbf{x}$ where η is chosen such that $\det D\mathbf{h}_0(\mathbf{0}) \neq 0$. Now let $\Omega_i \equiv \{\mathbf{x} \in \Omega : x_i \neq 0\}$. In other words, leave out the plane $x_i = 0$ from Ω in order to obtain Ω_i. A succession of modifications is about to take place on $\Omega_1, \Omega_1 \cup \Omega_2$, etc. Finally a function will be obtained on $\cup_{j=1}^n \Omega_j$ which is everything except $\mathbf{0}$.

Define $\mathbf{h}_1(\mathbf{x}) \equiv \mathbf{h}_0(\mathbf{x}) - \mathbf{y}^1 x_1^3$ where $\left|\mathbf{y}^1\right| < \eta$ and $\mathbf{y}^1 = \left(y_1^1, \cdots, y_n^1\right)$ is a regular value of the function, $\mathbf{x} \to \frac{\mathbf{h}_0(\mathbf{x})}{x_1^3}$ for $\mathbf{x} \in \Omega_1$. the existence of $\mathbf{y}^1$ follows from Sard's lemma because this function is in $C^2(\Omega_1; \mathbb{R}^n)$. Thus $\mathbf{h}_1(\mathbf{x}) = \mathbf{0}$ if and only if $\mathbf{y}^1 = \frac{\mathbf{h}_0(\mathbf{x})}{x_1^3}$. Since $\mathbf{y}^1$ is a regular value, it follows that for such $\mathbf{x}$,

$$\det\left(\frac{h_{0i,j}(\mathbf{x})\, x_1^3 - \frac{\partial}{\partial x_j}\left(x_1^3\right) h_{0i}(\mathbf{x})}{x_1^6}\right) =$$

$$\det\left(\frac{h_{0i,j}(\mathbf{x})\, x_1^3 - \frac{\partial}{\partial x_j}\left(x_1^3\right) y_i^1 x_1^3}{x_1^6}\right) \neq 0$$

implying that

$$\det\left(h_{0i,j}(\mathbf{x}) - \frac{\partial}{\partial x_j}\left(x_1^3\right) y_i^1\right) = \det\left(D\mathbf{h}_1(\mathbf{x})\right) \neq 0.$$

This shows $\mathbf{0}$ is a regular value of $\mathbf{h}_1$ on the set Ω_1 and it is clear $\mathbf{h}_1$ is an odd map in $C^2\left(\overline{\Omega}; \mathbb{R}^n\right)$ and $||\mathbf{h}_1 - \mathbf{g}||_\infty \leq C\eta$ where C depends only on the diameter of Ω.

Now suppose for some k such that $1 \leq k < n$ there exists an odd mapping $\mathbf{h}_k$ in $C^2\left(\overline{\Omega}; \mathbb{R}^n\right)$ such that $\mathbf{0}$ is a regular value of $\mathbf{h}_k$ on $\cup_{i=1}^k \Omega_i$ and $||\mathbf{h}_k - \mathbf{g}||_\infty \leq C\eta$. Sard's theorem implies there exists $\mathbf{y}^{k+1}$ a regular value of the function $\mathbf{x} \to \mathbf{h}_k(\mathbf{x}) / x_{k+1}^3$ defined on Ω_{k+1}such that $\left|\left|\mathbf{y}^{k+1}\right|\right| < \eta$ and let $\mathbf{h}_{k+1}(\mathbf{x}) = \mathbf{h}_k(\mathbf{x}) - \mathbf{y}^{k+1} x_{k+1}^3$. As before, $\mathbf{h}_{k+1}(\mathbf{x}) = \mathbf{0}$ if and only if $\mathbf{h}_k(\mathbf{x}) / x_{k+1}^3 = \mathbf{y}^{k+1}$, a regular value of $\mathbf{x} \to \mathbf{h}_k(\mathbf{x}) / x_{k+1}^3$. Consider such $\mathbf{x}$ for which $\mathbf{h}_{k+1}(\mathbf{x}) = \mathbf{0}$. First suppose $\mathbf{x} \in \Omega_{k+1}$. Then

$$\det\left(\frac{h_{ki,j}(\mathbf{x})\, x_{k+1}^3 - \frac{\partial}{\partial x_j}\left(x_{k+1}^3\right) y_i^{k+1} x_{k+1}^3}{x_{k+1}^6}\right) \neq 0$$

which implies that whenever $\mathbf{h}_{k+1}(\mathbf{x}) = \mathbf{0}$ and $\mathbf{x} \in \Omega_{k+1}$,

$$\det\left(h_{ki,j}(\mathbf{x}) - \frac{\partial}{\partial x_j}\left(x_{k+1}^3\right) y_i^{k+1}\right) = \det\left(D\mathbf{h}_{k+1}(\mathbf{x})\right) \neq 0. \tag{10.11}$$

However, if $\mathbf{x} \in \cup_{i=1}^{k}\Omega_k$ but $\mathbf{x} \notin \Omega_{k+1}$, then $x_{k+1} = 0$ and so the left side of 10.11 reduces to $\det(h_{ki,j}(\mathbf{x}))$ which is not zero because $\mathbf{0}$ is assumed a regular value of $\mathbf{h}_k$. Therefore, $\mathbf{0}$ is a regular value for $\mathbf{h}_{k+1}$ on $\cup_{i=1}^{k+1}\Omega_k$. (For $\mathbf{x} \in \cup_{i=1}^{k+1}\Omega_k$, either $\mathbf{x} \in \Omega_{k+1}$ or $\mathbf{x} \notin \Omega_{k+1}$. If $\mathbf{x} \in \Omega_{k+1}$ $\mathbf{0}$ is a regular value by the construction above. In the other case, $\mathbf{0}$ is a regular value by the induction hypothesis.) Also $\mathbf{h}_{k+1}$ is odd and in $C^2\left(\overline{\Omega};\mathbb{R}^n\right)$, and $||\mathbf{h}_{k+1} - \mathbf{g}||_\infty \leq C\eta$.

Let $\mathbf{h} \equiv \mathbf{h}_n$. Then $\mathbf{0}$ is a regular value of $\mathbf{h}$ for $\mathbf{x} \in \cup_{j=1}^{n}\Omega_j$. The point of Ω which is not in $\cup_{j=1}^{n}\Omega_j$ is $\mathbf{0}$. If $\mathbf{x} = \mathbf{0}$, then from the construction, $D\mathbf{h}(\mathbf{0}) = D\mathbf{h}_0(\mathbf{0})$ and so $\mathbf{0}$ is a regular value of $\mathbf{h}$ for $\mathbf{x} \in \Omega$. By choosing η small enough, it follows $||\mathbf{h} - \mathbf{g}||_\infty < \varepsilon$. This proves the lemma.

Theorem 10.3.3 *(Borsuk) Let $\mathbf{f} \in C\left(\overline{\Omega};\mathbb{R}^n\right)$ be odd and let Ω be symmetric with $\mathbf{0} \notin \mathbf{f}(\partial\Omega)$. Then $d(\mathbf{f},\Omega,\mathbf{0})$ equals an odd integer.*

Proof: Let δ be small enough that $B(\mathbf{0},3\delta) \cap \mathbf{f}(\partial\Omega) = \emptyset$. Let

$$\mathbf{g}_1 \in C^2\left(\overline{\Omega};\mathbb{R}^n\right)$$

be such that $||\mathbf{f} - \mathbf{g}_1||_\infty < \delta$ and let $\mathbf{g}$ denote the odd part of $\mathbf{g}_1$. Thus

$$\mathbf{g}(\mathbf{x}) \equiv \frac{1}{2}\left(\mathbf{g}_1(\mathbf{x}) - \mathbf{g}_1(-\mathbf{x})\right).$$

Then since $\mathbf{f}$ is odd, it follows that for $\mathbf{x} \in \overline{\Omega}$,

$$|\mathbf{f}(\mathbf{x}) - \mathbf{g}(\mathbf{x})| =$$

$$\left|\frac{1}{2}\left(\mathbf{f}(\mathbf{x}) - \mathbf{f}(-\mathbf{x})\right) - \frac{1}{2}\left(\mathbf{g}_1(\mathbf{x}) - \mathbf{g}_1(-\mathbf{x})\right)\right|$$

$$\leq \frac{1}{2}|\mathbf{f}(\mathbf{x}) - \mathbf{g}_1(\mathbf{x})| + \frac{1}{2}|\mathbf{f}(-\mathbf{x}) - \mathbf{g}_1(-\mathbf{x})| < \delta$$

Thus $||\mathbf{f} - \mathbf{g}||_\infty < \delta$ also. By Lemma 10.3.2 there exists odd $\mathbf{h} \in C^2\left(\overline{\Omega};\mathbb{R}^n\right)$ for which $\mathbf{0}$ is a regular value and $||\mathbf{h} - \mathbf{g}||_\infty < \delta$. Therefore,

$$||\mathbf{f} - \mathbf{h}||_\infty \leq ||\mathbf{f} - \mathbf{g}||_\infty + ||\mathbf{g} - \mathbf{h}||_\infty < 2\delta$$

and so for $t \in [0,1]$ and $\mathbf{x} \in \partial\Omega$,

$$\begin{aligned} |t\mathbf{h}(\mathbf{x}) + (1-t)\mathbf{f}(\mathbf{x}) - \mathbf{0}| &\geq |\mathbf{f}(\mathbf{x}) - \mathbf{0}| - t|\mathbf{h}(\mathbf{x}) - \mathbf{f}(\mathbf{x})| \\ &\geq 3\delta - \delta > 0 \end{aligned}$$

and so, from the from the definition of the degree, $d(\mathbf{f},\Omega,\mathbf{0}) = d(\mathbf{h},\Omega,\mathbf{0})$.

Since $\mathbf{0}$ is a regular point of $\mathbf{h}$, $\mathbf{h}^{-1}(\mathbf{0}) = \{\mathbf{x}_i, -\mathbf{x}_i, \mathbf{0}\}_{i=1}^{m}$, and since $\mathbf{h}$ is odd, $D\mathbf{h}(-\mathbf{x}_i) = D\mathbf{h}(\mathbf{x}_i)$ and so

$$d(\mathbf{h},\Omega,\mathbf{0}) \equiv \sum_{i=1}^{m} \operatorname{sgn}\det(D\mathbf{h}(\mathbf{x}_i)) + \sum_{i=1}^{m} \operatorname{sgn}\det(D\mathbf{h}(-\mathbf{x}_i)) + \operatorname{sgn}\det(D\mathbf{h}(\mathbf{0})),$$

an odd integer.

10.4 Applications

With these theorems it is possible to give easy proofs of some very important and difficult theorems.

Definition 10.4.1 *If $\mathbf{f} : U \subseteq \mathbb{R}^n \to \mathbb{R}^n$ where U is an open set. Then $\mathbf{f}$ is locally one to one if for every $\mathbf{x} \in U$, there exists $\delta > 0$ such that $\mathbf{f}$ is one to one on $B(\mathbf{x}, \delta)$.*

As a first application, consider the invariance of domain theorem. This result says that a one to one continuous map takes open sets to open sets. It is an amazing result which is essential to understand if you wish to study manifolds. In fact, the following theorem only requires $\mathbf{f}$ to be locally one to one. First here is a lemma which has the main idea.

Lemma 10.4.2 *Let $\mathbf{g} : \overline{B(\mathbf{0},r)} \to \mathbb{R}^n$ be one to one and continuous where here $B(\mathbf{0},r)$ is the ball centered at $\mathbf{0}$ of radius r in $\mathbb{R}^n$. Then there exists $\delta > 0$ such that*

$$\mathbf{g}(\mathbf{0}) + B(\mathbf{0},\delta) \subseteq \mathbf{g}(B(\mathbf{0},r)).$$

The symbol on the left means: $\{\mathbf{g}(\mathbf{0}) + \mathbf{x} : \mathbf{x} \in B(\mathbf{0},\delta)\}$.

Proof: For $t \in [0,1]$, let

$$\mathbf{h}(\mathbf{x},t) \equiv \mathbf{g}\left(\frac{\mathbf{x}}{1+t}\right) - \mathbf{g}\left(\frac{-t\mathbf{x}}{1+t}\right)$$

Then for $\mathbf{x} \in \partial B(\mathbf{0},r)$, $\mathbf{h}(\mathbf{0},t) \neq \mathbf{0}$ because if this were so, the fact $\mathbf{g}$ is one to one implies

$$\frac{\mathbf{x}}{1+t} = \frac{-t\mathbf{x}}{1+t}$$

and this requires $\mathbf{x} = \mathbf{0}$ which is not the case. Since $\partial B(\mathbf{0},r) \times [0,t]$ is compact, there exists $\delta > 0$ such that for all $t \in [0,1]$ and $\mathbf{x} \in \partial B(\mathbf{0},r)$,

$$B(\mathbf{0},\delta) \cap \mathbf{h}(\mathbf{x},t) = \emptyset.$$

In particular, when $t = 0$, $B(\mathbf{0},\delta)$ is contained in a single component of $\mathbb{R}^n \backslash (\mathbf{g} - \mathbf{g}(\mathbf{0}))(\partial B(\mathbf{0},r))$ and so

$$d(\mathbf{g} - \mathbf{g}(\mathbf{0}), B(\mathbf{0},r), \mathbf{z})$$

is constant for $\mathbf{z} \in B(\mathbf{0},\delta)$. Therefore, from the properties of the degree in Theorem 10.2.9,

$$\begin{aligned} d(\mathbf{g} - \mathbf{g}(\mathbf{0}), B(\mathbf{0},r), \mathbf{z}) &= d(\mathbf{g} - \mathbf{g}(\mathbf{0}), B(\mathbf{0},r), \mathbf{0}) \\ &= d(\mathbf{h}(\cdot,0), B(\mathbf{0},r), \mathbf{0}) \\ &= d(\mathbf{h}(\cdot,1), B(\mathbf{0},r), \mathbf{0}) \neq 0 \end{aligned}$$

the last assertion following from Borsuk's theorem, Theorem 10.3.3 and the observation that $\mathbf{h}(\cdot,1)$ is odd. From Theorem 10.2.9 again, it follows that for all $\mathbf{z} \in B(\mathbf{0},\delta)$ there exists $\mathbf{x} \in B(\mathbf{0},r)$ such that

$$\mathbf{z} = \mathbf{g}(\mathbf{x}) - \mathbf{g}(\mathbf{0})$$

which shows $\mathbf{g}(\mathbf{0}) + B(\mathbf{0},\delta) \subseteq \mathbf{g}(B(\mathbf{0},r))$ and this proves the lemma.

Now with this lemma, it is easy to prove the very important invariance of domain theorem.

Theorem 10.4.3 *(invariance of domain)Let Ω be any open subset of $\mathbb{R}^n$ and let $\mathbf{f} : \Omega \to \mathbb{R}^n$ be continuous and locally one to one. Then $\mathbf{f}$ maps open subsets of Ω to open sets in $\mathbb{R}^n$.*

Proof: Let $\overline{B(\mathbf{x}_0,r)} \subseteq U \subseteq \Omega$ where $\mathbf{f}$ is one to one on $\overline{B(\mathbf{x}_0,r)}$ and U is an open subset of Ω. Let $\mathbf{g}$ be defined on $\overline{B(\mathbf{0},r)}$ given by

$$\mathbf{g}(\mathbf{x}) \equiv \mathbf{f}(\mathbf{x}+\mathbf{x}_0)$$

Then $\mathbf{g}$ satisfies the conditions of Lemma 10.4.2, being one to one and continuous. It follows from that lemma there exists $\delta > 0$ such that

$$\begin{aligned} \mathbf{f}(U) &\supseteq \mathbf{f}(B(\mathbf{x}_0,r)) = \mathbf{f}(\mathbf{x}_0 + B(\mathbf{0},r)) \\ &= \mathbf{g}(B(\mathbf{0},r)) \supseteq \mathbf{g}(\mathbf{0}) + B(\mathbf{0},\delta) \\ &= \mathbf{f}(\mathbf{x}_0) + B(\mathbf{0},\delta) = B(\mathbf{f}(\mathbf{x}_0),\delta) \end{aligned}$$

This shows that for any $\mathbf{x}_0 \in U, \mathbf{f}(\mathbf{x}_0)$ is an interior point of $\mathbf{f}(U)$ which shows $\mathbf{f}(U)$ is open. This proves the theorem.

Corollary 10.4.4 *If $n > m$ there does not exist a continuous one to one map from $\mathbb{R}^n$ to $\mathbb{R}^m$.*

Proof: Suppose not and let $\mathbf{f}$ be such a continuous map, $\mathbf{f}(\mathbf{x}) \equiv (f_1(\mathbf{x}),\cdots,f_m(\mathbf{x}))^T$. Then let $\mathbf{g}(\mathbf{x}) \equiv (f_1(\mathbf{x}),\cdots,f_m(\mathbf{x}),0,\cdots,0)^T$ where there are $n-m$ zeros added in. Then $\mathbf{g}$ is a one to one continuous map from $\mathbb{R}^n$ to $\mathbb{R}^n$ and so $\mathbf{g}(\mathbb{R}^n)$ would have to be open from the invariance of domain theorem and this is not the case. This proves the corollary.

Corollary 10.4.5 *If $\mathbf{f}$ is locally one to one and continuous, $\mathbf{f}:\mathbb{R}^n \to \mathbb{R}^n$, and*

$$\lim_{|\mathbf{x}|\to\infty} |\mathbf{f}(\mathbf{x})| = \infty,$$

then $\mathbf{f}$ maps $\mathbb{R}^n$ onto $\mathbb{R}^n$.

Proof: By the invariance of domain theorem, $\mathbf{f}(\mathbb{R}^n)$ is an open set. If $\mathbf{f}(\mathbf{x}_k) \to \mathbf{y}$, the growth condition ensures that $\{\mathbf{x}_k\}$ is a bounded sequence. Taking a subsequence which converges to $\mathbf{x} \in \mathbb{R}^n$ and using the continuity of $\mathbf{f}$, we see that $\mathbf{f}(\mathbf{x}) = \mathbf{y}$. Thus $\mathbf{f}(\mathbb{R}^n)$ is both open and closed which implies $\mathbf{f}$ must be an onto map since otherwise, $\mathbb{R}^n$ would not be connected.

The next theorem is the famous Brouwer fixed point theorem.

Theorem 10.4.6 *(Brouwer fixed point) Let $B = \overline{B(\mathbf{0},r)} \subseteq \mathbb{R}^n$ and let $\mathbf{f}: B \to B$ be continuous. Then there exists a point, $\mathbf{x} \in B$, such that $\mathbf{f}(\mathbf{x}) = \mathbf{x}$.*

Proof: Consider $\mathbf{h}(\mathbf{x},t) \equiv t\mathbf{f}(\mathbf{x}) - \mathbf{x}$ for $t \in [0,1]$. Then if there is no fixed point in B for $\mathbf{f}$, it follows that $\mathbf{0} \notin \mathbf{h}(\partial B, t)$ for all t. When $t = 1$, this follows from having no fixed point for $\mathbf{f}$. If $t < 1$, then if this were not so, then for some $\mathbf{x} \in \partial B$,

$$t\mathbf{f}(\mathbf{x}) = \mathbf{x}$$

and taking norms,

$$r > tr = |\mathbf{x}| = r$$

a contradiction.

Therefore, by the homotopy invariance,

$$0 = d(\mathbf{f} - \mathrm{id}, B, \mathbf{0}) = d(-\mathrm{id}, B, \mathbf{0}) = (-1)^n,$$

a contradiction. This proves the theorem.

Definition 10.4.7 $\mathbf{f}$ *is a retraction of* $\overline{B(\mathbf{0},r)}$ *onto* $\partial B(0,r)$ *if* $\mathbf{f}$ *is continuous,* $\mathbf{f}\left(\overline{B(\mathbf{0},r)}\right) \subseteq \partial B(\mathbf{0},r)$, *and* $\mathbf{f}(\mathbf{x}) = \mathbf{x}$ *for all* $\mathbf{x} \in \partial B(\mathbf{0},r)$.

Theorem 10.4.8 *There does not exist a retraction of* $\overline{B(\mathbf{0},r)}$ *onto its boundary,* $\partial B(\mathbf{0},r)$.

Proof: Suppose $\mathbf{f}$ were such a retraction. Then for all $\mathbf{x} \in \partial B(\mathbf{0},r)$, $\mathbf{f}(\mathbf{x}) = \mathbf{x}$ and so from the properties of the degree, the one which says if two functions agree on $\partial\Omega$, then they have the same degree,

$$1 = d(\mathrm{id}, B(\mathbf{0},r), \mathbf{0}) = d(\mathbf{f}, B(\mathbf{0},r), \mathbf{0})$$

which is clearly impossible because $\mathbf{f}^{-1}(\mathbf{0}) = \emptyset$ which implies $d(\mathbf{f}, B(\mathbf{0},r), \mathbf{0}) = 0$. This proves the theorem.

You should now use this theorem to give another proof of the Brouwer fixed point theorem.

The proofs of the next two theorems make use of the Tietze extension theorem, Theorem 5.7.9.

Theorem 10.4.9 *Let* Ω *be a symmetric open set in* $\mathbb{R}^n$ *such that* $\mathbf{0} \in \Omega$ *and let* $\mathbf{f} : \partial\Omega \to V$ *be continuous where* V *is an* m *dimensional subspace of* $\mathbb{R}^n, m < n$. *Then* $\mathbf{f}(-\mathbf{x}) = \mathbf{f}(\mathbf{x})$ *for some* $\mathbf{x} \in \partial\Omega$.

Proof: Suppose not. Using the Tietze extension theorem, extend $\mathbf{f}$ to all of $\overline{\Omega}$, $\mathbf{f}\left(\overline{\Omega}\right) \subseteq V$. (Here the extended function is also denoted by $\mathbf{f}$.) Let $\mathbf{g}(\mathbf{x}) = \mathbf{f}(\mathbf{x}) - \mathbf{f}(-\mathbf{x})$. Then $\mathbf{0} \notin \mathbf{g}(\partial\Omega)$ and so for some $r > 0$, $B(\mathbf{0},r) \subseteq \mathbb{R}^n \setminus \mathbf{g}(\partial\Omega)$. For $\mathbf{z} \in B(\mathbf{0},r)$,

$$d(\mathbf{g}, \Omega, \mathbf{z}) = d(\mathbf{g}, \Omega, \mathbf{0}) \neq 0$$

because $B(\mathbf{0},r)$ is contained in a component of $\mathbb{R}^n \setminus \mathbf{g}(\partial\Omega)$ and Borsuk's theorem implies that $d(\mathbf{g}, \Omega, \mathbf{0}) \neq 0$ since $\mathbf{g}$ is odd. Hence

$$V \supseteq \mathbf{g}(\Omega) \supseteq B(\mathbf{0},r)$$

and this is a contradiction because V is m dimensional. This proves the theorem.

This theorem is called the Borsuk Ulam theorem. Note that it implies there exist two points on opposite sides of the surface of the earth which have the same atmospheric pressure and temperature, assuming the earth is symmetric and that pressure and temperature are continuous functions. The next theorem is an amusing result which is like combing hair. It gives the existence of a "cowlick".

Theorem 10.4.10 *Let* n *be odd and let* Ω *be an open bounded set in* $\mathbb{R}^n$ *with* $\mathbf{0} \in \Omega$. *Suppose* $\mathbf{f} : \partial\Omega \to \mathbb{R}^n \setminus \{\mathbf{0}\}$ *is continuous. Then for some* $\mathbf{x} \in \partial\Omega$ *and* $\lambda \neq 0$, $\mathbf{f}(\mathbf{x}) = \lambda\mathbf{x}$.

Proof: Using the Tietze extension theorem, extend $\mathbf{f}$ to all of $\overline{\Omega}$. Also denote the extended function by $\mathbf{f}$. Suppose for all $\mathbf{x} \in \partial\Omega$, $\mathbf{f}(\mathbf{x}) \neq \lambda\mathbf{x}$ for all $\lambda \in \mathbb{R}$. Then

$$\mathbf{0} \notin t\mathbf{f}(\mathbf{x}) + (1-t)\mathbf{x},\ (\mathbf{x},t) \in \partial\Omega \times [0,1]$$

$$\mathbf{0} \notin t\mathbf{f}(\mathbf{x}) - (1-t)\mathbf{x},\ (\mathbf{x},t) \in \partial\Omega \times [0,1].$$

Thus there exists a homotopy of $\mathbf{f}$ and id and a homotopy of $\mathbf{f}$ and $-\mathrm{id}$. Then by the homotopy invariance of degree,

$$d(\mathbf{f}, \Omega, \mathbf{0}) = d(\mathrm{id}, \Omega, \mathbf{0}),\ d(\mathbf{f}, \Omega, \mathbf{0}) = d(-\mathrm{id}, \Omega, \mathbf{0}).$$

But this is impossible because $d(\mathrm{id}, \Omega, \mathbf{0}) = 1$ but $d(-\mathrm{id}, \Omega, \mathbf{0}) = (-1)^n$. This proves the theorem.

10.5 The Product Formula

This section is on the product formula for the degree which is used to prove the Jordan separation theorem. To begin with here is a lemma.

Lemma 10.5.1 *Let $\mathbf{y}_1, \cdots, \mathbf{y}_r$ be points not in $\mathbf{f}(\partial\Omega)$ and let $\delta > 0$. Then there exists $\widetilde{\mathbf{f}} \in C^2\left(\overline{\Omega}; \mathbb{R}^n\right)$ such that $\left|\left|\widetilde{\mathbf{f}} - \mathbf{f}\right|\right|_\infty < \delta$ and $\mathbf{y}_i$ is a regular value for $\widetilde{\mathbf{f}}$ for each i.*

Proof: Let $\mathbf{f}_0 \in C^2\left(\overline{\Omega}; \mathbb{R}^n\right)$, $||\mathbf{f}_0 - \mathbf{f}||_\infty < \frac{\delta}{2}$. Let $\widetilde{\mathbf{y}}_1$ be a regular value for $\mathbf{f}_0$ and $|\widetilde{\mathbf{y}}_1 - \mathbf{y}_1| < \frac{\delta}{3r}$. Let $\mathbf{f}_1(\mathbf{x}) \equiv \mathbf{f}_0(\mathbf{x}) + \mathbf{y}_1 - \widetilde{\mathbf{y}}_1$. Thus $\mathbf{y}_1$ is a regular value of $\mathbf{f}_1$ because $D\mathbf{f}_1(\mathbf{x}) = D\mathbf{f}_0(\mathbf{x})$ and if $\mathbf{f}_1(\mathbf{x}) = \mathbf{y}_1$, this is the same as having $\mathbf{f}_0(\mathbf{x}) = \widetilde{\mathbf{y}}_1$ where $\widetilde{\mathbf{y}}_1$ is a regular value of $\mathbf{f}_0$. Then also

$$\begin{aligned} ||\mathbf{f} - \mathbf{f}_1||_\infty &\leq ||\mathbf{f} - \mathbf{f}_0||_\infty + ||\mathbf{f}_0 - \mathbf{f}_1||_\infty \\ &= ||\mathbf{f} - \mathbf{f}_0||_\infty + |\widetilde{\mathbf{y}}_1 - \mathbf{y}_1| \\ &< \frac{\delta}{3r} + \frac{\delta}{2}. \end{aligned}$$

Suppose now there exists $\mathbf{f}_k \in C^2\left(\overline{\Omega}; \mathbb{R}^n\right)$ with each of the $\mathbf{y}_i$ for $i = 1, \cdots, k$ a regular value of $\mathbf{f}_k$ and

$$||\mathbf{f} - \mathbf{f}_k||_\infty < \frac{\delta}{2} + \frac{k}{r}\left(\frac{\delta}{3}\right).$$

Then letting S_k denote the singular values of $\mathbf{f}_k$, Sard's theorem implies there exists $\widetilde{\mathbf{y}}_{k+1}$ such that

$$|\widetilde{\mathbf{y}}_{k+1} - \mathbf{y}_{k+1}| < \frac{\delta}{3r}$$

and

$$\widetilde{\mathbf{y}}_{k+1} \notin S_k \cup \cup_{i=1}^k \left(S_k + \mathbf{y}_{k+1} - \mathbf{y}_i\right). \tag{10.12}$$

Let

$$\mathbf{f}_{k+1}(\mathbf{x}) \equiv \mathbf{f}_k(\mathbf{x}) + \mathbf{y}_{k+1} - \widetilde{\mathbf{y}}_{k+1}. \tag{10.13}$$

If $\mathbf{f}_{k+1}(\mathbf{x}) = \mathbf{y}_i$ for some $i \leq k$, then

$$\mathbf{f}_k(\mathbf{x}) + \mathbf{y}_{k+1} - \mathbf{y}_i = \widetilde{\mathbf{y}}_{k+1}$$

and so $\mathbf{f}_k(\mathbf{x})$ is a regular value for $\mathbf{f}_k$ since by 10.12, $\widetilde{\mathbf{y}}_{k+1} \notin S_k + \mathbf{y}_{k+1} - \mathbf{y}_i$ and so $\mathbf{f}_k(\mathbf{x}) \notin S_k$. Therefore, for $i \leq k$, $\mathbf{y}_i$ is a regular value of $\mathbf{f}_{k+1}$ since by 10.13, $D\mathbf{f}_{k+1} = D\mathbf{f}_k$. Now suppose $\mathbf{f}_{k+1}(\mathbf{x}) = \mathbf{y}_{k+1}$. Then

$$\mathbf{y}_{k+1} = \mathbf{f}_k(\mathbf{x}) + \mathbf{y}_{k+1} - \widetilde{\mathbf{y}}_{k+1}$$

so $\mathbf{f}_k(\mathbf{x}) = \widetilde{\mathbf{y}}_{k+1}$ implying that $\mathbf{f}_k(\mathbf{x}) = \widetilde{\mathbf{y}}_{k+1} \notin S_k$. Hence $\det D\mathbf{f}_{k+1}(\mathbf{x}) = \det D\mathbf{f}_k(\mathbf{x}) \neq 0$. Thus $\mathbf{y}_{k+1}$ is also a regular value of $\mathbf{f}_{k+1}$. Also,

$$\begin{aligned} ||\mathbf{f}_{k+1} - \mathbf{f}|| &\leq ||\mathbf{f}_{k+1} - \mathbf{f}_k|| + ||\mathbf{f}_k - \mathbf{f}|| \\ &\leq \frac{\delta}{3r} + \frac{\delta}{2} + \frac{k}{r}\left(\frac{\delta}{3}\right) = \frac{\delta}{2} + \frac{k+1}{r}\left(\frac{\delta}{3}\right). \end{aligned}$$

Let $\widetilde{\mathbf{f}} \equiv \mathbf{f}_r$. Then

$$\left|\left|\widetilde{\mathbf{f}} - \mathbf{f}\right|\right|_\infty < \frac{\delta}{2} + \left(\frac{\delta}{3}\right) < \delta$$

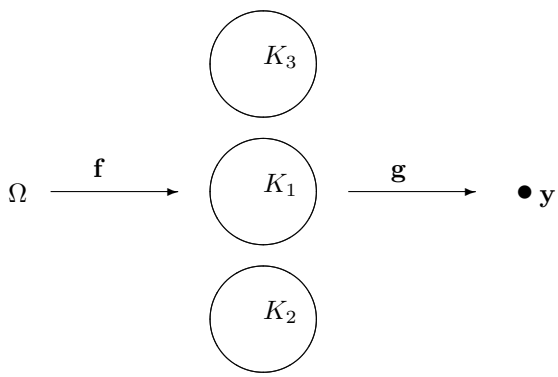
and each of the $\mathbf{y}_i$ is a regular value of $\widetilde{\mathbf{f}}$. This proves the lemma.

Definition 10.5.2 *Let the connected components of $\mathbb{R}^n \setminus \mathbf{f}(\partial\Omega)$ be denoted by K_i. From the properties of the degree listed in Theorem 10.2.9, $d(\mathbf{f},\Omega,\cdot)$ is constant on each of these components. Denote by $d(\mathbf{f},\Omega,K_i)$ the constant value on the component, K_i.*

The product formula considers the situation depicted in the following diagram in which $\mathbf{y} \notin \mathbf{g}(\mathbf{f}(\partial\Omega))$ and the K_i are the connected components of $\mathbb{R}^n \setminus \mathbf{f}(\partial\Omega)$.

$$\overline{\Omega} \xrightarrow{\mathbf{f}} \underset{\mathbb{R}^n\setminus\mathbf{f}(\partial\Omega)=\cup_i K_i}{\mathbf{f}\left(\overline{\Omega}\right)} \xrightarrow{\mathbf{g}} \underset{\mathbf{y}}{\mathbb{R}^n}$$

The following diagram may be helpful in remembering what it says.



Lemma 10.5.3 *Let $\mathbf{f} \in C\left(\overline{\Omega};\mathbb{R}^n\right), \mathbf{g} \in C^2(\mathbb{R}^n,\mathbb{R}^n)$, and $\mathbf{y} \notin \mathbf{g}(\mathbf{f}(\partial\Omega))$. Suppose also that $\mathbf{y}$ is a regular value of $\mathbf{g}$. Then the following product formula holds where K_i are the bounded components of $\mathbb{R}^n \setminus \mathbf{f}(\partial\Omega)$.*

$$d(\mathbf{g}\circ\mathbf{f},\Omega,\mathbf{y}) = \sum_{i=1}^{\infty} d(\mathbf{f},\Omega,K_i)\, d(\mathbf{g},K_i,\mathbf{y}).$$

All but finitely many terms in the sum are zero.

Proof: First note that if K_i is unbounded, $d(\mathbf{f},\Omega,K_i) = 0$ because there exists a point, $\mathbf{z} \in K_i$ such that $\mathbf{f}^{-1}(\mathbf{z}) = \emptyset$ due to the fact that $\mathbf{f}\left(\overline{\Omega}\right)$ is compact and is consequently bounded. Thus it makes no difference in the above formula whether the K_i are arbitrary components or only bounded components. Let $\left\{\mathbf{x}_j^i\right\}_{j=1}^{m_i}$ denote the points of $\mathbf{g}^{-1}(\mathbf{y})$ which are contained in K_i, the i^{th} bounded component of $\mathbb{R}^n \setminus \mathbf{f}(\partial\Omega)$. Then $m_i < \infty$ because if not, there would exist a limit point $\mathbf{x}$ for this sequence. Then $\mathbf{g}(\mathbf{x}) = \mathbf{y}$ and so $\mathbf{x} \notin \mathbf{f}(\partial\Omega)$. Thus $\det(D\mathbf{g}(\mathbf{x})) \neq 0$ and so by the inverse function theorem, $\mathbf{g}$ would be one to one on an open ball containing $\mathbf{x}$ which contradicts having $\mathbf{x}$ a limit point.

Note also that $\mathbf{g}^{-1}(\mathbf{y}) \cap \mathbf{f}\left(\overline{\Omega}\right)$ is a compact set covered by the components of $\mathbb{R}^n \setminus \mathbf{f}(\partial\Omega)$ because $\mathbf{g}^{-1}(\mathbf{y}) \cap \mathbf{f}(\partial\Omega) = \emptyset$. It follows $\mathbf{g}^{-1}(\mathbf{y}) \cap \mathbf{f}\left(\overline{\Omega}\right)$ is covered by finitely many of these components.

The only terms in the above sum which are nonzero are those corresponding to K_i having nonempty intersection with $\mathbf{g}^{-1}(\mathbf{y}) \cap \mathbf{f}\left(\overline{\Omega}\right)$. The other components contribute 0 to the above sum because if $K_i \cap \mathbf{g}^{-1}(\mathbf{y}) = \emptyset$, it follows from Theorem 10.2.9 that $d(\mathbf{g},K_i,\mathbf{y}) = 0$. If K_i does not intersect $\mathbf{f}\left(\overline{\Omega}\right)$, then $d(\mathbf{f},\Omega,K_i) = 0$. Therefore, the above sum is actually a finite sum since $\mathbf{g}^{-1}(\mathbf{y}) \cap \mathbf{f}\left(\overline{\Omega}\right)$, being a compact set, is covered by finitely many of the K_i. Thus there are no convergence problems. Now let $\varepsilon > 0$ be small enough that

$$B(\mathbf{y},3\varepsilon) \cap \mathbf{g}(\mathbf{f}(\partial\Omega)) = \emptyset,$$

and for each $\mathbf{x}_j^i \in \mathbf{g}^{-1}(\mathbf{y})$

$$B\left(\mathbf{x}_j^i, 3\varepsilon\right) \cap \mathbf{f}\left(\partial\Omega\right) = \emptyset.$$

By uniform continuity of $\mathbf{g}$ on the compact set $\mathbf{f}\left(\overline{\Omega}\right)$, there exists $\delta > 0$, $\delta < \varepsilon$ such that if $\mathbf{z}_1$ and $\mathbf{z}_2$ are any two points of $\mathbf{f}\left(\overline{\Omega}\right)$ with $|\mathbf{z}_1 - \mathbf{z}_2| < \delta$, it follows that $|\mathbf{g}(\mathbf{z}_1) - \mathbf{g}(\mathbf{z}_2)| < \varepsilon$.

Now using Lemma 10.5.1, choose $\widetilde{\mathbf{f}} \in C^2\left(\overline{\Omega}; \mathbb{R}^n\right)$ such that $\left\|\widetilde{\mathbf{f}} - \mathbf{f}\right\|_\infty < \delta$ and each point, $\mathbf{x}_j^i$ is a regular value of $\widetilde{\mathbf{f}}$. From the properties of the degree in Theorem 10.2.9

$$d\left(\mathbf{f}, \Omega, K_i\right) = d\left(\mathbf{f}, \Omega, \mathbf{x}_j^i\right)$$

for each $j = 1, \cdots, m_i$. For $\mathbf{x} \in \partial\Omega$, and $t \in [0,1]$,

$$\begin{aligned}\left|\mathbf{f}(\mathbf{x}) + t\left(\widetilde{\mathbf{f}}(\mathbf{x}) - \mathbf{f}(\mathbf{x})\right) - \mathbf{x}_j^i\right| &\geq \left|\mathbf{f}(\mathbf{x}) - \mathbf{x}_j^i\right| - t\left|\widetilde{\mathbf{f}}(\mathbf{x}) - \mathbf{f}(\mathbf{x})\right| \\ &> 3\varepsilon - t\varepsilon > 0\end{aligned}$$

and so by the homotopy invariance of the degree,

$$d\left(\widetilde{\mathbf{f}}, \Omega, \mathbf{x}_j^i\right) = d\left(\mathbf{f}, \Omega, \mathbf{x}_j^i\right) \equiv d\left(\mathbf{f}, \Omega, K_i\right) \tag{10.14}$$

independent of j. Also for $\mathbf{x} \in \partial\Omega$, and $t \in [0,1]$,

$$\left|\mathbf{g}(\mathbf{f}(\mathbf{x})) + t\left(\mathbf{g}\left(\widetilde{\mathbf{f}}(\mathbf{x})\right) - \mathbf{g}(\mathbf{f}(\mathbf{x}))\right) - \mathbf{y}\right| \geq 3\varepsilon - t\varepsilon > 0$$

and so by the homotopy invariance of the degree,

$$d\left(\mathbf{g} \circ \mathbf{f}, \Omega, \mathbf{y}\right) = d\left(\mathbf{g} \circ \widetilde{\mathbf{f}}, \Omega, \mathbf{y}\right). \tag{10.15}$$

Now $\widetilde{\mathbf{f}}^{-1}\left(\mathbf{x}_j^i\right)$ is a finite set because $\widetilde{\mathbf{f}}^{-1}\left(\mathbf{x}_j^i\right) \subseteq \Omega$, a bounded open set and $\mathbf{x}_j^i$ is a regular value. It follows from 10.15

$$d\left(\mathbf{g} \circ \mathbf{f}, \Omega, \mathbf{y}\right) = d\left(\mathbf{g} \circ \widetilde{\mathbf{f}}, \Omega, \mathbf{y}\right)$$

$$= \sum_{i=1}^{\infty} \sum_{j=1}^{m_i} \sum_{\mathbf{z} \in \widetilde{\mathbf{f}}^{-1}\left(\mathbf{x}_j^i\right)} \operatorname{sgn}\det D\mathbf{g}\left(\overbrace{\widetilde{\mathbf{f}}(\mathbf{z})}^{x_j^i}\right) \operatorname{sgn}\det D\widetilde{\mathbf{f}}(\mathbf{z})$$

$$= \sum_{i=1}^{\infty} \sum_{j=1}^{m_i} \operatorname{sgn}\det D\mathbf{g}\left(\mathbf{x}_j^i\right) d\left(\widetilde{\mathbf{f}}, \Omega, \mathbf{x}_j^i\right) = \sum_{i=1}^{\infty} d\left(\mathbf{g}, K_i, \mathbf{y}\right) d\left(\widetilde{\mathbf{f}}, \Omega, \mathbf{x}_j^i\right)$$

$$= \sum_{i=1}^{\infty} d\left(\mathbf{g}, K_i, \mathbf{y}\right) d\left(\mathbf{f}, \Omega, K_i\right).$$

This proves the lemma.

With this lemma, the following is the product formula.

Theorem 10.5.4 *(product formula) Let $\{K_i\}_{i=1}^{\infty}$ be the bounded components of $\mathbb{R}^n \setminus \mathbf{f}(\partial\Omega)$ for $\mathbf{f} \in C\left(\overline{\Omega}; \mathbb{R}^n\right)$, let $\mathbf{g} \in C\left(\mathbb{R}^n, \mathbb{R}^n\right)$, and suppose that $\mathbf{y} \notin \mathbf{g}(\mathbf{f}(\partial\Omega))$. Then*

$$d\left(\mathbf{g} \circ \mathbf{f}, \Omega, \mathbf{y}\right) = \sum_{i=1}^{\infty} d\left(\mathbf{g}, K_i, \mathbf{y}\right) d\left(\mathbf{f}, \Omega, K_i\right). \tag{10.16}$$

All but finitely many terms in the sum are zero.

Proof: Let $B(\mathbf{y},3\delta)\cap\mathbf{g}(\mathbf{f}(\partial\Omega))=\emptyset$ and let $\widetilde{\mathbf{g}}\in C^2(\mathbb{R}^n,\mathbb{R}^n)$ be such that

$$\sup\left\{|\widetilde{\mathbf{g}}(\mathbf{z})-\mathbf{g}(\mathbf{z})|:\mathbf{z}\in\mathbf{f}\left(\overline{\Omega}\right)\right\}<\delta$$

And also $\mathbf{y}$ is a regular value of $\widetilde{\mathbf{g}}$. Then from the above inequality, if $\mathbf{x}\in\partial\Omega$ and $t\in[0,1]$,

$$\begin{aligned}|\mathbf{g}(\mathbf{f}(\mathbf{x}))+t(\widetilde{\mathbf{g}}(\mathbf{f}(\mathbf{x}))-\mathbf{g}(\mathbf{f}(\mathbf{x})))-\mathbf{y}| &\geq |\mathbf{g}(\mathbf{f}(\mathbf{x}))-\mathbf{y}|-t|\widetilde{\mathbf{g}}(\mathbf{f}(\mathbf{x}))-\mathbf{g}(\mathbf{f}(\mathbf{x}))|\\ &\geq 3\delta-t\delta>0.\end{aligned}$$

It follows that

$$d(\mathbf{g}\circ\mathbf{f},\Omega,\mathbf{y})=d(\widetilde{\mathbf{g}}\circ\mathbf{f},\Omega,\mathbf{y}). \tag{10.17}$$

Now also, $\partial K_i\subseteq\mathbf{f}(\partial\Omega)$ and so if $\mathbf{z}\in\partial K_i$, then $\mathbf{g}(\mathbf{z})\in\mathbf{g}(\mathbf{f}(\partial\Omega))$. Consequently, for such $\mathbf{z}$,

$$|\mathbf{g}(\mathbf{z})+t(\widetilde{\mathbf{g}}(\mathbf{z})-\mathbf{g}(\mathbf{z}))-\mathbf{y}|\geq|\mathbf{g}(\mathbf{z})-\mathbf{y}|-t\delta>3\delta-t\delta>0$$

which shows that, by homotopy invariance,

$$d(\mathbf{g},K_i,\mathbf{y})=d(\widetilde{\mathbf{g}},K_i,\mathbf{y}). \tag{10.18}$$

Therefore, by Lemma 10.5.3,

$$\begin{aligned}d(\mathbf{g}\circ\mathbf{f},\Omega,\mathbf{y})=d(\widetilde{\mathbf{g}}\circ\mathbf{f},\Omega,\mathbf{y}) &= \sum_{i=1}^{\infty}d(\widetilde{\mathbf{g}},K_i,\mathbf{y})\,d(\mathbf{f},\Omega,K_i)\\ &= \sum_{i=1}^{\infty}d(\mathbf{g},K_i,\mathbf{y})\,d(\mathbf{f},\Omega,K_i)\end{aligned}$$

and the sum has only finitely many non zero terms. This proves the product formula.

Note there are no convergence problems because these sums are actually finite sums because, as in the previous lemma, $\mathbf{g}^{-1}(\mathbf{y})\cap\mathbf{f}\left(\overline{\Omega}\right)$ is a compact set covered by the components of $\mathbb{R}^n\setminus\mathbf{f}(\partial\Omega)$ and so it is covered by finitely many of these components. For the other components, $d(\mathbf{f},\Omega,K_i)=0$ or else $d(\mathbf{g},K_i,\mathbf{y})=0$.

The following theorem is the Jordan separation theorem, a major result. A homeomorphism is a function which is one to one onto and continuous having continuous inverse. Before the theorem, here is a helpful lemma.

Lemma 10.5.5 *Let Ω be a bounded open set in $\mathbb{R}^n$, $\mathbf{f}\in C\left(\overline{\Omega};\mathbb{R}^n\right)$, and suppose $\{\Omega_i\}_{i=1}^{\infty}$ are disjoint open sets contained in Ω such that*

$$\mathbf{y}\notin\mathbf{f}\left(\cup_{j=n}^{\infty}\Omega_j\right)$$

and

$$\mathbf{y}\notin\mathbf{f}\left(\overline{\Omega}\setminus\cup_{j=1}^{\infty}\Omega_j\right)$$

Then

$$d(\mathbf{f},\Omega,\mathbf{y})=\sum_{j=1}^{\infty}d(\mathbf{f},\Omega_j,\mathbf{y})$$

where the sum has all but finitely many terms equal to 0.

Proof: By Theorem 10.2.9 and letting $O=\cup_{j=n}^{\infty}\Omega_j$,

$$d(\mathbf{f},\Omega,\mathbf{y})=\sum_{j=1}^{n-1}d(\mathbf{f},\Omega_j,\mathbf{y})+d(\mathbf{f},O,\mathbf{y})=\sum_{j=1}^{\infty}d(\mathbf{f},\Omega_j,\mathbf{y})$$

because $d(\mathbf{f},O,\mathbf{y})=0$ as is $\sum_{j=n}^{\infty}d(\mathbf{f},\Omega_j,\mathbf{y})$. This proves the lemma.

Theorem 10.5.6 *(Jordan separation theorem) Let* $\mathbf{f}$ *be a homeomorphism of* C *and* $\mathbf{f}(C)$ *where* C *is a compact set in* $\mathbb{R}^n$. *Then* $\mathbb{R}^n \setminus C$ *and* $\mathbb{R}^n \setminus \mathbf{f}(C)$ *have the same number of connected components.*

Proof: Denote by $\mathcal{K}$ the bounded components of $\mathbb{R}^n \setminus C$ and denote by $\mathcal{L}$, the bounded components of $\mathbb{R}^n \setminus \mathbf{f}(C)$. Also using the Tietze extension theorem applied to the components there exists $\overline{\mathbf{f}}$ an extension of $\mathbf{f}$ to all of $\mathbb{R}^n$ which maps into a bounded set and let $\overline{\mathbf{f}^{-1}}$ be an extension of $\mathbf{f}^{-1}$ to all of $\mathbb{R}^n$ which also maps into a bounded set. Pick $K \in \mathcal{K}$ and take $\mathbf{y} \in K$. Let $\mathcal{H}$ denote the set of bounded components of $\mathbb{R}^n \setminus \mathbf{f}(\partial K)$ (note $\partial K \subseteq C$). Since $\overline{\mathbf{f}^{-1}} \circ \overline{\mathbf{f}}$ equals the identity, id, on ∂K, it follows from the properties of the degree that

$$1 = d(\mathrm{id}, K, \mathbf{y}) = d\left(\overline{\mathbf{f}^{-1}} \circ \overline{\mathbf{f}}, K, \mathbf{y}\right).$$

By the product formula,

$$1 = d\left(\overline{\mathbf{f}^{-1}} \circ \overline{\mathbf{f}}, K, \mathbf{y}\right) = \sum_{H \in \mathcal{H}} d\left(\overline{\mathbf{f}}, K, H\right) d\left(\overline{\mathbf{f}^{-1}}, H, \mathbf{y}\right), \tag{10.19}$$

the sum being a finite sum. It might help to consult the following diagram.

$$\begin{array}{ccc} \mathbb{R}^n \setminus C & \overset{\overline{\mathbf{f}}}{\underset{\overline{\mathbf{f}^{-1}}}{\rightleftarrows}} & \mathbb{R}^n \setminus \mathbf{f}(C) \\ \mathcal{K} & & \mathcal{L} \\ K & & \mathbb{R}^n \setminus \mathbf{f}(K) \\ \mathbf{y} \in K & & \mathcal{H}, \mathcal{H}_1 \\ & & H \\ & & \mathcal{G}_H \end{array}$$

Now letting $\mathbf{x} \in L \in \mathcal{L}$, if S is a connected set containing $\mathbf{x}$ and contained in $\mathbb{R}^n \setminus \mathbf{f}(C)$, then it follows S is contained in $\mathbb{R}^n \setminus \mathbf{f}(\partial K)$ because $\partial K \subseteq C$. Therefore, every set of $\mathcal{L}$ is contained in some set of $\mathcal{H}$. Letting $\mathcal{G}_H$ denote those sets of $\mathcal{L}$ which are contained in H, the following claim is needed.

Claim:

$$\overline{H} \setminus \cup \mathcal{G}_H \subseteq \mathbf{f}(C).$$

Proof of the claim: This is because if $\mathbf{z} \notin \cup \mathcal{G}_H$, then $\mathbf{z}$ cannot be contained in any set of $\mathcal{L}$ which has nonempty intersection with H since then, that whole set of $\mathcal{L}$ would be contained in H due to the fact that the sets of $\mathcal{H}$ are disjoint open sets and the sets of $\mathcal{L}$ are connected. Since the sets of $\mathcal{L}$ are open, $\mathbf{z}$ cannot be in any set of $\mathcal{L}$ which has empty intersection with H and still be in $\overline{H}$. It follows that $\mathbf{z}$ is not in any set of $\mathcal{L}$ and so it is in $\mathbf{f}(C)$. This proves the claim.

Claim: $\mathbf{y} \notin \overline{\mathbf{f}^{-1}}\left(\overline{H} \setminus \cup \mathcal{G}_H\right)$. Recall $\mathbf{y} \in K \in \mathcal{K}$ the bounded components of $\mathbb{R}^n \setminus C$.

Proof of the claim: If not, then $\overline{\mathbf{f}^{-1}}(\mathbf{z}) = \mathbf{y}$ where $\mathbf{z} \in \overline{H} \setminus \cup \mathcal{G}_H \subseteq \mathbf{f}(C)$ and so $\mathbf{f}^{-1}(\mathbf{z}) = \mathbf{y} \in C$. But $\mathbf{y} \notin C$ and this contradiction proves the claim.

Now every set of $\mathcal{L}$ is contained in some set of $\mathcal{H}$. What about those sets of $\mathcal{H}$ which contain no set of $\mathcal{L}$? Let H be one of these sets. Thus $\mathcal{G}_H = \emptyset$. From the claim, $\mathbf{y} \notin \overline{\mathbf{f}^{-1}}\left(\overline{H} \setminus \cup \mathcal{G}_H\right) = \overline{\mathbf{f}^{-1}}\left(\overline{H}\right)$ and so $d\left(\overline{\mathbf{f}^{-1}}, H, \mathbf{y}\right) = 0$. Therefore, letting $\mathcal{H}_1$ denote those sets of $\mathcal{H}$ which contain some set of $\mathcal{L}$, 10.19 is of the form

$$1 = \sum_{H \in \mathcal{H}_1} d\left(\overline{\mathbf{f}}, K, H\right) d\left(\overline{\mathbf{f}^{-1}}, H, \mathbf{y}\right).$$

I want to expand $d\left(\overline{\mathbf{f}^{-1}}, H, \mathbf{y}\right)$ as a sum of the form

$$\sum_{L\in\mathcal{G}_H} d\left(\overline{\mathbf{f}^{-1}}, L, \mathbf{y}\right)$$

To do this, consider whether $\mathbf{y} \in \overline{\mathbf{f}^{-1}}(L)$. If $\left(\overline{\mathbf{f}^{-1}}\right)^{-1}(\mathbf{y}) \in \mathbf{f}(C)$, then doing $\mathbf{f}^{-1}$ to both sides, it follows $\mathbf{y} \in C$ which is not the case. Therefore, the points of $\left(\overline{\mathbf{f}^{-1}}\right)^{-1}(\mathbf{y}) \cap \overline{H}$ are none of them in $\mathbf{f}(C)$. Thus, this compact set is covered by finitely many sets of $\mathcal{L}$. Consequently $\mathbf{y} \in \overline{\mathbf{f}^{-1}}(L)$ for only finitely many $L \in \mathcal{G}_H$. Also it was shown above that $\mathbf{y} \notin \overline{\mathbf{f}^{-1}}\left(\overline{H} \setminus \cup\mathcal{G}_H\right)$. By Lemma 10.5.5, I can write the above sum in place of $d\left(\overline{\mathbf{f}^{-1}}, H, \mathbf{y}\right)$. Therefore,

$$1 = \sum_{H\in\mathcal{H}_1} d\left(\overline{\mathbf{f}}, K, H\right) d\left(\overline{\mathbf{f}^{-1}}, H, \mathbf{y}\right) = \sum_{H\in\mathcal{H}_1} d\left(\overline{\mathbf{f}}, K, H\right) \sum_{L\in\mathcal{G}_H} d\left(\overline{\mathbf{f}^{-1}}, L, \mathbf{y}\right)$$

where that sum at the right end has only finitely many nonzero terms and the first sum also has only finitely many nonzero terms because it comes from the product formula. Now the above equals

$$= \sum_{H\in\mathcal{H}_1} \sum_{L\in\mathcal{G}_H} d\left(\overline{\mathbf{f}}, K, H\right) d\left(\overline{\mathbf{f}^{-1}}, L, \mathbf{y}\right)$$

Now by definition,

$$d\left(\overline{\mathbf{f}}, K, H\right) = d\left(\overline{\mathbf{f}}, K, \mathbf{x}\right)$$

where $\mathbf{x}$ is any point of H. In particular $d\left(\overline{\mathbf{f}}, K, H\right) = d\left(\overline{\mathbf{f}}, K, L\right)$ for any $L \in \mathcal{G}_H$. Therefore, the above reduces to

$$= \sum_{L\in\mathcal{L}} d\left(\overline{\mathbf{f}}, K, L\right) d\left(\overline{\mathbf{f}^{-1}}, L, \mathbf{y}\right)$$

$$= \sum_{L\in\mathcal{L}} d\left(\overline{\mathbf{f}}, K, L\right) d\left(\overline{\mathbf{f}^{-1}}, L, K\right)$$

and all but finitely many terms in the sum are 0. By the same argument,

$$1 = \sum_{K\in\mathcal{K}} d\left(\overline{\mathbf{f}}, K, L\right) d\left(\overline{\mathbf{f}^{-1}}, L, K\right)$$

and so letting $|\mathcal{K}|$ denote the number of elements in $\mathcal{K}$, similar for $\mathcal{L}$,

$$|\mathcal{K}| = \sum_{K\in\mathcal{K}} 1 = \sum_{K\in\mathcal{K}} \left(\sum_{L\in\mathcal{L}} d\left(\overline{\mathbf{f}}, K, L\right) d\left(\overline{\mathbf{f}^{-1}}, L, K\right) \right)$$

$$|\mathcal{L}| = \sum_{L\in\mathcal{L}} 1 = \sum_{L\in\mathcal{L}} \left(\sum_{K\in\mathcal{K}} d\left(\overline{\mathbf{f}}, K, L\right) d\left(\overline{\mathbf{f}^{-1}}, L, K\right) \right)$$

It follows that if either $|\mathcal{K}|$ or $|\mathcal{L}|$ is finite, then they are equal. Thus if one is infinite, so is the other. This proves the theorem because if $n > 1$ there is exactly one unbounded component to both $\mathbb{R}^n \setminus C$ and $\mathbb{R}^n \setminus \mathbf{f}(C)$ and if $n = 1$ there are exactly two unbounded components.

As an application, here is a very interesting little result. It has to do with $d(\mathbf{f}, \Omega, \mathbf{f}(\mathbf{x}))$ in the case where $\mathbf{f}$ is one to one and Ω is connected. You might imagine this should equal 1 or -1 based on one dimensional analogies. In fact this is the case and it is a nice application of the Jordan separation theorem and the product formula.

Proposition 10.5.7 *Let Ω be an open connected bounded set in $\mathbb{R}^n, n \geq 2$ such that $\mathbb{R}^n \setminus \partial\Omega$ consists of two connected components. Let $\mathbf{f} \in C\left(\overline{\Omega}; \mathbb{R}^n\right)$ be one to one. Then $\mathbf{f}(\Omega)$ is the bounded component of $\mathbb{R}^n \setminus \mathbf{f}(\partial\Omega)$ and for $\mathbf{y} \in \mathbf{f}(\Omega)$, $d(\mathbf{f}, \Omega, \mathbf{y})$ either equals 1 or -1.*

Proof: By the Jordan separation theorem, $\mathbb{R}^n \setminus \mathbf{f}(\partial\Omega)$ consists of two components, a bounded component B and an unbounded comonent U. Using the Tietze extention theorem, there exists $\mathbf{g}$ defined on $\mathbb{R}^n$ such that $\mathbf{g} = \mathbf{f}^{-1}$ on $\mathbf{f}\left(\overline{\Omega}\right)$. Thus on $\partial\Omega, \mathbf{g} \circ \mathbf{f} = \mathrm{id}$. It follows from this and the product formula that

$$1 = d(\mathrm{id}, \Omega, \mathbf{g}(\mathbf{y})) = d(\mathbf{g} \circ \mathbf{f}, \Omega, \mathbf{g}(\mathbf{y})) = d(\mathbf{g}, B, \mathbf{g}(\mathbf{y}))\, d(\mathbf{f}, \Omega, B)$$

Therefore, $d(\mathbf{f}, \Omega, B) \neq 0$ and so for every $\mathbf{z} \in B$, it follows $\mathbf{z} \in \mathbf{f}(\Omega)$. Thus $B \subseteq \mathbf{f}(\Omega)$. On the other hand, $\mathbf{f}(\Omega)$ cannot have points in both U and B because it is a connected set. Therefore $\mathbf{f}(\Omega) \subseteq B$ and this shows $B = \mathbf{f}(\Omega)$. Thus $d(\mathbf{f}, \Omega, B) = d(\mathbf{f}, \Omega, \mathbf{y})$ and the above formula shows this equals either 1 or -1 because the degree is an integer. This proves the proposition.

10.6 Exercises

1. Show the Brouwer fixed point theorem is equivalent to the nonexistence of a continuous retraction onto the boundary of $B(\mathbf{0}, r)$.

2. Using the Jordan separation theorem, prove the invariance of domain theorem. **Hint:** You might consider $B(\mathbf{x}, r)$ and show $\mathbf{f}$ maps the inside to one of two components of $\mathbb{R}^n \setminus \mathbf{f}(\partial B(\mathbf{x}, r))$. Thus an open ball goes to some open set.

3. Give a version of Proposition 10.5.7 which is valid for the case where $n = 1$.

4. Suppose $n > m$. Does there exists a continuous one to one map, $\mathbf{f}$ which maps $\mathbb{R}^m$ onto $\mathbb{R}^n$? This is a very interesting question because there do exist continuous maps of $[0,1]$ which cover a square for example. **Hint:** First show that if K is compact and $\mathbf{f} : K \to \mathbf{f}(K)$ is one to one, then $\mathbf{f}^{-1}$ must be continuous. Now consider the increasing sequence of compact sets $\left\{\overline{B(\mathbf{0}, k)}\right\}_{k=1}^{\infty}$ whose union is $\mathbb{R}^m$ and the increasing sequence $\left\{\mathbf{f}\left(\overline{B(\mathbf{0}, k)}\right)\right\}_{k=1}^{\infty}$ which you might assume covers $\mathbb{R}^n$. You might use this to argue that if such a function exists, then $\mathbf{f}^{-1}$ must be continuous and then apply Corollary 10.4.4.

5. Can there exists a one to one onto continuous map, $\mathbf{f}$ which takes the unit interval to the unit disk? **Hint:** Think in terms of invariance of domain and use the hint to Problem 4.

6. Consider the unit disk,
$$\left\{(x, y) : x^2 + y^2 \leq 1\right\} \equiv D$$
and the annulus
$$\left\{(x, y) : \frac{1}{2} \leq x^2 + y^2 \leq 1\right\} \equiv A$$
Is it possible there exists a one to one onto continuous map $\mathbf{f}$ such that $\mathbf{f}(D) = A$? Thus D has no holes and A is really like D but with one hole punched out. Can you generalize to different numbers of holes? **Hint:** Consider the invariance of domain theorem. The interior of D would need to be mapped to the interior of A. Where do the points of the boundary of A come from? Consider Theorem 5.3.5.

7. Suppose C is a compact set in $\mathbb{R}^n$ which has empty interior and $\mathbf{f} : C \to \Gamma \subseteq \mathbb{R}^n$ is one to one onto and continuous with continuous inverse. Could Γ have nonempty interior? Show also that if $\mathbf{f}$ is one to one and onto Γ then if it is continuous, so is $\mathbf{f}^{-1}$.

8. Let C denote the unit circle, $\left\{(x,y) \in \mathbb{R}^2 : x^2 + y^2 = 1\right\}$. Suppose $\mathbf{f} : C \to \Gamma \subseteq \mathbb{R}^2$ is one to one and onto having continuous inverse. The Jordan curve theorem says that under these conditions $\mathbb{R}^2 \setminus \Gamma$ consists of two components, a bounded component (called the inside) and an unbounded component (called the outside). Prove the Jordan curve theorem using the Jordan separation theorem. Also show that the boundary of each of these two components of $\mathbb{R}^2 \setminus \Gamma$ is Γ. **Hint:** Let U_1 and U_2 be the two components of $\mathbb{R}^2 \setminus \Gamma$. Explain why none of the limit points of U_1 are are in U_2 so they must all be in Γ. Similarly no limit point of U_2 can be in U_1. Use Problem 7 to show Γ has empty interior. Use this observation to argue $\overline{U_1} \cup \overline{U_2} = \mathbb{R}^2$. Suppose $\mathbf{p} \in \partial U_1 \setminus \partial U_2$. Then explain why, since $\mathbf{p} \notin \partial U_2$, it is not a limit point of U_2 and so there is a ball $B(\mathbf{p}, r)$ which contains no points of $\overline{U_2}$. Now where is $B(\mathbf{p}, r)$? Doesn't this contradict $\mathbf{p} \in \partial U_1$? Similarly $\partial U_2 \setminus \partial U_1 = \emptyset$ and so $\partial U_1 = \partial U_2$. Now justify the statement

$$\Gamma \subseteq \partial U_1 \cup \partial U_2 = \partial U_1 = \partial U_2 \subseteq \Gamma.$$

Thus $\partial U_i = \Gamma$.

9. Let K be a nonempty closed and convex subset of $\mathbb{R}^n$. Recall K is convex means that if $\mathbf{x}, \mathbf{y} \in K$, then for all $t \in [0,1]$, $t\mathbf{x} + (1-t)\mathbf{y} \in K$. Show that if $\mathbf{x} \in \mathbb{R}^n$ there exists a unique $\mathbf{z} \in K$ such that

$$|\mathbf{x} - \mathbf{z}| = \min\{|\mathbf{x} - \mathbf{y}| : \mathbf{y} \in K\}.$$

This $\mathbf{z}$ will be denoted as $P\mathbf{x}$. **Hint:** First note you do not know K is compact. Establish the parallelogram identity if you have not already done so,

$$|\mathbf{u} - \mathbf{v}|^2 + |\mathbf{u} + \mathbf{v}|^2 = 2|\mathbf{u}|^2 + 2|\mathbf{v}|^2.$$

Then let $\{\mathbf{z}_k\}$ be a minimizing sequence,

$$\lim_{k \to \infty} |\mathbf{z}_k - \mathbf{x}|^2 = \inf\{|\mathbf{x} - \mathbf{y}| : \mathbf{y} \in K\} \equiv \lambda.$$

Now using convexity, explain why

$$\left|\frac{\mathbf{z}_k - \mathbf{z}_m}{2}\right|^2 + \left|\mathbf{x} - \frac{\mathbf{z}_k + \mathbf{z}_m}{2}\right|^2 = 2\left|\frac{\mathbf{x} - \mathbf{z}_k}{2}\right|^2 + 2\left|\frac{\mathbf{x} - \mathbf{z}_m}{2}\right|^2$$

and then use this to argue $\{\mathbf{z}_k\}$ is a Cauchy sequence. Then if $\mathbf{z}_i$ works for $i = 1, 2$, consider $(\mathbf{z}_1 + \mathbf{z}_2)/2$ to get a contradiction.

10. In Problem 9 show that $P\mathbf{x}$ satisfies the following variational inequality.

$$(\mathbf{x} - P\mathbf{x}) \cdot (\mathbf{y} - P\mathbf{x}) \leq 0$$

for all $\mathbf{y} \in K$. Then show that $|P\mathbf{x}_1 - P\mathbf{x}_2| \leq |\mathbf{x}_1 - \mathbf{x}_2|$. **Hint:** For the first part note that if $\mathbf{y} \in K$, the function $t \to |\mathbf{x} - (P\mathbf{x} + t(\mathbf{y} - P\mathbf{x}))|^2$ achieves its minimum on $[0,1]$ at $t = 0$. For the second part,

$$(\mathbf{x}_1 - P\mathbf{x}_1) \cdot (P\mathbf{x}_2 - P\mathbf{x}_1) \leq 0, \ (\mathbf{x}_2 - P\mathbf{x}_2) \cdot (P\mathbf{x}_1 - P\mathbf{x}_2) \leq 0.$$

Explain why

$$(\mathbf{x}_2 - P\mathbf{x}_2 - (\mathbf{x}_1 - P\mathbf{x}_1)) \cdot (P\mathbf{x}_2 - P\mathbf{x}_1) \geq 0$$

and then use a some manipulations and the Cauchy Schwarz inequality to get the desired inequality.

11. Establish the Brouwer fixed point theorem for any convex compact set in $\mathbb{R}^n$. **Hint:** If K is a compact and convex set, let R be large enough that the closed ball, $D(\mathbf{0}, R) \supseteq K$. Let P be the projection onto K as in Problem 10 above. If $\mathbf{f}$ is a continuous map from K to K, consider $\mathbf{f} \circ P$. You want to show $\mathbf{f}$ has a fixed point in K.

12. Suppose D is a set which is homeomorphic to $\overline{B(\mathbf{0}, 1)}$. This means there exists a continuous one to one map, $\mathbf{h}$ such that $\mathbf{h}\left(\overline{B(\mathbf{0}, 1)}\right) = D$ such that $\mathbf{h}^{-1}$ is also one to one. Show that if $\mathbf{f}$ is a continuous function which maps D to D then $\mathbf{f}$ has a fixed point. Now show that it suffices to say that $\mathbf{h}$ is one to one and continuous. In this case the continuity of $\mathbf{h}^{-1}$ is automatic. Sets which have the property that continuous functions taking the set to itself have at least one fixed point are said to have the fixed point property. Work Problem 6 using this notion of fixed point property. What about a solid ball and a donut?

13. There are many different proofs of the Brouwer fixed point theorem. Let l be a line segment. Label one end with A and the other end B. Now partition the segment into n little pieces and label each of these partition points with either A or B. Show there is an odd number of little segments with one end labeled with A and the other labeled with B. If $\mathbf{f} : l \to l$ is continuous, use the fact it is uniformly continuous and this little labeling result to give a proof for the Brouwer fixed point theorem for a one dimensional segment. Next consider a triangle. Label the vertices with A, B, C and subdivide this triangle into little triangles, $T_1, \cdots, T_m$ in such a way that any pair of these little triangles intersects either along an entire edge or a vertex. Now label the unlabeled vertices of these little triangles with either A, B, or C in any way. Show there is an odd number of little triangles having their vertices labeled as A, B, C. Use this to show the Brouwer fixed point theorem for any triangle. This approach generalizes to higher dimensions and you will see how this would take place if you are successful in going this far. This is an outline of the Sperner's lemma approach to the Brouwer fixed point theorem. Are there other sets besides compact convex sets which have the fixed point property?

14. Using the definition of the derivative and the Vitali covering theorem, show that if $\mathbf{f} \in C^1\left(\overline{U}, \mathbb{R}^n\right)$ and ∂U has n dimensional measure zero then $\mathbf{f}(\partial U)$ also has measure zero. (This problem has little to do with this chapter. It is a review.)

15. Suppose Ω is any open bounded subset of $\mathbb{R}^n$ which contains $\mathbf{0}$ and that $\mathbf{f} : \overline{\Omega} \to \mathbb{R}^n$ is continuous with the property that

$$\mathbf{f}(\mathbf{x}) \cdot \mathbf{x} \geq 0$$

for all $\mathbf{x} \in \partial\Omega$. Show that then there exists $\mathbf{x} \in \Omega$ such that $\mathbf{f}(\mathbf{x}) = \mathbf{0}$. Give a similar result in the case where the above inequality is replaced with $\leq$. **Hint:** You might consider the function

$$\mathbf{h}(t, \mathbf{x}) \equiv t\mathbf{f}(\mathbf{x}) + (1 - t)\mathbf{x}.$$

16. Suppose Ω is an open set in $\mathbb{R}^n$ containing $\mathbf{0}$ and suppose that $\mathbf{f} : \overline{\Omega} \to \mathbb{R}^n$ is continuous and $|\mathbf{f}(\mathbf{x})| \leq |\mathbf{x}|$ for all $\mathbf{x} \in \partial\Omega$. Show $\mathbf{f}$ has a fixed point in $\overline{\Omega}$. **Hint:** Consider $\mathbf{h}(t, \mathbf{x}) \equiv t(\mathbf{x} - \mathbf{f}(\mathbf{x})) + (1 - t)\mathbf{x}$ for $t \in [0, 1]$. If $t = 1$ and some $\mathbf{x} \in \partial\Omega$ is sent to $\mathbf{0}$, then you are done. Suppose therefore, that no fixed point exists on $\partial\Omega$. Consider $t < 1$ and use the given inequality.

17. Let Ω be an open bounded subset of $\mathbb{R}^n$ and let $\mathbf{f}, \mathbf{g} : \overline{\Omega} \to \mathbb{R}^n$ both be continuous such that

$$|\mathbf{f}(\mathbf{x})| - |\mathbf{g}(\mathbf{x})| > 0$$

for all $\mathbf{x} \in \partial\Omega$. Show that then

$$d(\mathbf{f}-\mathbf{g},\Omega,\mathbf{0}) = d(\mathbf{f},\Omega,\mathbf{0})$$

Show that if there exists $\mathbf{x} \in \mathbf{f}^{-1}(\mathbf{0})$, then there exists $\mathbf{x} \in (\mathbf{f}-\mathbf{g})^{-1}(\mathbf{0})$. **Hint:** You might consider $\mathbf{h}(t,\mathbf{x}) \equiv (1-t)\mathbf{f}(\mathbf{x}) + t(\mathbf{f}(\mathbf{x}) - \mathbf{g}(\mathbf{x}))$ and argue $\mathbf{0} \notin \mathbf{h}(t,\partial\Omega)$ for $t \in [0,1]$.

18. Let $f : \mathbb{C} \to \mathbb{C}$ where $\mathbb{C}$ is the field of complex numbers. Thus f has a real and imaginary part. Letting $z = x+iy$,

$$f(z) = u(x,y) + iv(x,y)$$

Recall that the norm in $\mathbb{C}$ is given by $|x+iy| = \sqrt{x^2+y^2}$ and this is the usual norm in $\mathbb{R}^2$ for the ordered pair (x,y). Thus complex valued functions defined on $\mathbb{C}$ can be considered as $\mathbb{R}^2$ valued functions defined on some subset of $\mathbb{R}^2$. Such a complex function is said to be analytic if the usual definition holds. That is

$$f'(z) = \lim_{h\to 0} \frac{f(z+h)-f(z)}{h}.$$

In other words,

$$f(z+h) = f(z) + f'(z)h + o(h) \tag{10.20}$$

at a point z where the derivative exists. Let $f(z) = z^n$ where n is a positive integer. Thus $z^n = p(x,y) + iq(x,y)$ for p,q suitable polynomials in x and y. Show this function is analytic. Next show that for an analytic function and u and v the real and imaginary parts, the Cauchy Riemann equations hold.

$$u_x = v_y,\ u_y = -v_x.$$

In terms of mappings show 10.20 has the form

$$\begin{aligned}\begin{pmatrix} u(x+h_1,y+h_2) \\ v(x+h_1,y+h_2)\end{pmatrix} &= \begin{pmatrix} u(x,y) \\ v(x,y)\end{pmatrix} + \begin{pmatrix} u_x(x,y) & u_y(x,y) \\ v_x(x,y) & v_y(x,y)\end{pmatrix}\begin{pmatrix} h_1 \\ h_2\end{pmatrix} + \mathbf{o}(\mathbf{h}) \\ &= \begin{pmatrix} u(x,y) \\ v(x,y)\end{pmatrix} + \begin{pmatrix} u_x(x,y) & -v_x(x,y) \\ v_x(x,y) & u_x(x,y)\end{pmatrix}\begin{pmatrix} h_1 \\ h_2\end{pmatrix} + \mathbf{o}(\mathbf{h})\end{aligned}$$

where $\mathbf{h} = (h_1,h_2)^T$ and h is given by $h_1 + ih_2$. Thus the determinant of the above matrix is always nonnegative. Letting B_r denote the ball $B(0,r) = B((0,0),r)$ show

$$d(f,B_r,\mathbf{0}) = n.$$

where $f(z) = z^n$. In terms of mappings on $\mathbb{R}^2$,

$$\mathbf{f}(x,y) = \begin{pmatrix} u(x,y) \\ v(x,y)\end{pmatrix}.$$

Thus show

$$d(\mathbf{f},B_r,\mathbf{0}) = n.$$

Hint: You might consider

$$g(z) \equiv \prod_{j=1}^{n}(z-a_j)$$

where the a_j are small real distinct numbers and argue that both this function and f are analytic but that $\mathbf{0}$ is a regular value for $\mathbf{g}$ although it is not so for $\mathbf{f}$. However, for each a_j small but distinct $d(\mathbf{f},B_r,\mathbf{0}) = d(\mathbf{g},B_r,\mathbf{0})$.

19. Using Problem 18, prove the fundamental theorem of algebra as follows. Let $p(z)$ be a nonconstant polynomial of degree n,

$$p(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots$$

Show that for large enough $r, |p(z)| > |p(z) - a_n z^n|$ for all $z \in \partial B(0,r)$. Now from Problem 17 you can conclude $d(p, B_r, 0) = d(f, B_r, 0) = n$ where $f(z) = a_n z^n$.

Line Integrals

11.1 Basic Properties

11.1.1 Length

I will give a discussion of what is meant by a line integral which is independent of the earlier material on Lebesgue integration. Line integrals are of fundamental importance in physics and in the theory of functions of a complex variable.

Definition 11.1.1 *Let $\boldsymbol{\gamma} : [a,b] \to \mathbb{R}^n$ be a function. Then $\boldsymbol{\gamma}$ is of bounded variation if*

$$\sup\left\{\sum_{i=1}^{n} |\boldsymbol{\gamma}(t_i) - \boldsymbol{\gamma}(t_{i-1})| : a = t_0 < \cdots < t_n = b\right\} \equiv V(\boldsymbol{\gamma}, [a,b]) < \infty$$

where the sums are taken over all possible lists, $\{a = t_0 < \cdots < t_n = b\}$. The set of points traced out will be denoted by $\boldsymbol{\gamma}^* \equiv \boldsymbol{\gamma}([a,b])$. The function $\boldsymbol{\gamma}$ is called a parameterization of $\boldsymbol{\gamma}^*$. The set of points $\boldsymbol{\gamma}^*$ is called a rectifiable curve. If a set of points $\boldsymbol{\gamma}^* = \boldsymbol{\gamma}([a,b])$ where $\boldsymbol{\gamma}$ is continuous and $\boldsymbol{\gamma}$ is one to one on (a,b) such that also $\boldsymbol{\gamma}(t) \neq \boldsymbol{\gamma}(a)$ if $t \in (a,b)$ and $\boldsymbol{\gamma}(t) \neq \boldsymbol{\gamma}(b)$ if $t \in (a,b)$, then $\boldsymbol{\gamma}^*$ is called a simple curve. A closed curve is one which has a parameterization $\boldsymbol{\gamma}$ defined on an interval $[a,b]$ such that $\boldsymbol{\gamma}(a) = \boldsymbol{\gamma}(b)$.

The case of most interest is for simple curves. It turns out that in this case, the above concept of length is a property which $\boldsymbol{\gamma}^*$ possesses independent of the parameterization $\boldsymbol{\gamma}$ used to describe the set of points $\boldsymbol{\gamma}^*$. To show this, it is helpful to use the following lemma.

Lemma 11.1.2 *Let $\phi : [a,b] \to \mathbb{R}$ be a continuous function and suppose ϕ is $1-1$ on (a,b). Then ϕ is either strictly increasing or strictly decreasing on $[a,b]$. Furthermore, ϕ^{-1} is continuous.*

Proof: First it is shown that ϕ is either strictly increasing or strictly decreasing on (a,b).

If ϕ is not strictly decreasing on (a,b), then there exists $x_1 < y_1$, $x_1, y_1 \in (a,b)$ such that

$$(\phi(y_1) - \phi(x_1))(y_1 - x_1) > 0.$$

If for some other pair of points, $x_2 < y_2$ with $x_2, y_2 \in (a,b)$, the above inequality does not hold, then since ϕ is $1-1$,

$$(\phi(y_2) - \phi(x_2))(y_2 - x_2) < 0.$$

Let $x_t \equiv tx_1 + (1-t)x_2$ and $y_t \equiv ty_1 + (1-t)y_2$. Then $x_t < y_t$ for all $t \in [0,1]$ because

$$tx_1 \leq ty_1 \text{ and } (1-t)x_2 \leq (1-t)y_2$$

with strict inequality holding for at least one of these inequalities since not both t and $(1-t)$ can equal zero. Now define

$$h(t) \equiv (\phi(y_t) - \phi(x_t))(y_t - x_t).$$

Since h is continuous and $h(0) < 0$, while $h(1) > 0$, there exists $t \in (0,1)$ such that $h(t) = 0$. Therefore, both x_t and y_t are points of (a,b) and $\phi(y_t) - \phi(x_t) = 0$ contradicting the assumption that ϕ is one to one. It follows ϕ is either strictly increasing or strictly decreasing on (a,b).

This property of being either strictly increasing or strictly decreasing on (a,b) carries over to $[a,b]$ by the continuity of ϕ. Suppose ϕ is strictly increasing on (a,b), a similar argument holding for ϕ strictly decreasing on (a,b). If $x > a$, then pick $y \in (a,x)$ and from the above, $\phi(y) < \phi(x)$. Now by continuity of ϕ at a,

$$\phi(a) = \lim_{x \to a+} \phi(z) \leq \phi(y) < \phi(x).$$

Therefore, $\phi(a) < \phi(x)$ whenever $x \in (a,b)$. Similarly $\phi(b) > \phi(x)$ for all $x \in (a,b)$.

It only remains to verify ϕ^{-1} is continuous. Suppose then that $s_n \to s$ where s_n and s are points of $\phi([a,b])$. It is desired to verify that $\phi^{-1}(s_n) \to \phi^{-1}(s)$. If this does not happen, there exists $\varepsilon > 0$ and a subsequence, still denoted by s_n such that $\left|\phi^{-1}(s_n) - \phi^{-1}(s)\right| \geq \varepsilon$. Using the sequential compactness of $[a,b]$ there exists a further subsequence, still denoted by n, such that $\phi^{-1}(s_n) \to t_1 \in [a,b], t_1 \neq \phi^{-1}(s)$. Then by continuity of ϕ, it follows $s_n \to \phi(t_1)$ and so $s = \phi(t_1)$. Therefore, $t_1 = \phi^{-1}(s)$ after all. This proves the lemma.

Now suppose $\boldsymbol{\gamma}$ and $\boldsymbol{\eta}$ are two parameterizations of the simple curve $\boldsymbol{\gamma}^*$ as described above. Thus $\boldsymbol{\gamma}([a,b]) = \boldsymbol{\gamma}^* = \boldsymbol{\eta}([c,d])$ and the two continuous functions $\boldsymbol{\gamma}, \boldsymbol{\eta}$ are of bounded variation and one to one on their respective open intervals. I need to show the two definitions of length yield the same thing with either parameterization. Since $\boldsymbol{\gamma}^*$ is compact, it follows from Theorem 5.1.3 on Page 84, both $\boldsymbol{\gamma}^{-1}$ and $\boldsymbol{\eta}^{-1}$ are continuous. Thus $\boldsymbol{\gamma}^{-1} \circ \boldsymbol{\eta} : [c,d] \to [a,b]$ is continuous. It is also uniformly continuous because $[c,d]$ is compact. Let $\mathcal{P} \equiv \{t_0, \cdots, t_n\}$ be a partition of $[a,b], t_0 < t_1 < \cdots$, such that

$$0 \leq V(\boldsymbol{\gamma}, [a,b]) - \sum_{k=1}^{n} |\boldsymbol{\gamma}(t_k) - \boldsymbol{\gamma}(t_{k-1})| < \varepsilon$$

Note the sums approximating the total variation are all no larger than the total variation because when another point is added in to the partition, it is an easy exercise in the triangle inequality to show the corresponding sum either becomes larger or stays the same.

Let $\boldsymbol{\gamma}^{-1} \circ \boldsymbol{\eta}(s_k) = t_k$ so that $\{s_0, \cdots, s_n\}$ is a partition of $[c,d]$. By the lemma, the s_k are either strictly decreasing or strictly increasing as a function of k, depending on whether $\boldsymbol{\gamma}^{-1} \circ \boldsymbol{\eta}$ is increasing or decreasing. Thus $\boldsymbol{\gamma}(t_k) = \boldsymbol{\eta}(s_k)$ and so

$$V(\boldsymbol{\gamma}, [a,b]) - V(\boldsymbol{\eta}, [c,d]) \leq V(\boldsymbol{\gamma}, [a,b]) - \sum_{k=1}^{n} |\boldsymbol{\eta}(s_k) - \boldsymbol{\eta}(s_{k-1})| < \varepsilon$$

It follows

$$V(\boldsymbol{\gamma}, [a,b]) \leq V(\boldsymbol{\eta}, [c,d]) + \varepsilon$$

and since ε is arbitrary, this shows $V(\boldsymbol{\gamma}, [a,b]) \leq V(\boldsymbol{\eta}, [c,d])$. Turning the argument around reverses the inequality. This proves the following fundamental theorem.

Theorem 11.1.3 *Let Γ be a simple curve and let $\boldsymbol{\gamma}$ be a parameterization for Γ where $\boldsymbol{\gamma}$ is one to one on (a,b), continuous on $[a,b]$ and of bounded variation. Then the total variation*

$$V(\boldsymbol{\gamma}, [a,b])$$

can be used as a definition for the length of Γ *in the sense that if* $\Gamma = \boldsymbol{\eta}([c,d])$ *where* $\boldsymbol{\eta}$ *is a bounded variation continuous function which is one to one on* (c,d) *with* $\boldsymbol{\eta}([c,d]) = \Gamma$,

$$V(\boldsymbol{\gamma},[a,b]) = V(\boldsymbol{\eta},[c,d]).$$

This common value can be denoted by $V(\Gamma)$ *and is called the length of* Γ.

The length is not dependent on parameterization. Simple curves which have such parameterizations are called rectifiable.

11.1.2 Orientation

There is another notion called orientation. For simple rectifiable curves, you can think of it as a direction of motion over the curve but what does this really mean for a wriggly curve? A precise description is needed.

Definition 11.1.4 *Let* $\boldsymbol{\eta},\boldsymbol{\gamma}$ *be continuous one to one parameterizations for a simple rectifiable curve. If* $\boldsymbol{\eta}^{-1}\circ\boldsymbol{\gamma}$ *is increasing, then* $\boldsymbol{\gamma}$ *and* $\boldsymbol{\eta}$ *are said to be equivalent parameterizations and this is written as* $\boldsymbol{\gamma}\sim\boldsymbol{\eta}$. *It is also said that the two parameterizations give the same orientation for the curve when* $\boldsymbol{\gamma}\sim\boldsymbol{\eta}$.

When the parameterizations are equivalent, they preserve the direction of motion along the curve and this also shows there are exactly two orientations of the curve since either $\boldsymbol{\eta}^{-1}\circ\boldsymbol{\gamma}$ is increasing or it is decreasing thanks to Lemma 11.1.2. In simple language, the message is that there are exactly two directions of motion along a simple curve.

Lemma 11.1.5 *The following hold for* $\sim$.

$$\boldsymbol{\gamma}\sim\boldsymbol{\gamma}, \tag{11.1}$$

$$\text{If } \boldsymbol{\gamma}\sim\boldsymbol{\eta} \text{ then } \boldsymbol{\eta}\sim\boldsymbol{\gamma}, \tag{11.2}$$

$$\text{If } \boldsymbol{\gamma}\sim\boldsymbol{\eta} \text{ and } \boldsymbol{\eta}\sim\boldsymbol{\theta}, \text{ then } \boldsymbol{\gamma}\sim\boldsymbol{\theta}. \tag{11.3}$$

Proof: Formula 11.1 is obvious because $\boldsymbol{\gamma}^{-1}\circ\boldsymbol{\gamma}(t) = t$ so it is clearly an increasing function. If $\boldsymbol{\gamma}\sim\boldsymbol{\eta}$ then $\boldsymbol{\gamma}^{-1}\circ\boldsymbol{\eta}$ is increasing. Now $\boldsymbol{\eta}^{-1}\circ\boldsymbol{\gamma}$ must also be increasing because it is the inverse of $\boldsymbol{\gamma}^{-1}\circ\boldsymbol{\eta}$. This verifies 11.2. To see 11.3, $\boldsymbol{\gamma}^{-1}\circ\boldsymbol{\theta} = \left(\boldsymbol{\gamma}^{-1}\circ\boldsymbol{\eta}\right)\circ\left(\boldsymbol{\eta}^{-1}\circ\boldsymbol{\theta}\right)$ and so since both of these functions are increasing, it follows $\boldsymbol{\gamma}^{-1}\circ\boldsymbol{\theta}$ is also increasing. This proves the lemma.

Definition 11.1.6 *Let* Γ *be a simple rectifiable curve and let* $\boldsymbol{\gamma}$ *be a parameterization for* Γ. *Denoting by* $[\boldsymbol{\gamma}]$ *the equivalence class of parameterizations determined by the above equivalence relation, the following pair will be called an oriented curve.*

$$(\Gamma,[\boldsymbol{\gamma}])$$

In simple language, an oriented curve is one which has a direction of motion specified.

Actually, people usually just write Γ and there is understood a direction of motion or orientation on Γ. How can you identify which orientation is being considered?

Proposition 11.1.7 *Let* $(\Gamma,[\boldsymbol{\gamma}])$ *be an oriented simple curve and let* $\mathbf{p},\mathbf{q}$ *be any two distinct points of* Γ. *Then* $[\boldsymbol{\gamma}]$ *is determined by the order of* $\boldsymbol{\gamma}^{-1}(\mathbf{p})$ *and* $\boldsymbol{\gamma}^{-1}(\mathbf{q})$. *This means that* $\boldsymbol{\eta}\in[\boldsymbol{\gamma}]$ *if and only if* $\boldsymbol{\eta}^{-1}(\mathbf{p})$ *and* $\boldsymbol{\eta}^{-1}(\mathbf{q})$ *occur in the same order as* $\boldsymbol{\gamma}^{-1}(\mathbf{p})$ *and* $\boldsymbol{\gamma}^{-1}(\mathbf{q})$.

Proof: Suppose $\boldsymbol{\gamma}^{-1}(\mathbf{p}) < \boldsymbol{\gamma}^{-1}(\mathbf{q})$ and let $\boldsymbol{\eta} \in [\boldsymbol{\gamma}]$. Is it true that $\boldsymbol{\eta}^{-1}(\mathbf{p}) < \boldsymbol{\eta}^{-1}(\mathbf{q})$? Of course it is because $\boldsymbol{\gamma}^{-1} \circ \boldsymbol{\eta}$ is increasing. Therefore, if $\boldsymbol{\eta}^{-1}(\mathbf{p}) > \boldsymbol{\eta}^{-1}(\mathbf{q})$ it would follow

$$\boldsymbol{\gamma}^{-1}(\mathbf{p}) = \boldsymbol{\gamma}^{-1} \circ \boldsymbol{\eta}\left(\boldsymbol{\eta}^{-1}(\mathbf{p})\right) > \boldsymbol{\gamma}^{-1} \circ \boldsymbol{\eta}\left(\boldsymbol{\eta}^{-1}(\mathbf{q})\right) = \boldsymbol{\gamma}^{-1}(\mathbf{q})$$

which is a contradiction. Thus if $\boldsymbol{\gamma}^{-1}(\mathbf{p}) < \boldsymbol{\gamma}^{-1}(\mathbf{q})$ for one $\boldsymbol{\gamma} \in [\boldsymbol{\gamma}]$, then this is true for all $\boldsymbol{\eta} \in [\boldsymbol{\gamma}]$.

Now suppose $\boldsymbol{\eta}$ is a parameterization for Γ defined on $[c,d]$ which has the property that

$$\boldsymbol{\eta}^{-1}(\mathbf{p}) < \boldsymbol{\eta}^{-1}(\mathbf{q})$$

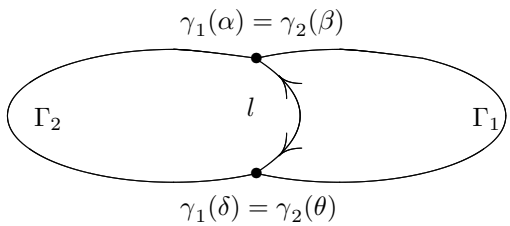
Does it follow $\boldsymbol{\eta} \in [\boldsymbol{\gamma}]$? Is $\boldsymbol{\gamma}^{-1} \circ \boldsymbol{\eta}$ increasing? By Lemma 11.1.2 it is either increasing or decreasing. Thus it suffices to test it on two points of $[c,d]$. Pick the two points $\boldsymbol{\eta}^{-1}(\mathbf{p}), \boldsymbol{\eta}^{-1}(\mathbf{q})$. Is

$$\boldsymbol{\gamma}^{-1} \circ \boldsymbol{\eta}\left(\boldsymbol{\eta}^{-1}(\mathbf{p})\right) < \boldsymbol{\gamma}^{-1} \circ \boldsymbol{\eta}\left(\boldsymbol{\eta}^{-1}(\mathbf{q})\right)?$$

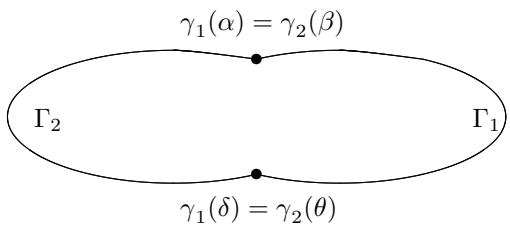
Yes because these reduce to $\boldsymbol{\gamma}^{-1}(\mathbf{p})$ on the left and $\boldsymbol{\gamma}^{-1}(\mathbf{q})$ on the right. It is given that $\boldsymbol{\gamma}^{-1}(\mathbf{p}) < \boldsymbol{\gamma}^{-1}(\mathbf{q})$. This proves the lemma.

This shows that the direction of motion on the curve is determined by any two points and the determination of which is encountered first by any parameterization in the equivalence class of parameterizations which determines the orientation. Sometimes people indicate this direction of motion by drawing an arrow.

Now here is an interesting observation relative to two simple closed rectifiable curves. The situation is illustrated by the following picture.

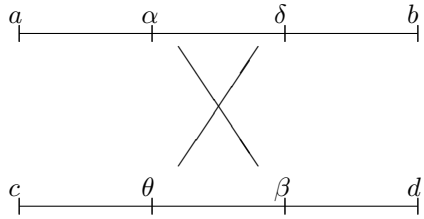


Proposition 11.1.8 *Let Γ_1 and Γ_2 be two simple closed rectifiable oriented curves and let their intersection be l. Suppose also that l is itself a simple curve. Also suppose the orientation of l when considered a part of Γ_1 is opposite its orientation when considered a part of Γ_2. Then if the open segment (l except for its endpoints) of l is removed, the result is a simple closed rectifiable curve Γ. This curve has a parameterization $\boldsymbol{\gamma}$ with the property that on $\boldsymbol{\gamma}_j^{-1}(\Gamma \cap \Gamma_j)$, $\boldsymbol{\gamma}^{-1}\boldsymbol{\gamma}_j$ is increasing. In other words, Γ has an orientation consistent with that of Γ_1 and Γ_2. Furthermore, if Γ has such a consistent orientation, then the orientations of l as part of the two simple closed curves, Γ_1 and Γ_2 are opposite.*



Proof: Let $\Gamma_1 = \boldsymbol{\gamma}_1([a,b]), \boldsymbol{\gamma}_1(a) = \boldsymbol{\gamma}_1(b)$, and $\Gamma_2 = \boldsymbol{\gamma}_2([c,d]), \boldsymbol{\gamma}_2(c) = \boldsymbol{\gamma}_2(d)$, with $l = \boldsymbol{\gamma}_1([\alpha,\delta]) = \boldsymbol{\gamma}_2([\theta,\beta])$. (Recall continuous images of connected sets are connected and the connected sets on the real line are intervals.) By the assumption the two orientations

are opposite, someting can be said about the relationship of $\alpha, \delta, \theta, \beta$. Suppose without loss of generality that $\alpha < \delta$. Then because of this assumption it follows $\theta < \beta$. The following diagram might be useful to summarize what was just said.



Note the first of the interval $[\beta, d]$ matches the last of the interval $[a, \alpha]$ and the first of $[\delta, \beta]$ matches the last of $[c, \theta]$, all this in terms of where these points are sent. If the orientations for l were not opposite, such a thing would not happen.

Now I need to describe the parameterization of $\Gamma \equiv \Gamma_1 \cup \Gamma_2$. To verify it is a simple closed curve, I must produce an interval and a mapping from this interval to Γ which satisfies the conditions needed for $\boldsymbol{\gamma}$ to be a simple closed rectifiable curve. The following is the definition as well as a description of which part of Γ_j is being obtained. Then $\boldsymbol{\gamma}(t)$ is given by

$$\boldsymbol{\gamma}(t) \equiv$$

$$\begin{cases} \boldsymbol{\gamma}_1(t), t \in [a, \alpha], \boldsymbol{\gamma}_1(a) \to \boldsymbol{\gamma}_1(\alpha) = \boldsymbol{\gamma}_2(\beta) \\ \boldsymbol{\gamma}_2(t + \beta - \alpha), t \in [\alpha, \alpha + d - \beta], \boldsymbol{\gamma}_2(\beta) \to \boldsymbol{\gamma}_2(d) = \boldsymbol{\gamma}_2(c) \\ \boldsymbol{\gamma}_2(t + c - \alpha - d + \beta), t \in [\alpha + d - \beta, \alpha + d - \beta + \theta - c], \\ \boldsymbol{\gamma}_2(c) = \boldsymbol{\gamma}_2(d) \to \boldsymbol{\gamma}_2(\theta) = \boldsymbol{\gamma}_1(\delta) \\ \boldsymbol{\gamma}_1(t - \alpha - d + \beta - \theta + c + \delta), t \in [\alpha + d - \beta + \theta - c, \alpha + d - \beta + \theta - c + b - \delta], \\ \boldsymbol{\gamma}_1(\delta) \to \boldsymbol{\gamma}_1(b) = \boldsymbol{\gamma}_1(a) \end{cases}$$

The construction shows $\boldsymbol{\gamma}$ is one to one on

$$(a, \alpha + d - \beta + \theta - c + b - \delta)$$

and if t is in this open interval, then

$$\boldsymbol{\gamma}(t) \neq \boldsymbol{\gamma}(a) = \boldsymbol{\gamma}_1(a)$$

and

$$\boldsymbol{\gamma}(t) \neq \boldsymbol{\gamma}(\alpha + d - \beta + \theta - c + b - \delta) = \boldsymbol{\gamma}_1(b).$$

Also

$$\boldsymbol{\gamma}(a) = \boldsymbol{\gamma}_1(a) = \boldsymbol{\gamma}(\alpha + d - \beta + \theta - c + b - \delta) = \boldsymbol{\gamma}_1(b)$$

so it is a simple closed curve. The claim about preserving the orientation is also obvious from the formula. Note that t is never subtracted.

It only remains to prove the last claim. Suppose then that it is not so and l has the same orientation as part of each Γ_j. Then from a repeat of the above argument, you could change the orientation of l relative to Γ_2 and obtain an orientation of Γ which is consistent with that of Γ_1 and Γ_2. Call a parameterization which has this new orientation $\boldsymbol{\gamma}_n$ while $\boldsymbol{\gamma}$ is the one which is assumed to exist. This new orientation of l changes the orientation of Γ_2 because there are two points in l. Therefore on $\boldsymbol{\gamma}_2^{-1}(\Gamma \cap \Gamma_2)$, $\boldsymbol{\gamma}_n^{-1}\boldsymbol{\gamma}_2$ is decreasing while $\boldsymbol{\gamma}^{-1}\boldsymbol{\gamma}_2$ is assumed to be increasing. Hence $\boldsymbol{\gamma}$ and $\boldsymbol{\gamma}_n$ are not equivalent. However, the above construction would leave the orientation of both $\boldsymbol{\gamma}_1([a, \alpha])$ and $\boldsymbol{\gamma}_1([\delta, b])$ unchanged and at least one of these must have at least two points. Thus the orientation of Γ must be the same for $\boldsymbol{\gamma}_n$ as for $\boldsymbol{\gamma}$. That is, $\boldsymbol{\gamma} \sim \boldsymbol{\gamma}_n$. This is a contradiction. This proves the proposition.

There is a slightly different aspect of the above proposition which is interesting. It involves using the shared segment to orient the simple closed curve Γ.

Corollary 11.1.9 *Let the intersection of simple closed rectifiable curves, Γ_1 and Γ_2 consist of the simple curve l. Then place opposite orientations on l, and use these two different orientations to specify orientations of Γ_1 and Γ_2. Then letting Γ denote the simple closed curve which is obtained from deleting the open segment of l, there exists an orientation for Γ which is consistent with the orientations of Γ_1 and Γ_2 obtained from the given specification of opposite orientations on l.*

11.2 The Line Integral

Now I will return to considering the more general notion of bounded variation parameterizations without worrying about whether $\boldsymbol{\gamma}$ is one to one on the open interval. The line integral and its properties are presented next.

Definition 11.2.1 *Let $\boldsymbol{\gamma} : [a,b] \to \mathbb{R}^n$ be of bounded variation and let $\mathbf{f} : \boldsymbol{\gamma}^* \to \mathbb{R}^n$. Letting $\mathcal{P} \equiv \{t_0, \cdots, t_n\}$ where $a = t_0 < t_1 < \cdots < t_n = b$, define*

$$||\mathcal{P}|| \equiv \max\{|t_j - t_{j-1}| : j = 1, \cdots, n\}$$

and the Riemann Stieltjes sum by

$$S(\mathcal{P}) \equiv \sum_{j=1}^{n} \mathbf{f}(\boldsymbol{\gamma}(\tau_j)) \cdot (\boldsymbol{\gamma}(t_j) - \boldsymbol{\gamma}(t_{j-1}))$$

where $\tau_j \in [t_{j-1}, t_j]$. (Note this notation is a little sloppy because it does not identify the specific point, τ_j used. It is understood that this point is arbitrary.) Define $\int_{\boldsymbol{\gamma}} \mathbf{f} \cdot d\boldsymbol{\gamma}$ as the unique number which satisfies the following condition. For all $\varepsilon > 0$ there exists a $\delta > 0$ such that if $||\mathcal{P}|| \leq \delta$, then

$$\left| \int_{\boldsymbol{\gamma}} \mathbf{f} \cdot d\boldsymbol{\gamma} - S(\mathcal{P}) \right| < \varepsilon.$$

Sometimes this is written as

$$\int_{\boldsymbol{\gamma}} \mathbf{f} \cdot d\boldsymbol{\gamma} \equiv \lim_{||\mathcal{P}|| \to 0} S(\mathcal{P}).$$

Then $\boldsymbol{\gamma}^*$ is a set of points in $\mathbb{R}^n$ and as t moves from a to b, $\boldsymbol{\gamma}(t)$ moves from $\boldsymbol{\gamma}(a)$ to $\boldsymbol{\gamma}(b)$. Thus $\boldsymbol{\gamma}^*$ has a first point and a last point. (In the case of a closed curve these are the same point.) If $\phi : [c,d] \to [a,b]$ is a continuous nondecreasing function, then $\boldsymbol{\gamma} \circ \phi : [c,d] \to \mathbb{R}^n$ is also of bounded variation and yields the same set of points in $\mathbb{R}^n$ with the same first and last points.

Theorem 11.2.2 *Let ϕ and $\boldsymbol{\gamma}$ be as just described. Then assuming that*

$$\int_{\boldsymbol{\gamma}} \mathbf{f} \cdot d\boldsymbol{\gamma}$$

exists, so does

$$\int_{\boldsymbol{\gamma} \circ \phi} \mathbf{f} \cdot d(\boldsymbol{\gamma} \circ \phi)$$

and

$$\int_{\boldsymbol{\gamma}} \mathbf{f} \cdot d\boldsymbol{\gamma} = \int_{\boldsymbol{\gamma} \circ \phi} \mathbf{f} \cdot d(\boldsymbol{\gamma} \circ \phi). \tag{11.4}$$

Proof: There exists $\delta > 0$ such that if $\mathcal{P}$ is a partition of $[a,b]$ such that $||\mathcal{P}|| < \delta$, then

$$\left| \int_{\boldsymbol{\gamma}} \mathbf{f} \cdot d\boldsymbol{\gamma} - S(\mathcal{P}) \right| < \varepsilon.$$

By continuity of ϕ, there exists $\sigma > 0$ such that if $\mathcal{Q}$ is a partition of $[c,d]$ with $||\mathcal{Q}|| < \sigma, \mathcal{Q} = \{s_0, \cdots, s_n\}$, then $|\phi(s_j) - \phi(s_{j-1})| < \delta$. Thus letting $\mathcal{P}$ denote the points in $[a,b]$ given by $\phi(s_j)$ for $s_j \in \mathcal{Q}$, it follows that $||\mathcal{P}|| < \delta$ and so

$$\left| \int_{\boldsymbol{\gamma}} \mathbf{f} \cdot d\boldsymbol{\gamma} - \sum_{j=1}^{n} \mathbf{f}(\boldsymbol{\gamma}(\phi(\tau_j))) \cdot (\boldsymbol{\gamma}(\phi(s_j)) - \boldsymbol{\gamma}(\phi(s_{j-1}))) \right| < \varepsilon$$

where $\tau_j \in [s_{j-1}, s_j]$. Therefore, from the definition 11.4 holds and

$$\int_{\boldsymbol{\gamma}\circ\phi} \mathbf{f} \cdot d(\boldsymbol{\gamma} \circ \phi)$$

exists. This proves the theorem.

This theorem shows that $\int_{\boldsymbol{\gamma}} \mathbf{f} \cdot d\boldsymbol{\gamma}$ is independent of the particular parameterization $\boldsymbol{\gamma}$ used in its computation to the extent that if ϕ is any nondecreasing continuous function from another interval, $[c,d]$, mapping to $[a,b]$, then the same value is obtained by replacing $\boldsymbol{\gamma}$ with $\boldsymbol{\gamma} \circ \phi$. In other words, this line integral depends only on $\boldsymbol{\gamma}^*$ and the order in which $\boldsymbol{\gamma}(t)$ encounters the points of $\boldsymbol{\gamma}^*$ as t moves from one end to the other of the interval. For the case of an oriented rectifiable curve Γ this shows the line integral is dependent only on the set of points and the orientation of Γ.

The fundamental result in this subject is the following theorem.

Theorem 11.2.3 *Let $\mathbf{f} : \boldsymbol{\gamma}^* \to \mathbb{R}^n$ be continuous and let $\boldsymbol{\gamma} : [a,b] \to \mathbb{R}^n$ be continuous and of bounded variation. Then $\int_{\boldsymbol{\gamma}} \mathbf{f} \cdot d\boldsymbol{\gamma}$ exists. Also letting $\delta_m > 0$ be such that $|t-s| < \delta_m$ implies $|\mathbf{f}(\boldsymbol{\gamma}(t)) - \mathbf{f}(\boldsymbol{\gamma}(s))| < \frac{1}{m}$,*

$$\left| \int_{\boldsymbol{\gamma}} \mathbf{f} d\boldsymbol{\gamma} - S(\mathcal{P}) \right| \leq \frac{2V(\boldsymbol{\gamma}, [a,b])}{m}$$

whenever $||\mathcal{P}|| < \delta_m$.

Proof: The function, $\mathbf{f} \circ \boldsymbol{\gamma}$, is uniformly continuous because it is defined on a compact set. Therefore, there exists a decreasing sequence of positive numbers, $\{\delta_m\}$ such that if $|s-t| < \delta_m$, then

$$|\mathbf{f}(\boldsymbol{\gamma}(t)) - \mathbf{f}(\boldsymbol{\gamma}(s))| < \frac{1}{m}.$$

Let

$$F_m \equiv \overline{\{S(\mathcal{P}) : ||\mathcal{P}|| < \delta_m\}}.$$

Thus F_m is a closed set. (The symbol, $S(\mathcal{P})$ in the above definition, means to include all sums corresponding to $\mathcal{P}$ for any choice of τ_j.) It is shown that

$$\operatorname{diam}(F_m) \leq \frac{2V(\boldsymbol{\gamma}, [a,b])}{m} \tag{11.5}$$

and then it will follow there exists a unique point, $I \in \cap_{m=1}^{\infty} F_m$. This is because $\mathbb{R}$ is complete. It will then follow $I = \int_{\boldsymbol{\gamma}} \mathbf{f}(t)\, d\boldsymbol{\gamma}(t)$. To verify 11.5, it suffices to verify that whenever $\mathcal{P}$ and $\mathcal{Q}$ are partitions satisfying $||\mathcal{P}|| < \delta_m$ and $||\mathcal{Q}|| < \delta_m$,

$$|S(\mathcal{P}) - S(\mathcal{Q})| \leq \frac{2}{m} V(\boldsymbol{\gamma}, [a,b]). \tag{11.6}$$

Suppose $||\mathcal{P}|| < \delta_m$ and $\mathcal{Q} \supseteq \mathcal{P}$. Then also $||\mathcal{Q}|| < \delta_m$. To begin with, suppose that $\mathcal{P} \equiv \{t_0, \cdots, t_p, \cdots, t_n\}$ and $\mathcal{Q} \equiv \{t_0, \cdots, t_{p-1}, t^*, t_p, \cdots, t_n\}$. Thus $\mathcal{Q}$ contains only one more point than $\mathcal{P}$. Letting $S(\mathcal{Q})$ and $S(\mathcal{P})$ be Riemann Stieltjes sums,

$$S(\mathcal{Q}) \equiv \sum_{j=1}^{p-1} \mathbf{f}(\boldsymbol{\gamma}(\sigma_j)) \cdot (\boldsymbol{\gamma}(t_j) - \boldsymbol{\gamma}(t_{j-1})) + \mathbf{f}(\boldsymbol{\gamma}(\sigma_*)) (\boldsymbol{\gamma}(t^*) - \boldsymbol{\gamma}(t_{p-1}))$$

$$+\mathbf{f}(\boldsymbol{\gamma}(\sigma^*)) \cdot (\boldsymbol{\gamma}(t_p) - \boldsymbol{\gamma}(t^*)) + \sum_{j=p+1}^{n} \mathbf{f}(\boldsymbol{\gamma}(\sigma_j)) \cdot (\boldsymbol{\gamma}(t_j) - \boldsymbol{\gamma}(t_{j-1})),$$

$$S(\mathcal{P}) \equiv \sum_{j=1}^{p-1} \mathbf{f}(\boldsymbol{\gamma}(\tau_j)) \cdot (\boldsymbol{\gamma}(t_j) - \boldsymbol{\gamma}(t_{j-1})) +$$

$$\overbrace{\mathbf{f}(\boldsymbol{\gamma}(\tau_p)) \cdot (\boldsymbol{\gamma}(t^*) - \boldsymbol{\gamma}(t_{p-1})) + \mathbf{f}(\boldsymbol{\gamma}(\tau_p)) \cdot (\boldsymbol{\gamma}(t_p) - \boldsymbol{\gamma}(t^*))}^{=\mathbf{f}(\boldsymbol{\gamma}(\tau_p)) \cdot (\boldsymbol{\gamma}(t_p) - \boldsymbol{\gamma}(t_{p-1}))}$$

$$+ \sum_{j=p+1}^{n} \mathbf{f}(\boldsymbol{\gamma}(\tau_j)) \cdot (\boldsymbol{\gamma}(t_j) - \boldsymbol{\gamma}(t_{j-1})).$$

Therefore,

$$|S(\mathcal{P}) - S(\mathcal{Q})| \leq \sum_{j=1}^{p-1} \frac{1}{m} |\boldsymbol{\gamma}(t_j) - \boldsymbol{\gamma}(t_{j-1})| + \frac{1}{m} |\boldsymbol{\gamma}(t^*) - \boldsymbol{\gamma}(t_{p-1})| +$$

$$\frac{1}{m} |\boldsymbol{\gamma}(t_p) - \boldsymbol{\gamma}(t^*)| + \sum_{j=p+1}^{n} \frac{1}{m} |\boldsymbol{\gamma}(t_j) - \boldsymbol{\gamma}(t_{j-1})| \leq \frac{1}{m} V(\boldsymbol{\gamma}, [a,b]). \tag{11.7}$$

Clearly the extreme inequalities would be valid in 11.7 if $\mathcal{Q}$ had more than one extra point. You simply do the above trick more than one time. Let $S(\mathcal{P})$ and $S(\mathcal{Q})$ be Riemann Stieltjes sums for which $||\mathcal{P}||$ and $||\mathcal{Q}||$ are less than δ_m and let $\mathcal{R} \equiv \mathcal{P} \cup \mathcal{Q}$. Then from what was just observed,

$$|S(\mathcal{P}) - S(\mathcal{Q})| \leq |S(\mathcal{P}) - S(\mathcal{R})| + |S(\mathcal{R}) - S(\mathcal{Q})| \leq \frac{2}{m} V(\boldsymbol{\gamma}, [a,b]).$$

and this shows 11.6 which proves 11.5. Therefore, there exists a unique number, $I \in \cap_{m=1}^{\infty} F_m$ which satisfies the definition of $\int_{\boldsymbol{\gamma}} \mathbf{f} \cdot d\boldsymbol{\gamma}$. This proves the theorem.

Note this is a general sort of result. It is not assumed that $\boldsymbol{\gamma}$ is one to one anywhere in the proof. The following theorem follows easily from the above definitions and theorem. This theorem is used to establish estimates.

Theorem 11.2.4 *Let* $\mathbf{f}$ *be a continuous function defined on* $\boldsymbol{\gamma}^*$, *denoted as* $C(\boldsymbol{\gamma}^*)$ *where* $\boldsymbol{\gamma} : [a,b] \to \mathbb{R}^n$ *is of bounded variation and continuous. Let*

$$M \geq \max\{|\mathbf{f} \circ \boldsymbol{\gamma}(t)| : t \in [a,b]\}. \tag{11.8}$$

Then

$$\left| \int_{\boldsymbol{\gamma}} \mathbf{f} \cdot d\boldsymbol{\gamma} \right| \leq MV(\boldsymbol{\gamma}, [a,b]). \tag{11.9}$$

Also if $\{\mathbf{f}_n\}$ *is a sequence of functions of* $C(\boldsymbol{\gamma}^*)$ *which is converging uniformly to the function,* $\mathbf{f}$ *on* $\boldsymbol{\gamma}^*$, *then*

$$\lim_{n \to \infty} \int_{\boldsymbol{\gamma}} \mathbf{f}_n \cdot d\boldsymbol{\gamma} = \int_{\boldsymbol{\gamma}} \mathbf{f} \cdot d\boldsymbol{\gamma}. \tag{11.10}$$

In case $\boldsymbol{\gamma}(a) = \boldsymbol{\gamma}(b)$ *so the curve is a closed curve and for* f_k *the* k^{th} *component of* $\mathbf{f}$,

$$m_k \leq f_k(\mathbf{x}) \leq M_k$$

for all $\mathbf{x} \in \boldsymbol{\gamma}^*$, *it also follows*

$$\left| \int_{\boldsymbol{\gamma}} \mathbf{f} \cdot d\boldsymbol{\gamma} \right| \leq \frac{1}{2} \left(\sum_{k=1}^{n} (M_k - m_k)^2 \right)^{1/2} V(\boldsymbol{\gamma}, [a,b]) \tag{11.11}$$

Proof: Let 11.8 hold. From the proof of Theorem 11.2.3, when $||\mathcal{P}|| < \delta_m$,

$$\left| \int_{\boldsymbol{\gamma}} \mathbf{f} \cdot d\boldsymbol{\gamma} - S(\mathcal{P}) \right| \leq \frac{2}{m} V(\boldsymbol{\gamma}, [a,b])$$

and so

$$\left| \int_{\boldsymbol{\gamma}} \mathbf{f} \cdot d\boldsymbol{\gamma} \right| \leq |S(\mathcal{P})| + \frac{2}{m} V(\boldsymbol{\gamma}, [a,b])$$

Using the Cauchy Schwarz inequality and the above estimate in $S(\mathcal{P})$,

$$\begin{aligned} &\leq \sum_{j=1}^{n} M \left| \boldsymbol{\gamma}(t_j) - \boldsymbol{\gamma}(t_{j-1}) \right| + \frac{2}{m} V(\boldsymbol{\gamma}, [a,b]) \\ &\leq MV(\boldsymbol{\gamma}, [a,b]) + \frac{2}{m} V(\boldsymbol{\gamma}, [a,b]). \end{aligned}$$

This proves 11.9 since m is arbitrary. To verify 11.10 use the above inequality to write

$$\left| \int_{\boldsymbol{\gamma}} \mathbf{f} \cdot d\boldsymbol{\gamma} - \int_{\boldsymbol{\gamma}} \mathbf{f}_n \cdot d\boldsymbol{\gamma} \right| = \left| \int_{\boldsymbol{\gamma}} (\mathbf{f} - \mathbf{f}_n) \cdot d\boldsymbol{\gamma}(t) \right|$$

$$\leq \max \{ |\mathbf{f} \circ \boldsymbol{\gamma}(t) - \mathbf{f}_n \circ \boldsymbol{\gamma}(t)| : t \in [a,b] \} V(\boldsymbol{\gamma}, [a,b]).$$

Since the convergence is assumed to be uniform, this proves 11.10.

It only remains to verify 11.11. In this case $\boldsymbol{\gamma}(a) = \boldsymbol{\gamma}(b)$ and so for each vector $\mathbf{c}$

$$\int_{\boldsymbol{\gamma}} \mathbf{f} \cdot d\boldsymbol{\gamma} = \int_{\boldsymbol{\gamma}} (\mathbf{f} - \mathbf{c}) \cdot d\boldsymbol{\gamma}$$

for any constant vector $\mathbf{c}$. Let

$$c_k = \frac{1}{2}(M_k + m_k)$$

Then for $t \in [a,b]$

$$\begin{aligned} |\mathbf{f}(\boldsymbol{\gamma}(t)) - \mathbf{c}|^2 &= \sum_{k=1}^{n} \left| f_k(\boldsymbol{\gamma}(t)) - \frac{1}{2}(M_k + m_k) \right|^2 \\ &\leq \sum_{k=1}^{n} \left(\frac{1}{2}(M_k - m_k) \right)^2 = \frac{1}{4} \sum_{k=1}^{n} (M_k - m_k)^2 \end{aligned}$$

Then with this choice of $\mathbf{c}$, it follows from 11.9 that

$$\left| \int_{\boldsymbol{\gamma}} \mathbf{f} \cdot d\boldsymbol{\gamma} \right| = \left| \int_{\boldsymbol{\gamma}} (\mathbf{f} - \mathbf{c}) \cdot d\boldsymbol{\gamma} \right|$$

$$\leq \frac{1}{2}\left(\sum_{k=1}^{n}\left(M_k - m_k\right)^2\right)^{1/2} V\left(\boldsymbol{\gamma}, [a,b]\right)$$

This proves the lemma.

It turns out to be much easier to evaluate such integrals in the case where $\boldsymbol{\gamma}$ is also $C^1\left([a,b]\right)$. The following theorem about approximation will be very useful but first here is an easy lemma.

Lemma 11.2.5 *Let $\boldsymbol{\gamma} : [a,b] \to \mathbb{R}^n$ be in $C^1\left([a,b]\right)$. Then $V\left(\boldsymbol{\gamma}, [a,b]\right) < \infty$ so $\boldsymbol{\gamma}$ is of bounded variation.*

Proof: This follows from the following

$$\begin{aligned}
\sum_{j=1}^{n} \left|\boldsymbol{\gamma}\left(t_j\right) - \boldsymbol{\gamma}\left(t_{j-1}\right)\right| &= \sum_{j=1}^{n} \left|\int_{t_{j-1}}^{t_j} \boldsymbol{\gamma}'(s)\, ds\right| \\
&\leq \sum_{j=1}^{n} \int_{t_{j-1}}^{t_j} \left|\boldsymbol{\gamma}'(s)\right| ds \\
&\leq \sum_{j=1}^{n} \int_{t_{j-1}}^{t_j} \left|\left|\boldsymbol{\gamma}'\right|\right|_\infty ds \\
&= \left|\left|\boldsymbol{\gamma}'\right|\right|_\infty (b-a).
\end{aligned}$$

where

$$\left|\left|\boldsymbol{\gamma}'\right|\right|_\infty \equiv \max\left\{\left|\boldsymbol{\gamma}'(t)\right| : t \in [a,b]\right\}.$$

Therefore it follows $V\left(\boldsymbol{\gamma}, [a,b]\right) \leq \left|\left|\boldsymbol{\gamma}'\right|\right|_\infty (b-a)$.

The following is a useful theorem for reducing bounded variation curves to ones which have a C^1 parameterization.

Theorem 11.2.6 *Let $\boldsymbol{\gamma} : [a,b] \to \mathbb{R}^n$ be continuous and of bounded variation. Let Ω be an open set containing $\boldsymbol{\gamma}^*$ and let $\mathbf{f} : \Omega \to \mathbb{R}^n$ be continuous, and let $\varepsilon > 0$ be given. Then there exists $\boldsymbol{\eta} : [a,b] \to \mathbb{R}^n$ such that $\boldsymbol{\eta}(a) = \boldsymbol{\gamma}(a)$, $\boldsymbol{\gamma}(b) = \boldsymbol{\eta}(b)$, $\boldsymbol{\eta} \in C^1\left([a,b]\right)$, and*

$$\left|\left|\boldsymbol{\gamma} - \boldsymbol{\eta}\right|\right| < \varepsilon, \tag{11.12}$$

where $\left|\left|\boldsymbol{\gamma} - \boldsymbol{\eta}\right|\right| \equiv \max\left\{\left|\boldsymbol{\gamma}(t) - \boldsymbol{\eta}(t)\right| : t \in [a,b]\right\}$. Also

$$\left|\int_{\boldsymbol{\gamma}} \mathbf{f} \cdot d\boldsymbol{\gamma} - \int_{\boldsymbol{\eta}} \mathbf{f} \cdot d\boldsymbol{\eta}\right| < \varepsilon, \tag{11.13}$$

$$V\left(\boldsymbol{\eta}, [a,b]\right) \leq V\left(\boldsymbol{\gamma}, [a,b]\right), \tag{11.14}$$

Proof: Extend $\boldsymbol{\gamma}$ to be defined on all $\mathbb{R}$ according to the rule $\boldsymbol{\gamma}(t) = \boldsymbol{\gamma}(a)$ if $t < a$ and $\boldsymbol{\gamma}(t) = \boldsymbol{\gamma}(b)$ if $t > b$. Now define

$$\boldsymbol{\gamma}_h(t) \equiv \frac{1}{2h} \int_{-2h + t + \frac{2h}{(b-a)}(t-a)}^{t + \frac{2h}{(b-a)}(t-a)} \boldsymbol{\gamma}(s)\, ds.$$

where the integral is defined in the obvious way, that is componentwise. Since $\boldsymbol{\gamma}$ is continuous, this is certainly possible. Then

$$\boldsymbol{\gamma}_h(b) \equiv \frac{1}{2h} \int_b^{b+2h} \boldsymbol{\gamma}(s)\, ds = \frac{1}{2h} \int_b^{b+2h} \boldsymbol{\gamma}(b)\, ds = \boldsymbol{\gamma}(b),$$

$$\boldsymbol{\gamma}_h(a) \equiv \frac{1}{2h}\int_{a-2h}^{a} \boldsymbol{\gamma}(s)\,ds = \frac{1}{2h}\int_{a-2h}^{a} \boldsymbol{\gamma}(a)\,ds = \boldsymbol{\gamma}(a).$$

Also, because of continuity of $\boldsymbol{\gamma}$ and the fundamental theorem of calculus,

$$\boldsymbol{\gamma}_h'(t) = \frac{1}{2h}\left\{\boldsymbol{\gamma}\left(t+\frac{2h}{b-a}(t-a)\right)\left(1+\frac{2h}{b-a}\right) - \boldsymbol{\gamma}\left(-2h+t+\frac{2h}{b-a}(t-a)\right)\left(1+\frac{2h}{b-a}\right)\right\}$$

and so $\boldsymbol{\gamma}_h \in C^1([a,b])$. The following lemma is significant.

Lemma 11.2.7 *$V(\boldsymbol{\gamma}_h,[a,b]) \le V(\boldsymbol{\gamma},[a,b])$.*

Proof: Let $a = t_0 < t_1 < \cdots < t_n = b$. Then using the definition of $\boldsymbol{\gamma}_h$ and changing the variables to make all integrals over $[0,2h]$,

$$\sum_{j=1}^{n} |\boldsymbol{\gamma}_h(t_j) - \boldsymbol{\gamma}_h(t_{j-1})| =$$

$$\sum_{j=1}^{n}\left|\frac{1}{2h}\int_0^{2h}\left[\boldsymbol{\gamma}\left(s-2h+t_j+\frac{2h}{b-a}(t_j-a)\right) - \boldsymbol{\gamma}\left(s-2h+t_{j-1}+\frac{2h}{b-a}(t_{j-1}-a)\right)\right]\right|$$

$$\le \frac{1}{2h}\int_0^{2h}\sum_{j=1}^{n}\left|\boldsymbol{\gamma}\left(s-2h+t_j+\frac{2h}{b-a}(t_j-a)\right) - \boldsymbol{\gamma}\left(s-2h+t_{j-1}+\frac{2h}{b-a}(t_{j-1}-a)\right)\right| ds.$$

For a given $s \in [0,2h]$, the points, $s-2h+t_j+\frac{2h}{b-a}(t_j-a)$ for $j=1,\cdots,n$ form an increasing list of points in the interval $[a-2h,b+2h]$ and so the integrand is bounded above by $V(\boldsymbol{\gamma},[a-2h,b+2h]) = V(\boldsymbol{\gamma},[a,b])$. It follows

$$\sum_{j=1}^{n} |\boldsymbol{\gamma}_h(t_j) - \boldsymbol{\gamma}_h(t_{j-1})| \le V(\boldsymbol{\gamma},[a,b])$$

which proves the lemma.

With this lemma the proof of the theorem can be completed without too much trouble. Let H be an open set containing $\boldsymbol{\gamma}^*$ such that $\overline{H}$ is a compact subset of Ω. Let $0 < \varepsilon < \operatorname{dist}\left(\boldsymbol{\gamma}^*, H^C\right)$. Then there exists δ_1 such that if $h < \delta_1$, then for all t,

$$\begin{aligned} |\boldsymbol{\gamma}(t) - \boldsymbol{\gamma}_h(t)| &\le \frac{1}{2h}\int_{-2h+t+\frac{2h}{(b-a)}(t-a)}^{t+\frac{2h}{(b-a)}(t-a)} |\boldsymbol{\gamma}(s) - \boldsymbol{\gamma}(t)|\,ds \\ &< \frac{1}{2h}\int_{-2h+t+\frac{2h}{(b-a)}(t-a)}^{t+\frac{2h}{(b-a)}(t-a)} \varepsilon ds = \varepsilon \end{aligned} \tag{11.15}$$

due to the uniform continuity of $\boldsymbol{\gamma}$. This proves 11.12.

Using the estimate from Theorem 11.2.3, 11.5, the uniform continuity of $\mathbf{f}$ on H, and the above lemma, there exists δ such that if $||\mathcal{P}|| < \delta$, then

$$\left|\int_{\boldsymbol{\gamma}} \mathbf{f}\cdot d\boldsymbol{\gamma}\,(t) - S(\mathcal{P})\right| < \frac{\varepsilon}{3},\ \left|\int_{\boldsymbol{\gamma}_h} \mathbf{f}\cdot d\boldsymbol{\gamma}_h\,(t) - S_h(\mathcal{P})\right| < \frac{\varepsilon}{3}$$

for all $h < 1$. Here $S(\mathcal{P})$ is a Riemann Stieltjes sum of the form

$$\sum_{i=1}^{n} \mathbf{f}(\boldsymbol{\gamma}(\tau_i)) \cdot (\boldsymbol{\gamma}(t_i) - \boldsymbol{\gamma}(t_{i-1}))$$

and $S_h(\mathcal{P})$ is a similar Riemann Stieltjes sum taken with respect to $\boldsymbol{\gamma}_h$ instead of $\boldsymbol{\gamma}$. Because of 11.15 $\boldsymbol{\gamma}_h(t)$ has values in $H \subseteq \Omega$. Therefore, fix the partition, $\mathcal{P}$, and choose h small enough that in addition to this, the following inequality is valid.

$$|S(\mathcal{P}) - S_h(\mathcal{P})| < \frac{\varepsilon}{3}$$

This is possible because of 11.15 and the uniform continuity of $\mathbf{f}$ on $\overline{H}$. It follows

$$\begin{aligned}\left|\int_{\boldsymbol{\gamma}} \mathbf{f}\cdot d\boldsymbol{\gamma}\,(t) - \int_{\boldsymbol{\gamma}_h} \mathbf{f}\cdot d\boldsymbol{\gamma}_h\,(t)\right| \leq \\ \left|\int_{\boldsymbol{\gamma}} \mathbf{f}\cdot d\boldsymbol{\gamma}\,(t) - S(\mathcal{P})\right| + |S(\mathcal{P}) - S_h(\mathcal{P})| \\ + \left|S_h(\mathcal{P}) - \int_{\boldsymbol{\gamma}_h} \mathbf{f}\cdot d\boldsymbol{\gamma}_h\,(t)\right| < \varepsilon.\end{aligned}$$

Let $\boldsymbol{\eta} \equiv \boldsymbol{\gamma}_h$. Formula 11.14 follows from the lemma. This proves the theorem.

This is a very useful theorem because if $\boldsymbol{\gamma}$ is $C^1([a,b])$, it is easy to calculate $\int_{\boldsymbol{\gamma}} \mathbf{f} d\boldsymbol{\gamma}$ and the above theorem allows a reduction to the case where $\boldsymbol{\gamma}$ is C^1. The next theorem shows how easy it is to compute these integrals in the case where $\boldsymbol{\gamma}$ is C^1. First note that if $\mathbf{f}$ is continuous and $\boldsymbol{\gamma} \in C^1([a,b])$, then by Lemma 11.2.5 and the fundamental existence theorem, Theorem 11.2.3, $\int_{\boldsymbol{\gamma}} \mathbf{f}\cdot d\boldsymbol{\gamma}$ exists.

Theorem 11.2.8 *If $\mathbf{f} : \boldsymbol{\gamma}^* \to X$ is continuous and $\boldsymbol{\gamma} : [a,b] \to \mathbb{R}^n$ is in $C^1([a,b])$, then*

$$\int_{\boldsymbol{\gamma}} \mathbf{f}\cdot d\boldsymbol{\gamma} = \int_a^b \mathbf{f}(\boldsymbol{\gamma}(t)) \cdot \boldsymbol{\gamma}'(t)\, dt. \tag{11.16}$$

Proof: Let $\mathcal{P}$ be a partition of $[a,b]$, $\mathcal{P} = \{t_0, \cdots, t_n\}$ and $||\mathcal{P}||$ is small enough that whenever $|t-s| < ||\mathcal{P}||$,

$$|\mathbf{f}(\boldsymbol{\gamma}(t)) - \mathbf{f}(\boldsymbol{\gamma}(s))| < \varepsilon \tag{11.17}$$

and

$$\left|\int_{\boldsymbol{\gamma}} \mathbf{f}\cdot d\boldsymbol{\gamma} - \sum_{j=1}^{n} \mathbf{f}(\boldsymbol{\gamma}(\tau_j)) \cdot (\boldsymbol{\gamma}(t_j) - \boldsymbol{\gamma}(t_{j-1}))\right| < \varepsilon.$$

Now

$$\begin{aligned}&\sum_{j=1}^{n} \mathbf{f}(\boldsymbol{\gamma}(\tau_j)) \cdot (\boldsymbol{\gamma}(t_j) - \boldsymbol{\gamma}(t_{j-1})) \\ = &\int_a^b \sum_{j=1}^{n} \mathbf{f}(\boldsymbol{\gamma}(\tau_j)) \cdot \mathcal{X}_{[t_{j-1},t_j]}(s)\, \boldsymbol{\gamma}'(s)\, ds\end{aligned}$$

where here

$$\mathcal{X}_{[p,q]}(s) \equiv \begin{cases} 1 \text{ if } s \in [p,q] \\ 0 \text{ if } s \notin [p,q] \end{cases}.$$

Also,

$$\int_a^b \mathbf{f}(\boldsymbol{\gamma}(s)) \cdot \boldsymbol{\gamma}'(s)\, ds = \int_a^b \sum_{j=1}^n \mathbf{f}(\boldsymbol{\gamma}(s)) \cdot \mathcal{X}_{[t_{j-1},t_j]}(s)\, \boldsymbol{\gamma}'(s)\, ds$$

and thanks to 11.17,

$$\left| \overbrace{\int_a^b \sum_{j=1}^n \mathbf{f}(\boldsymbol{\gamma}(\tau_j)) \cdot \mathcal{X}_{[t_{j-1},t_j]}(s)\, \boldsymbol{\gamma}'(s)\, ds}^{=\sum_{j=1}^n \mathbf{f}(\boldsymbol{\gamma}(\tau_j))\cdot(\boldsymbol{\gamma}(t_j)-\boldsymbol{\gamma}(t_{j-1}))} - \overbrace{\int_a^b \sum_{j=1}^n \mathbf{f}(\boldsymbol{\gamma}(s)) \cdot \mathcal{X}_{[t_{j-1},t_j]}(s)\, \boldsymbol{\gamma}'(s)\, ds}^{=\int_a^b \mathbf{f}(\boldsymbol{\gamma}(s))\cdot\boldsymbol{\gamma}'(s)ds} \right|$$

$$\begin{aligned} &\leq \sum_{j=1}^n \int_{t_{j-1}}^{t_j} |\mathbf{f}(\boldsymbol{\gamma}(\tau_j)) - \mathbf{f}(\boldsymbol{\gamma}(s))|\, |\boldsymbol{\gamma}'(s)|\, ds \\ &\leq ||\boldsymbol{\gamma}'||_\infty \sum_j \varepsilon (t_j - t_{j-1}) \\ &= \varepsilon\, ||\boldsymbol{\gamma}'||_\infty (b-a). \end{aligned}$$

It follows that

$$\begin{aligned} &\left| \int_{\boldsymbol{\gamma}} \mathbf{f} \cdot d\boldsymbol{\gamma} - \int_a^b \mathbf{f}(\boldsymbol{\gamma}(s)) \cdot \boldsymbol{\gamma}'(s)\, ds \right| \\ \leq\ &\left| \int_{\boldsymbol{\gamma}} \mathbf{f} \cdot d\boldsymbol{\gamma} - \sum_{j=1}^n \mathbf{f}(\boldsymbol{\gamma}(\tau_j)) \cdot (\boldsymbol{\gamma}(t_j) - \boldsymbol{\gamma}(t_{j-1})) \right| \\ &+ \left| \sum_{j=1}^n \mathbf{f}(\boldsymbol{\gamma}(\tau_j)) \cdot (\boldsymbol{\gamma}(t_j) - \boldsymbol{\gamma}(t_{j-1})) - \int_a^b \mathbf{f}(\boldsymbol{\gamma}(s)) \cdot \boldsymbol{\gamma}'(s)\, ds \right| \\ \leq\ &\varepsilon\, ||\boldsymbol{\gamma}'||_\infty (b-a) + \varepsilon. \end{aligned}$$

Since ε is arbitrary, this verifies 11.16.

You can piece bounded variation curves together to get another bounded variation curve. You can also take the integral in the opposite direction along a given curve. There is also something called a potential.

Definition 11.2.9 *A function $\mathbf{f} : \Omega \to \mathbb{R}^n$ for Ω an open set in $\mathbb{R}^n$ has a potential if there exists a function, F, the potential, such that $\nabla F = \mathbf{f}$. Also if $\boldsymbol{\gamma}_k : [a_k, b_k] \to \mathbb{R}^n$ is continuous and of bounded variation, for $k = 1, \cdots, m$ and $\boldsymbol{\gamma}_k(b_k) = \boldsymbol{\gamma}_{k+1}(a_k)$, define*

$$\int_{\sum_{k=1}^m \boldsymbol{\gamma}_k} \mathbf{f} \cdot d\boldsymbol{\gamma}_k \equiv \sum_{k=1}^m \int_{\boldsymbol{\gamma}_k} \mathbf{f} \cdot d\boldsymbol{\gamma}_k. \tag{11.18}$$

In addition to this, for $\boldsymbol{\gamma} : [a,b] \to \mathbb{R}^n$, *define* $-\boldsymbol{\gamma} : [a,b] \to \mathbb{R}^n$ *by* $-\boldsymbol{\gamma}(t) \equiv \boldsymbol{\gamma}(b+a-t)$. *Thus* $\boldsymbol{\gamma}$ *simply traces out the points of* $\boldsymbol{\gamma}^*$ *in the opposite order.*

The following lemma is useful and follows quickly from Theorem 11.2.2.

Lemma 11.2.10 *In the above definition, there exists a continuous bounded variation function,* $\boldsymbol{\gamma}$ *defined on some closed interval,* $[c,d]$, *such that* $\boldsymbol{\gamma}([c,d]) = \cup_{k=1}^m \boldsymbol{\gamma}_k([a_k,b_k])$ *and* $\boldsymbol{\gamma}(c) = \boldsymbol{\gamma}_1(a_1)$ *while* $\boldsymbol{\gamma}(d) = \boldsymbol{\gamma}_m(b_m)$. *Furthermore,*

$$\int_{\boldsymbol{\gamma}} \mathbf{f} \cdot d\boldsymbol{\gamma} = \sum_{k=1}^m \int_{\boldsymbol{\gamma}_k} \mathbf{f} \cdot d\boldsymbol{\gamma}_k.$$

If $\boldsymbol{\gamma} : [a,b] \to \mathbb{R}^n$ *is of bounded variation and continuous, then*

$$\int_{\boldsymbol{\gamma}} \mathbf{f} \cdot d\boldsymbol{\gamma} = -\int_{-\boldsymbol{\gamma}} \mathbf{f} \cdot d\boldsymbol{\gamma}.$$

The following theorem shows that it is very easy to compute a line integral when the function has a potential.

Theorem 11.2.11 *Let* $\boldsymbol{\gamma} : [a,b] \to \mathbb{R}^n$ *be continuous and of bounded variation. Also suppose* $\nabla F = \mathbf{f}$ *on* Ω, *an open set containing* $\boldsymbol{\gamma}^*$ *and* $\mathbf{f}$ *is continuous on* Ω. *Then*

$$\int_{\boldsymbol{\gamma}} \mathbf{f} \cdot d\boldsymbol{\gamma} = F(\boldsymbol{\gamma}(b)) - F(\boldsymbol{\gamma}(a)).$$

Proof: By Theorem 11.2.6 there exists $\boldsymbol{\eta} \in C^1([a,b])$ such that $\boldsymbol{\gamma}(a) = \boldsymbol{\eta}(a)$, and $\boldsymbol{\gamma}(b) = \boldsymbol{\eta}(b)$ such that

$$\left| \int_{\boldsymbol{\gamma}} \mathbf{f} \cdot d\boldsymbol{\gamma} - \int_{\boldsymbol{\eta}} \mathbf{f} \cdot d\boldsymbol{\eta} \right| < \varepsilon.$$

Then from Theorem 11.2.8, since $\boldsymbol{\eta}$ is in $C^1([a,b])$, it follows from the chain rule and the fundamental theorem of calculus that

$$\begin{aligned} \int_{\boldsymbol{\eta}} \mathbf{f} \cdot d\boldsymbol{\eta} &= \int_a^b \mathbf{f}(\boldsymbol{\eta}(t)) \boldsymbol{\eta}'(t)\, dt = \int_a^b \frac{d}{dt} F(\boldsymbol{\eta}(t))\, dt \\ &= F(\boldsymbol{\eta}(b)) - F(\boldsymbol{\eta}(a)) = F(\boldsymbol{\gamma}(b)) - F(\boldsymbol{\gamma}(a)). \end{aligned}$$

Therefore,

$$\left\| (F(\boldsymbol{\gamma}(b)) - F(\boldsymbol{\gamma}(a))) - \int_{\boldsymbol{\gamma}} \mathbf{f}(z)\, dz \right\| < \varepsilon$$

and since $\varepsilon > 0$ is arbitrary, this proves the theorem.

Corollary 11.2.12 *If* $\boldsymbol{\gamma} : [a,b] \to \mathbb{R}^n$ *is continuous, has bounded variation, is a closed curve,* $\boldsymbol{\gamma}(a) = \boldsymbol{\gamma}(b)$, *and* $\boldsymbol{\gamma}^* \subseteq \Omega$ *where* Ω *is an open set on which* $\nabla F = \mathbf{f}$, *then*

$$\int_{\boldsymbol{\gamma}} \mathbf{f} \cdot d\boldsymbol{\gamma} = 0.$$

Theorem 11.2.13 *Let* Ω *be a connected open set and let* $\mathbf{f} : \Omega \to \mathbb{R}^n$ *be continuous. Then* $\mathbf{f}$ *has a potential* F *if and only if*

$$\int_{\boldsymbol{\gamma}} \mathbf{f} \cdot d\boldsymbol{\gamma}$$

is path independent for all $\boldsymbol{\gamma}$ *a bounded variation curve such that* $\boldsymbol{\gamma}^*$ *is contained in* Ω. *This means the above line integral depends only on* $\boldsymbol{\gamma}(a)$ *and* $\boldsymbol{\gamma}(b)$.

Proof: The first part was proved in Theorem 11.2.11. It remains to verify the existence of a potential in the situation of path independence.

Let $x_0 \in \Omega$ be fixed. Let S be the points $\mathbf{x}$ of Ω which have the property there is a bounded variation curve joining $\mathbf{x}_0$ to $\mathbf{x}$. Let $\boldsymbol{\gamma}_{\mathbf{x}_0\mathbf{x}}$ denote such a curve. Note first that S is nonempty. To see this, $B(\mathbf{x}_0, r) \subseteq \Omega$ for r small enough. Every $\mathbf{x} \in B(\mathbf{x}_0, r)$ is in S. Then S is open because if $\mathbf{x} \in S$, then $B(\mathbf{x}, r) \subseteq \Omega$ for small enough r and if $\mathbf{y} \in B(\mathbf{x}, r)$, you could go take $\boldsymbol{\gamma}_{\mathbf{x}_0\mathbf{x}}$ and from $\mathbf{x}$ follow the straight line segment joining $\mathbf{x}$ to $\mathbf{y}$. In addition to this, $\Omega \setminus S$ must also be open because if $\mathbf{x} \in \Omega \setminus S$, then choosing $B(\mathbf{x}, r) \subseteq \Omega$, no point of $B(\mathbf{x}, r)$ can be in S because then you could take the straight line segment from that point to $\mathbf{x}$ and conclude that $\mathbf{x} \in S$ after all. Therefore, since Ω is connected, it follows $\Omega \setminus S = \emptyset$. Thus for every $\mathbf{x} \in S$, there exists $\boldsymbol{\gamma}_{\mathbf{x}_0\mathbf{x}}$, a bounded variation curve from $\mathbf{x}_0$ to $\mathbf{x}$.

Define

$$F(\mathbf{x}) \equiv \int_{\boldsymbol{\gamma}_{\mathbf{x}_0\mathbf{x}}} \mathbf{f} \cdot d\boldsymbol{\gamma}_{\mathbf{x}_0\mathbf{x}}$$

F is well defined by assumption. Now let $l_{\mathbf{x}(\mathbf{x}+t\mathbf{e}_k)}$ denote the linear segment from $\mathbf{x}$ to $\mathbf{x} + t\mathbf{e}_k$. Thus to get to $\mathbf{x} + t\mathbf{e}_k$ you could first follow $\boldsymbol{\gamma}_{\mathbf{x}_0\mathbf{x}}$ to $\mathbf{x}$ and from there follow $l_{\mathbf{x}(\mathbf{x}+t\mathbf{e}_k)}$ to $\mathbf{x} + t\mathbf{e}_k$. Hence

$$\frac{F(\mathbf{x}+t\mathbf{e}_k) - F(\mathbf{x})}{t} = \frac{1}{t}\int_{l_{\mathbf{x}(\mathbf{x}+t\mathbf{e}_k)}} \mathbf{f} \cdot dl_{\mathbf{x}(\mathbf{x}+t\mathbf{e}_k)}$$

$$= \frac{1}{t}\int_0^t \mathbf{f}(\mathbf{x} + s\mathbf{e}_k) \cdot \mathbf{e}_k ds \to f_k(\mathbf{x})$$

by continuity of $\mathbf{f}$. Thus $\nabla F = \mathbf{f}$ and this proves the theorem.

Corollary 11.2.14 *Let Ω be a connected open set and $\mathbf{f} : \Omega \to \mathbb{R}^n$. Then $\mathbf{f}$ has a potential if and only if every closed, $\boldsymbol{\gamma}(a) = \boldsymbol{\gamma}(b)$, bounded variation curve has the property that*

$$\int_{\boldsymbol{\gamma}} \mathbf{f} \cdot d\boldsymbol{\gamma} = 0$$

Proof: Using Lemma 11.2.10, this condition about closed curves is equivalent to the condition that the line integrals of the above theorem are path independent. This proves the corollary.

Such a vector valued function is called conservative.

11.3 Simple Closed Rectifiable Curves

There are examples of space filling continuous curves. However, bounded variation curves are not like this. In fact, one can even say the two dimensional Lebesgue measure of a bounded variation curve is 0.

Theorem 11.3.1 *Let $\boldsymbol{\gamma} : [a, b] \to \boldsymbol{\gamma}^* \subseteq \mathbb{R}^n$ where $n \geq 2$ is a continuous bounded variation curve. Then*

$$m_n(\boldsymbol{\gamma}^*) = 0$$

where m_n denotes n dimensional Lebesgue measure.

Proof: Let $\varepsilon > 0$ be given. Let $t_0 \equiv a$ and if $t_0, \cdots, t_k$ have been chosen, let t_{k+1} be the first number larger than t_k such that

$$|\boldsymbol{\gamma}(t_{k+1}) - \boldsymbol{\gamma}(t_k)| = \varepsilon.$$

If the set of t such that $|\boldsymbol{\gamma}(t) - \boldsymbol{\gamma}(t_k)| = \varepsilon$ is nonempty, then this set is clearly closed and so such a t_{k+1} exists until k is such that

$$\boldsymbol{\gamma}^* \subseteq \cup_{j=0}^{k} B(\boldsymbol{\gamma}(t_j), \varepsilon)$$

Let m be the last index of this process where t_{m+1} does not exist. How large is m? This can be estimated because

$$V(\boldsymbol{\gamma}, [a,b]) \geq \sum_{k=0}^{m} |\boldsymbol{\gamma}(t_{k+1}) - \boldsymbol{\gamma}(t_k)| = m\varepsilon$$

and so $m \leq V(\boldsymbol{\gamma}, [a,b])/\varepsilon$. Since $\boldsymbol{\gamma}^* \subseteq \cup_{j=0}^{m} B(\boldsymbol{\gamma}(t_j), \varepsilon)$,

$$\begin{aligned} m_n(\boldsymbol{\gamma}^*) &\leq \sum_{j=0}^{m} m_n(B(\boldsymbol{\gamma}(t_j), \varepsilon)) \\ &\leq \frac{V(\boldsymbol{\gamma}, [a,b])}{\varepsilon} c_n \varepsilon^n = c_n V(\boldsymbol{\gamma}, [a,b]) \varepsilon^{n-1} \end{aligned}$$

Since ε was arbitrary, this proves the theorem.

Since a ball has positive measure, this proves the following corollary.

Corollary 11.3.2 *Let $\boldsymbol{\gamma} : [a,b] \to \boldsymbol{\gamma}^* \subseteq \mathbb{R}^n$ where $n \geq 2$ is a continuous bounded variation curve. Then $\boldsymbol{\gamma}^*$ has empty interior.*

Lemma 11.3.3 *Let Γ be a simple closed curve. Then there exists a mapping $\boldsymbol{\theta} : C \to \Gamma$ where C is the unit circle*

$$\left\{(x,y) : x^2 + y^2 = 1\right\},$$

such that $\boldsymbol{\theta}$ is one to one and continuous.

Proof: Since Γ is a simple closed curve, there is a parameterization $\boldsymbol{\gamma}$ and an interval $[a,b]$ such that $\boldsymbol{\gamma}$ is continuous and one to one on (a,b) and $\boldsymbol{\gamma}(a) = \boldsymbol{\gamma}(b)$. Also $\boldsymbol{\gamma}(t) \neq \boldsymbol{\gamma}(a) = \boldsymbol{\gamma}(b)$ if $t \neq a$ and if $t \neq b$. Define $\boldsymbol{\theta}^{-1} : \Gamma \to C$ by

$$\boldsymbol{\theta}^{-1}(\mathbf{x}) \equiv \left(\cos\left(\frac{2\pi}{b-a}\left(\boldsymbol{\gamma}^{-1}(\mathbf{x}) - a\right)\right), \sin\left(\frac{2\pi}{b-a}\left(\boldsymbol{\gamma}^{-1}(\mathbf{x}) - a\right)\right)\right)$$

Note that $\boldsymbol{\theta}^{-1}$ is onto C. The function is well defined because it sends the point $\boldsymbol{\gamma}(a) = \boldsymbol{\gamma}(b)$ to the same point, $(1,0)$. It is also one to one. To see this note $\boldsymbol{\gamma}^{-1}$ is one to one on $\Gamma \setminus \{\boldsymbol{\gamma}(a), \boldsymbol{\gamma}(b)\}$. What about the case where $\mathbf{x} \neq \boldsymbol{\gamma}(a) = \boldsymbol{\gamma}(b)$? Could $\boldsymbol{\theta}^{-1}(\mathbf{x}) = \boldsymbol{\theta}^{-1}(\boldsymbol{\gamma}(a))$? In this case, $\boldsymbol{\gamma}^{-1}(\mathbf{x})$ is in (a,b) while $\boldsymbol{\gamma}^{-1}(\boldsymbol{\gamma}(a)) = a$ so

$$\boldsymbol{\theta}^{-1}(\mathbf{x}) \neq \boldsymbol{\theta}^{-1}(\boldsymbol{\gamma}(a)) = (1,0).$$

Thus $\boldsymbol{\theta}^{-1}$ is one to one on Γ.

Why is $\boldsymbol{\theta}^{-1}$ continuous? Suppose $\mathbf{x}_n \to \boldsymbol{\gamma}(a) = \boldsymbol{\gamma}(b)$ first. Why does $\boldsymbol{\theta}^{-1}(\mathbf{x}_n) \to (1,0) = \boldsymbol{\theta}^{-1}(\boldsymbol{\gamma}(a))$? Let $\{\mathbf{x}_n\}$ denote any subsequence of the given sequence. Then by compactness of $[a,b]$ there exists a further subsequence, still denoted by $\mathbf{x}_n$ such that

$$\boldsymbol{\gamma}^{-1}(\mathbf{x}_n) \to t \in [a,b]$$

Hence by continuity of $\boldsymbol{\gamma}$, $\mathbf{x}_n \to \boldsymbol{\gamma}(t)$ and so $\boldsymbol{\gamma}(t)$ must equal $\boldsymbol{\gamma}(a) = \boldsymbol{\gamma}(b)$. It follows from the assumption of what a simple curve is that $t \in \{a,b\}$. Hence $\boldsymbol{\theta}^{-1}(\mathbf{x}_n)$ converges to either

$$\left(\cos\left(\frac{2\pi}{b-a}(a-a)\right), \sin\left(\frac{2\pi}{b-a}(a-a)\right)\right)$$

or

$$\left(\cos\left(\frac{2\pi}{b-a}(b-a)\right),\sin\left(\frac{2\pi}{b-a}(b-a)\right)\right)$$

but these are the same point. This has shown that if $\mathbf{x}_n \to \boldsymbol{\gamma}(a) = \boldsymbol{\gamma}(b)$, there is a subsequence such that $\boldsymbol{\theta}^{-1}(\mathbf{x}_n) \to \boldsymbol{\theta}^{-1}(\boldsymbol{\gamma}(a))$. Thus $\boldsymbol{\theta}^{-1}$ is continuous at $\boldsymbol{\gamma}(a) = \boldsymbol{\gamma}(b)$. Next suppose $\mathbf{x}_n \to \mathbf{x} \neq \boldsymbol{\gamma}(a) \equiv \mathbf{p}$. Then there exists $B(\mathbf{p}, r)$ such that for all n large enough, $\mathbf{x}_n$ and $\mathbf{x}$ are contained in the compact set $\Gamma \setminus B(\mathbf{p}, r) \equiv K$. Then $\boldsymbol{\gamma}$ is continuous and one to one on the compact set $\boldsymbol{\gamma}^{-1}(K) \subseteq (a, b)$ and so by Theorem 5.1.3 $\boldsymbol{\gamma}^{-1}$ is continuous on K. In particular it is continuous at $\mathbf{x}$ so $\boldsymbol{\theta}^{-1}(\mathbf{x}_n) \to \boldsymbol{\theta}^{-1}(\mathbf{x})$. This proves the lemma.

11.3.1 The Jordan Curve Theorem

The following theorem includes the Jordan Curve theorem, a major result for simple closed curves in the plane. In this theorem and in what follows U_i will denote the inside of a simple closed curve and U_o will denote the outside.

Theorem 11.3.4 *Let C denote the unit circle, $\{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$. Suppose $\boldsymbol{\gamma} : C \to \Gamma \subseteq \mathbb{R}^2$ is one to one onto and continuous. Then $\mathbb{R}^n \setminus \Gamma$ consists of two components, a bounded component (called the inside) U_i and an unbounded component (called the outside), U_o. Also the boundary of each of these two components of $\mathbb{R}^n \setminus \Gamma$ is Γ and Γ has empty interior.*

Proof: That $\mathbb{R}^n \setminus \Gamma$ consists of two components, U_o and U_i follows from the Jordan separation theorem. There is exactly one unbounded component because Γ is bounded and so U_i is defined as the bounded component. It remains to verify the assertion about Γ being the boundary. Let $\mathbf{x}$ be a limit point of U_i. Then it can't be in U_o because these are both open sets. Therefore, all the limit points of U_i are in $U_i \cup \Gamma$. Similarly all the limit points of U_o are in $U_o \cup \Gamma$. Thus $\partial U_i \subseteq \Gamma$ and $\partial U_o \subseteq \Gamma$.

I claim Γ has empty interior. This follows because by Theorem 5.1.3 on Page 84, $\boldsymbol{\gamma}$ and $\boldsymbol{\gamma}^{-1}$ must both be continuous since C is compact. Thus if B is an open ball contained in Γ, it follows from invariance of domain that $\boldsymbol{\gamma}^{-1}(B)$ is an open set in $\mathbb{R}^2$. But this needs to be contained in C which is a contradiction because C has empty interior obviously.

Now let $\mathbf{x} \in \mathbb{R}^n$ such that $\mathbf{x} \notin U_o \cup U_i$. Then $\mathbf{x} \in \Gamma$ and must be a limit point of either U_o or U_i since if this were not so, Γ would be forced to have nonempty interior. Hence $\overline{U_1} \cup \overline{U_2} = \mathbb{R}^2$. Next I will show $\partial U_i = \partial U_o$. Suppose then that

$$\mathbf{p} \in \partial U_i \setminus \partial U_o$$

Then $\mathbf{p} \notin U_o$ because $\mathbf{p} \in \partial U_i \subseteq \Gamma$ which is disjoint from U_o. Thus $\mathbf{p}$ is not in $\overline{U_o}$ because it is given to not be in ∂U_o. Hence there is a ball centered at $\mathbf{p}$, $B(\mathbf{p}, r)$ which contains no points of $\overline{U_o}$. Thus $B(\mathbf{p}, r) \subseteq \overline{U_i}$ and so $\mathbf{p}$ is an interior point of $\overline{U_i}$ which implies $\mathbf{p}$ is actually in U_i, a contradiction to $\mathbf{p} \in \partial U_i$. Thus $\partial U_i \setminus \partial U_o = \emptyset$ and a similar argument shows $\partial U_o \setminus \partial U_i = \emptyset$. Thus $\partial U_i = \partial U_o$ and so if $\mathbf{x} \in \Gamma$, then it is in either ∂U_i or ∂U_o and these are equal. Thus

$$\partial U_i = \partial U_o \subseteq \Gamma \subseteq \partial U_i \cup \partial U_o = \partial U_i = \partial U_o.$$

This proves the theorem.

The following lemma will be of importance in what follows. There are of course more general versions of this lemma. I am only presenting what will be needed. To say two sets are homeomorphic is to say there is a one to one continuous onto mapping from one to the other which also has continuous inverse. Clearly the statement that two sets are homeomorphic defines an equivalence relation.

Lemma 11.3.5 *In the situation of Theorem 11.3.4, let* Γ *be a simple closed curve and let* $\boldsymbol{\gamma}^*$ *be a straight line segment such that the open segment,* $\boldsymbol{\gamma}^*$ *without its endpoints, is contained in* U_i *such that the intersection of* $\boldsymbol{\gamma}^*$ *with* Γ *equals* $\{\mathbf{p},\mathbf{q}\}$. *Then this line segment divides* U_i *into two connected open sets which are the interiors of two simple closed curves.*

Proof: Suppose $\boldsymbol{\theta}$ is a one to one onto continuous mapping of C, the unit circle, to Γ. Say $\boldsymbol{\theta}(\mathbf{a}) = \mathbf{p}$ and $\boldsymbol{\theta}(\mathbf{b}) = \mathbf{q}$ where $\mathbf{a},\mathbf{b}$ are points of C. Then $\Gamma \cup \boldsymbol{\gamma}^*$ is homeomorphic to the $C \cup l^*$ where l^* is the straight line joining $\mathbf{a}$ and $\mathbf{b}$. This is easy to see because l^* is clearly homeomorphic to $\boldsymbol{\gamma}^*$ so all that is required is to extend $\boldsymbol{\theta}$ to all of $C \cup l^*$. By the Jordan separation theorem, $\mathbb{R}^2 \setminus (\Gamma \cup \boldsymbol{\gamma}^*)$ is the union of three disjoint open connected sets, exactly one of which is unbounded. This is because this is obviously true for the circle and secant line l^*. Also, denoting one of the arcs of C joining $\mathbf{a}$ to $\mathbf{b}$ by C_1 and the other arc by C_2, it follows $\boldsymbol{\theta}(C_j) \cup \boldsymbol{\gamma}^*$ is a simple closed curve because it is homeomorphic to the half of the circle $C_j \cup l^*$ which is clearly homeomorphic to the unit circle. Let $\Gamma_j \equiv \boldsymbol{\theta}(C_j)$. The two connected open sets are the insides of $\Gamma_1 \cup \boldsymbol{\gamma}^*$ and $\Gamma_2 \cup \boldsymbol{\gamma}^*$ respectively and $\Gamma \cup \boldsymbol{\gamma}^* = (\Gamma_1 \cup \boldsymbol{\gamma}^*) \cup (\Gamma_2 \cup \boldsymbol{\gamma}^*)$.

Here is why. Denote them by U_{1i} and U_{2i} respectively. First note they can have no point in common for if $\mathbf{x}$ were such a point, then since both of these sets are open connected components, this would require them to coincide. Hence $\mathbb{R}^2 \setminus (\Gamma \cup \boldsymbol{\gamma}^*)$ would only have two components, a bounded and an unbounded component contrary to what was shown above.

If $\mathbf{x}$ is a point of U_i which is not on $\boldsymbol{\gamma}^*$, why must it be in one of U_{1i} or U_{2i}? If it is in neither, then it is in $U_{1o} \cap U_{2o}$ the intersection of the two unbounded components. I claim this intersection equals U_o. To see this, note the unbounded component of $\mathbb{R}^2 \setminus (\Gamma \cup \boldsymbol{\gamma}^*)$ equals the unbounded component of $\mathbb{R}^2 \setminus \Gamma$ which equals U_o because $\boldsymbol{\gamma}^*$ is contained entirely in $U_i \cup \Gamma$. But the unbounded component of $\mathbb{R}^2 \setminus (\Gamma \cup \boldsymbol{\gamma}^*)$ equals everything which is not in the union of the interior components and the boundary. Thus this unbounded component equals

$$\begin{aligned} & ((\Gamma_1 \cup \boldsymbol{\gamma}^*) \cup (\Gamma_2 \cup \boldsymbol{\gamma}^*) \cup U_{1i} \cup U_{2i})^C \\ = \;& ((\Gamma_1 \cup \boldsymbol{\gamma}^*) \cup U_{1i})^C \cap ((\Gamma_2 \cup \boldsymbol{\gamma}^*) \cup U_{2i})^C \\ = \;& U_{1o} \cap U_{2o} \end{aligned}$$

Now this is a contradiction because $\mathbf{x} \in U_i$ and so is not in U_o. This proves the lemma.

The following lemma has to do with decomposing the inside and boundary of a simple closed rectifiable curve into small pieces. The argument is like one given in Apostol [3]. In doing this I will refer to a region as the union of a connected open set with its boundary. Also, two regions will be said to be non overlapping if they either have empty intersection or the intersection is contained in the intersection of their boundaries.The height of a set A equals $\sup\{|y_1 - y_2| : (x_1,y_1),(x_2,y_2) \in A\}$. The width of A will be defined similarly.

Lemma 11.3.6 *Let* Γ *be a simple closed rectifiable curve. Also let* $\delta > 0$ *be given such that* 2δ *is smaller than both the height and width of* Γ. *Then there exist finitely many non overlapping regions* $\{R_k\}_{k=1}^n$ *consisting of simple closed curves along with their interiors whose union equals* $U_i \cup \Gamma$. *These regions consist of two kinds, those contained in* U_i *and those with nonempty intersection with* Γ. *These latter regions are called "border" regions. The boundary of a border region consists of straight line segments parallel to the coordinate axes of the form* $x = m\delta$ *or* $y = k\delta$ *for* m,k *integers along with arcs from* Γ. *The regions contained in* U_i *consist of rectangles. Thus all of these regions have boundaries which are rectifiable simple closed curves. Also all regions are contained in a square having sides of length no more than* 2δ. *There are at most*

$$4\left(\frac{V(\Gamma)}{\delta} + 1\right)$$

border regions. The construction also yields an orientation for Γ *and for all these regions and the orientations for any segment shared by two regions are opposite.*

Proof: Let $\Gamma = \boldsymbol{\gamma}([a,b])$ where $\boldsymbol{\gamma} = (\boldsymbol{\gamma}_1, \boldsymbol{\gamma}_2)$. Let

$$y_1 \equiv \max\{\boldsymbol{\gamma}_2(t) : t \in [a,b]\}$$

and let

$$y_2 \equiv \min\{\boldsymbol{\gamma}_2(t) : t \in [a,b]\}.$$

Thus (x_1, y_1) is the "top" point of Γ while (x_2, y_2) is the "bottom" point of Γ. Consider the lines $y = y_1$ and $y = y_2$. By assumption $|y_1 - y_2| > 2\delta$. Consider the line l given by $y = m\delta$ where m is chosen to make $m\delta$ as close as possible to $(y_1 + y_2)/2$. Thus $y_1 > m\delta > y_2$. By Theorem 11.3.4 (x_j, y_j) $j = 1, 2$ are neither of them interior points of Γ and so by Theorem 11.3.4 again, there exist points $\mathbf{p}_j \in U_i$ such that $\mathbf{p}_1$ is above l and $\mathbf{p}_2$ is below l. (Simply pick $\mathbf{p}_j$ very close to (x_j, y_j) and yet in U_i and this will take place.) Therefore, the horizontal line l must have nonempty intersection with U_i because U_i is connected. If it had empty intersection it would be possible to separate U_i into two nonempty open sets, one containing $\mathbf{p}_1$ and the other containing $\mathbf{p}_2$.

Let $\mathbf{q}$ be a point of U_i which is also in l. Then there exists a maximal segment of the line l containing $\mathbf{q}$ which is contained in $U_i \cup \Gamma$. This segment, $\boldsymbol{\gamma}^*$ satisfies the conditions of Lemma 11.3.5 and so it divides U_i into disjoint open connected sets whose boundaries are simple rectifiable closed curves. Note the line segment has finite length. Letting Γ_j be the simple closed curve which contains $\mathbf{p}_j$, orient $\boldsymbol{\gamma}^*$ as part of Γ_2 such that motion is from right to left. As part of Γ_1 the motion along the curve is from left to right. By Proposition 11.1.7 this provides an orientation to each Γ_j. By Proposition 11.1.8 there exists an orientation for Γ which is consistent with these two orientations on the Γ_j.

Now do the same process to the two simple closed curves just obtained and continue till all regions have height less than 2δ. Each application of the process yields two new non overlapping regions of the desired sort in place of an earlier region of the desired sort except possibly the regions might have excessive height. The orientation of a new line segment in the construction is determined from the orientations of the simple closed curves obtained earlier. By Proposition 11.1.7 the orientations of the segments shared by two regions are opposite so the line integrals over these segments cancel. Eventually this process ends because all regions have "height" less than 2δ. The reason for this is that if it did not end, the curve Γ could not have finite total variation because there would exist an arbitrarily large number of non overlapping regions each of which have a pair of points which are farther apart than 2δ. This takes care of finding the subregions so far as height is concerned.

Now follow the same process just described on each of the non overlapping "short" regions just obtained using vertical rather than horizontal lines, letting the orientation of the vertical edges be determined from the orientation already obtained, but this time feature width instead of height and let the lines be vertical of the form $x = k\delta$ where k is an integer.

How many border regions are there? Denote by $V(\Gamma)$ the length of Γ. Now decompose Γ into N arcs of length δ with maybe one having length less than δ. Thus $N - 1 \leq \frac{V(\Gamma)}{\delta}$ and so

$$N \leq \frac{V(\Gamma)}{\delta} + 1$$

Each of these N arcs can't intersect any more than four of the boxes in the construction. Therefore, at most $4N$ boxes of the construction can intersect Γ. Thus there are no more than

$$4\left(\frac{V(\Gamma)}{\delta} + 1\right)$$

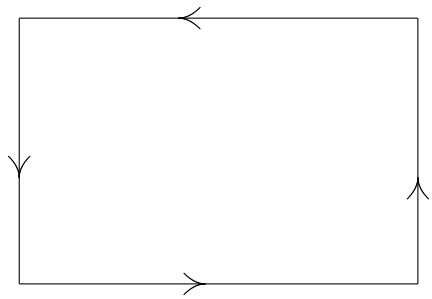
border regions. This proves the lemma.

11.3.2 Orientation And Green's Formula

How do you describe the orientation of a simple closed rectifiable curve analytically? The above process did it but I want another way to identify this which is more geometrically appealing. For simple examples, this is not too hard but it becomes less obvious when you consider the general case. The problem is the simple closed curve could be very wriggly.

The orientation of a rectifiable simple closed curve will be defined in terms of a very important formula known as Green's formula. First I will present Green's formula for a rectangle. In this lemma, it is very easy to understand the orientation of the bounding curve. The direction of motion is counter clockwise. As described in Proposition 11.1.7 it suffices to describe a direction of motion along the curve using any two points.

Lemma 11.3.7 *Let $R = [a,b] \times [c,d]$ be a rectangle and let P, Q be functions which are C^1 in some open set containing R. Orient the boundary of R as shown in the following picture. This is called the counter clockwise direction or the positive orientation*



Then letting $\boldsymbol{\gamma}$ denote the oriented boundary of R as shown,

$$\int_R \left(Q_x(x,y) - P_y(x,y)\right) dm_2 = \int_{\boldsymbol{\gamma}} \mathbf{f} \cdot d\boldsymbol{\gamma}$$

where

$$\mathbf{f}(x,y) \equiv \left(P(x,y), Q(x,y)\right).$$

In this context the line integral is usually written using the notation

$$\int_{\partial R} P dx + Q dy.$$

Proof: This follows from direct computation. A parameterization for the bottom line of R is

$$\boldsymbol{\gamma}_B(t) = \left(a + t(b-a), c\right), \ t \in [0,1]$$

A parameterization for the top line of R with the given orientation is

$$\boldsymbol{\gamma}_T(t) = \left(b + t(a-b), d\right), \ t \in [0,1]$$

A parameterization for the line on the right side is

$$\boldsymbol{\gamma}_R(t) = \left(b, c + t(d-c)\right), \ t \in [0,1]$$

and a parameterization for the line on the left side is

$$\boldsymbol{\gamma}_L(t) = \left(a, d + t(c-d)\right), \ t \in [0,1]$$

Now it is time to do the computations using Theorem 11.2.8.

$$\begin{aligned}\int_{\boldsymbol{\gamma}} \mathbf{f}\cdot d\boldsymbol{\gamma} &= \int_0^1 P\left(a+t\left(b-a\right),c\right)\left(b-a\right)dt \\ &\quad + \int_0^1 P\left(b+t\left(a-b\right),d\right)\left(a-b\right)dt \\ + \int_0^1 Q\left(b,c+t\left(d-c\right)\right)\left(d-c\right)dt &+ \int_0^1 Q\left(a,d+t\left(c-d\right)\right)\left(c-d\right)dt\end{aligned}$$

Changing the variables and combining the integrals, this equals

$$= \int_a^b P\left(x,c\right)dx - \int_a^b P\left(x,d\right)dx + \int_c^d Q\left(b,y\right)dy - \int_c^d Q\left(a,y\right)dy$$

$$\begin{aligned}&= -\int_a^b\int_c^d P_y\left(x,y\right)dydx + \int_c^d\int_a^b Q_x\left(x,y\right)dxdy \\ &= \int_R \left(Q_x - P_y\right)dm_2\end{aligned}$$

By Fubini's theorem, Theorem 9.2.3 on Page 204. (To use this theorem you can extend the functions to equal 0 off R.) This proves the lemma.

Note that if the rectangle were oriented in the opposite way, you would get

$$\int_{\boldsymbol{\gamma}} \mathbf{f}\cdot d\boldsymbol{\gamma} = \int_R \left(P_y - Q_x\right)dm_2$$

With this lemma, it is possible to prove Green's theorem and also give an analytic criterion which will distinguish between different orientations of a simple closed rectifiable curve. First here is a discussion which amounts to a computation.

Let Γ be a rectifiable simple closed curve with inside U_i and outside U_o. Let $\{R_k\}_{k=1}^{n_\delta}$ denote the non overlapping regions of Lemma 11.3.6 all oriented as explained there and let Γ also be oriented as explained there. It could be shown that all the regions contained in U_i have positive orientation but this will not be fussed over here. What can be said with no fussing is that since the shared edges have opposite orientations, all these interior regions are either oriented positively or they are all oriented negatively.

Let $\mathcal{B}_\delta$ be the set of border regions and let $\mathcal{I}_\delta$ be the rectangles contained in U_i. Thus in taking the sum of the line integrals over the boundaries of the interior rectangles, the integrals over the "interior edges" cancel out and you are left with a line integral over the exterior edges of a polygon which is composed of the union of the squares in $\mathcal{I}_\delta$.

Now let $\mathbf{f}\left(x,y\right) = \left(P\left(x,y\right),Q\left(x,y\right)\right)$ be a vector field which is C^1 on U_i, and suppose also that both P_y and Q_x are in $L^1\left(U_i\right)$ and that P, Q are continuous on $U_i \cup \Gamma$. (An easy way to get all this to happen is to let P, Q be restrictions to $U_i \cup \Gamma$ of functions which are C^1 on some open set containing $U_i \cup \Gamma$.) Note that

$$\cup_{\delta>0}\left\{R : R \in \mathcal{I}_\delta\right\} = U_i$$

and that for

$$I_\delta \equiv \cup\left\{R : R \in \mathcal{I}_\delta\right\},$$

the following pointwise convergence holds.

$$\lim_{\delta\to 0} \mathcal{X}_{I_\delta}\left(\mathbf{x}\right) = \mathcal{X}_{U_i}\left(\mathbf{x}\right).$$

By the dominated convergence theorem,

$$\begin{aligned}\lim_{\delta\to 0}\int_{I_\delta}(Q_x-P_y)\,dm_2 &= \int_{U_i}(Q_x-P_y)\,dm_2\\ \lim_{\delta\to 0}\int_{I_\delta}(P_y-Q_x)\,dm_2 &= \int_{U_i}(P_y-Q_x)\,dm_2\end{aligned}$$

Let ∂R denote the boundary of R for R one of these regions of Lemma 11.3.6 oriented as described. Let $w_\delta(R)^2$ denote

$$\begin{aligned}&\left(\max\{Q(\mathbf{x}):\mathbf{x}\in\partial R\}-\min\{Q(\mathbf{x}):\mathbf{x}\in\partial R\}\right)^2\\ &+\left(\max\{P(\mathbf{x}):\mathbf{x}\in\partial R\}-\min\{P(\mathbf{x}):\mathbf{x}\in\partial R\}\right)^2\end{aligned}$$

By uniform continuity of P,Q on the compact set $U_i\cup\Gamma$, if δ is small enough, $w_\delta(R)<\varepsilon$ for all $R\in\mathcal{B}_\delta$. Then for $R\in\mathcal{B}_\delta$, it follows from Theorem 11.2.4

$$\left|\int_{\partial R}\mathbf{f}\cdot d\boldsymbol{\gamma}\right|\le\frac{1}{2}w_\delta(R)\left(V(\partial R)\right)<\varepsilon\left(V(\partial R)\right) \tag{11.19}$$

whenever δ is small enough. Always let δ be this small.

Also since the line integrals cancel on shared edges

$$\sum_{R\in\mathcal{I}_\delta}\int_{\partial R}\mathbf{f}\cdot d\boldsymbol{\gamma}+\sum_{R\in\mathcal{B}_\delta}\int_{\partial R}\mathbf{f}\cdot d\boldsymbol{\gamma}=\int_\Gamma\mathbf{f}\cdot d\boldsymbol{\gamma} \tag{11.20}$$

Consider the second sum on the left. From 11.19

$$\left|\sum_{R\in\mathcal{B}_\delta}\int_{\partial R}\mathbf{f}\cdot d\boldsymbol{\gamma}\right|\le\sum_{R\in\mathcal{B}_\delta}\left|\int_{\partial R}\mathbf{f}\cdot d\boldsymbol{\gamma}\right|\le\varepsilon\sum_{R\in\mathcal{B}_\delta}\left(V(\partial R)\right)$$

Denote by Γ_R the part of Γ which is contained in $R\in\mathcal{B}_\delta$ and $V(\Gamma_R)$ is its length. Then the above sum equals

$$\varepsilon\left(\sum_{R\in\mathcal{B}_\delta}V(\Gamma_R)+B_\delta\right)=\varepsilon\left(V(\Gamma)+B_\delta\right)$$

where B_δ is the sum of the lengths of the straight edges. This is easy to estimate. Recall from 11.3.6 there are no more than

$$4\left(\frac{V(\Gamma)}{\delta}+1\right)$$

of these border regions. Furthermore, the sum of the lengths of all four edges of one of these is no more than 8δ and so

$$B_\delta\le 4\left(\frac{V(\Gamma)}{\delta}+1\right)8\delta=32V(\Gamma)+32\delta.$$

Thus the absolute value of the second sum on the right in 11.20 is dominated by

$$\varepsilon\left(33V(\Gamma)+32\delta\right)$$

Since ε was arbitrary, this formula implies with Green's theorem proved above for squares

$$\int_\Gamma\mathbf{f}\cdot d\boldsymbol{\gamma}=\lim_{\delta\to 0}\sum_{R\in\mathcal{I}_\delta}\int_{\partial R}\mathbf{f}\cdot d\boldsymbol{\gamma}+\lim_{\delta\to 0}\sum_{R\in\mathcal{B}_\delta}\int_{\partial R}\mathbf{f}\cdot d\boldsymbol{\gamma}$$

$$= \lim_{\delta \to 0} \sum_{R \in \mathcal{I}_\delta} \int_{\partial R} \mathbf{f} \cdot d\boldsymbol{\gamma} = \lim_{\delta \to 0} \int_{I_\delta} \pm (Q_x - P_y)\, dm_2 = \int_{U_i} \pm (Q_x - P_y)\, dm_2$$

where the $\pm$ adusts for whether the interior rectangles are all oriented positively or all oriented negatively. This has proved the general form of Green's theorem which is stated in the following theorem.

Theorem 11.3.8 *Let Γ be a rectifiable simple closed curve in $\mathbb{R}^2$ having inside U_i and outside U_o. Let P, Q be functions with the property that*

$$Q_x, P_y \in L^1(U_i)$$

and P, Q are C^1 on U_i. Assume also P, Q are continuous on $\Gamma \cup U_i$. Then there exists an orientation for Γ (Remember there are only two.) such that for

$$\mathbf{f}(x,y) = (P(x,y), Q(x,y)),$$

$$\int_\Gamma \mathbf{f} \cdot d\boldsymbol{\gamma} = \int_{U_i} (Q_x - P_y)\, dm_2.$$

Proof: In the construction of the regions, an orientation was imparted to Γ. The above computation shows

$$\int_\Gamma \mathbf{f} \cdot d\boldsymbol{\gamma} = \int_{U_i} \pm (Q_x - P_y)\, dm_2$$

If the area integral equals

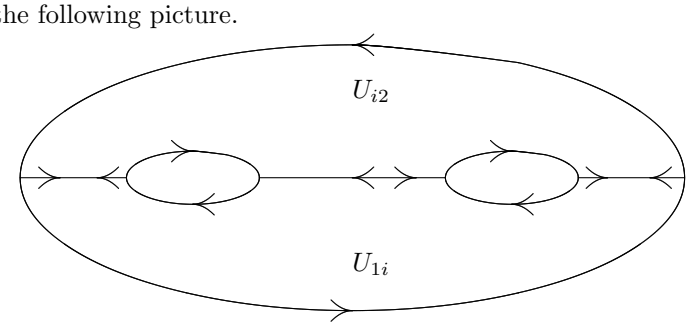
$$\int_{U_i} -(Q_x - P_y)\, dm_2,$$

just take the other orientation for Γ. This proves the theorem.

With this wonderful theorem, it is possible to give an analytic description of the two different orientations of a rectifiable simple closed curve. The positive orientation is the one for which Greens theorem holds and the other one, called the negative orientation is the one for which

$$\int_\Gamma \mathbf{f} \cdot d\boldsymbol{\gamma} = \int_{U_i} (P_y - Q_x)\, dm_2.$$

There are other regions for which Green's theorem holds besides just the inside and boundary of a simple closed curve. For Γ a simple closed curve and U_i its inside, lets refer to $U_i \cup \Gamma$ as a Jordan region. When you have two non overlapping Jordan regions which intersect in a finite number of simple curves, you can delete the interiors of these simple curves and what results will also be a region for which Green's theorem holds. This is illustrated in the following picture.



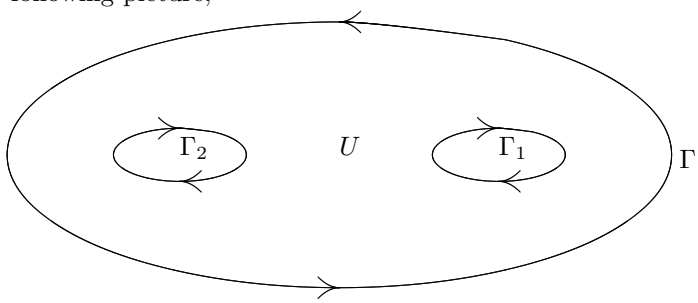
There are two Jordan regions here with insides U_{1i} and U_{2i} and these regions intersect in three simple curves. As indicated in the picture, opposite orientations are given to each

of these three simple curves. Then the line integrals over these cancel. The area integrals add. Recall the two dimensional area of a bounded variation curve equals 0.

Denote by Γ the curve on the outside of the whole thing and Γ_1 and Γ_2 the oriented boundaries of the two holes which result when the curves of intersection are removed, the orientations as shown. Then letting $\mathbf{f}(x,y) = (P(x,y), Q(x,y))$, and

$$U = U_{1i} \cup U_{2i} \cup \{\text{Open segments of intersection}\}$$

as shown in the following picture,



it follows from applying Green's theorem to both of the Jordan regions,

$$\begin{aligned}\int_\Gamma \mathbf{f}\cdot d\boldsymbol{\gamma} + \int_{\Gamma_1} \mathbf{f}\cdot d\boldsymbol{\gamma}_1 + \int_{\Gamma_2} \mathbf{f}\cdot d\gamma_2 &= \int_{U_{1i}\cup U_{2i}} (Q_x - P_y)\, dm_2 \\ &= \int_U (Q_x - P_y)\, dm_2\end{aligned}$$

To make this simpler, just write it in the form

$$\int_{\partial U} \mathbf{f}\cdot d\boldsymbol{\gamma} = \int_U (Q_x - P_y)\, dm_2$$

where ∂U is oriented as indicated in the picture and involves the three oriented curves $\Gamma, \Gamma_1, \Gamma_2$.

11.4 Stoke's Theorem

Stokes theorem is usually presented in calculus courses under far more restrictive assumptions than will be used here. It turns out that all the hard questions are related to Green's theorem and that when you have the general version of Green's theorem this can be used to obtain a general version of Stoke's theorem using a simple identity. This is because Stoke's theorem is really just a three dimensional version of the two dimensional Green's theorem. This will be made more precise below.

To begin with suppose Γ is a rectifiable curve in $\mathbb{R}^2$ having parameterization $\boldsymbol{\alpha} : [a,b] \to \Gamma$ for $\boldsymbol{\alpha}$ a continuous function. Let $\mathbf{R} : U \to \mathbb{R}^n$ be a C^1 function where U contains $\boldsymbol{\alpha}^*$. Then one could define a curve

$$\boldsymbol{\gamma}(t) \equiv \mathbf{R}(\boldsymbol{\alpha}(t)),\ t \in [a,b].$$

Lemma 11.4.1 *The curve $\boldsymbol{\gamma}^*$ where $\boldsymbol{\gamma}$ is as just described is a rectifiable curve. If $\mathbf{F}$ is defined and continuous on $\boldsymbol{\gamma}^*$ then*

$$\int_{\boldsymbol{\gamma}} \mathbf{F}\cdot d\boldsymbol{\gamma} = \int_{\boldsymbol{\alpha}} ((\mathbf{F}\circ\mathbf{R})\cdot\mathbf{R}_u, (\mathbf{F}\circ\mathbf{R})\cdot\mathbf{R}_v)\cdot d\boldsymbol{\alpha}$$

where $\mathbf{R}_u$ signifies the partial derivative of $\mathbf{R}$ with respect to the variable u.

Proof: Let

$$K \equiv \left\{ \mathbf{y} \in \mathbb{R}^2 : \mathrm{dist}\left(\mathbf{y}, \boldsymbol{\alpha}^*\right) \leq r \right\}$$

where r is small enough that $K \subseteq U$. This is easily done because $\boldsymbol{\alpha}^*$ is compact. Let

$$C_K \equiv \max\left\{ ||D\mathbf{R}\left(\mathbf{x}\right)|| : \mathbf{x} \in K \right\}$$

Consider

$$\sum_{j=0}^{n-1} \left|\mathbf{R}\left(\boldsymbol{\alpha}\left(t_{j+1}\right)\right) - \mathbf{R}\left(\boldsymbol{\alpha}\left(t_j\right)\right)\right| \tag{11.21}$$

where $\{t_0, \cdots, t_n\}$ is a partition of $[a,b]$. Since $\boldsymbol{\alpha}$ is continuous, there exists a δ such that if $||\mathcal{P}|| < \delta$, then the segment

$$\left\{ \boldsymbol{\alpha}\left(t_j\right) + s\left(\boldsymbol{\alpha}\left(t_{j+1}\right) - \boldsymbol{\alpha}\left(t_j\right)\right) : s \in [0,1] \right\}$$

is contained in K. Therefore, by the mean value inequality, Theorem 6.4.2,

$$\sum_{j=0}^{n-1} \left|\mathbf{R}\left(\boldsymbol{\alpha}\left(t_{j+1}\right)\right) - \mathbf{R}\left(\boldsymbol{\alpha}\left(t_j\right)\right)\right| \leq \sum_{j=0}^{n-1} C_K \left|\boldsymbol{\alpha}\left(t_{j+1}\right) - \boldsymbol{\alpha}\left(t_j\right)\right|$$

Now if $\mathcal{P}$ is any partition, 11.21 can always be made larger by adding in points to $\mathcal{P}$ till $||\mathcal{P}|| < \delta$ and so this shows

$$V\left(\boldsymbol{\gamma}, [a,b]\right) \leq C_K V\left(\boldsymbol{\alpha}, [a,b]\right).$$

This proves the first part.

Next consider the claim about the integral. Let

$$G\left(\mathbf{v}, \mathbf{x}\right) \equiv \mathbf{R}\left(\mathbf{x} + \mathbf{v}\right) - \mathbf{R}\left(\mathbf{x}\right) - D\mathbf{R}\left(\mathbf{x}\right)\left(\mathbf{v}\right).$$

Then

$$D_1 G\left(\mathbf{v}, \mathbf{x}\right) = D\mathbf{R}\left(\mathbf{x} + \mathbf{v}\right) - D\mathbf{R}\left(\mathbf{x}\right)$$

and so by uniform continuity of $D\mathbf{R}$ on the compact set K, it follows there exists $\delta > 0$ such that if $|\mathbf{v}| < \delta$, then for all $\mathbf{x} \in \boldsymbol{\alpha}^*$,

$$||D\mathbf{R}\left(\mathbf{x} + \mathbf{v}\right) - D\mathbf{R}\left(\mathbf{x}\right)|| = ||D_1 G\left(\mathbf{v}, \mathbf{x}\right)|| < \varepsilon.$$

By Theorem 6.4.2 again it follows that for all $\mathbf{x} \in \boldsymbol{\alpha}^*$ and $|\mathbf{v}| < \delta$,

$$\left|G\left(\mathbf{v}, \mathbf{x}\right)\right| = \left|\mathbf{R}\left(\mathbf{x} + \mathbf{v}\right) - \mathbf{R}\left(\mathbf{x}\right) - D\mathbf{R}\left(\mathbf{x}\right)\left(\mathbf{v}\right)\right| \leq \varepsilon \left|\mathbf{v}\right| \tag{11.22}$$

Letting $||\mathcal{P}||$ be small enough, it follows from the continuity of $\boldsymbol{\alpha}$ that

$$\left|\boldsymbol{\alpha}\left(t_{j+1}\right) - \boldsymbol{\alpha}\left(t_j\right)\right| < \delta$$

Therefore for such $\mathcal{P}$,

$$\sum_{j=0}^{n-1} \mathbf{F}\left(\boldsymbol{\gamma}\left(t_j\right)\right) \cdot \left(\boldsymbol{\gamma}\left(t_{j+1}\right) - \boldsymbol{\gamma}\left(t_j\right)\right)$$

$$= \sum_{j=0}^{n-1} \mathbf{F}\left(\mathbf{R}\left(\boldsymbol{\alpha}\left(t_j\right)\right)\right) \cdot \left(\mathbf{R}\left(\boldsymbol{\alpha}\left(t_{j+1}\right)\right) - \mathbf{R}\left(\boldsymbol{\alpha}\left(t_j\right)\right)\right)$$

$$= \sum_{j=0}^{n-1} \mathbf{F}(\mathbf{R}(\boldsymbol{\alpha}(t_j))) \cdot [D\mathbf{R}(\boldsymbol{\alpha}(t_j))(\boldsymbol{\alpha}(t_{j+1}) - \boldsymbol{\alpha}(t_j)) + \mathbf{o}(\boldsymbol{\alpha}(t_{j+1}) - \boldsymbol{\alpha}(t_j))]$$

where

$$\mathbf{o}(\boldsymbol{\alpha}(t_{j+1}) - \boldsymbol{\alpha}(t_j)) = \mathbf{R}(\boldsymbol{\alpha}(t_{j+1})) - \mathbf{R}(\boldsymbol{\alpha}(t_j)) - D\mathbf{R}(\boldsymbol{\alpha}(t_j))(\boldsymbol{\alpha}(t_{j+1}) - \boldsymbol{\alpha}(t_j))$$

and by 11.22,

$$|\mathbf{o}(\boldsymbol{\alpha}(t_{j+1}) - \boldsymbol{\alpha}(t_j))| < \varepsilon |\boldsymbol{\alpha}(t_{j+1}) - \boldsymbol{\alpha}(t_j)|$$

It follows

$$\left| \sum_{j=0}^{n-1} \mathbf{F}(\boldsymbol{\gamma}(t_j)) \cdot (\boldsymbol{\gamma}(t_{j+1}) - \boldsymbol{\gamma}(t_j)) - \sum_{j=0}^{n-1} \mathbf{F}(\mathbf{R}(\boldsymbol{\alpha}(t_j))) \cdot D\mathbf{R}(\boldsymbol{\alpha}(t_j))(\boldsymbol{\alpha}(t_{j+1}) - \boldsymbol{\alpha}(t_j)) \right| \tag{11.23}$$

$$\leq \sum_{j=0}^{n-1} |\mathbf{o}(\boldsymbol{\alpha}(t_{j+1}) - \boldsymbol{\alpha}(t_j))| \leq \sum_{j=0}^{n-1} \varepsilon |\boldsymbol{\alpha}(t_{j+1}) - \boldsymbol{\alpha}(t_j)| \leq \varepsilon V(\boldsymbol{\alpha}, [a,b])$$

Consider the second sum in 11.23. A term in the sum equals

$$\mathbf{F}(\mathbf{R}(\boldsymbol{\alpha}(t_j))) \cdot (\mathbf{R}_u(\boldsymbol{\alpha}(t_j))(\alpha_1(t_{j+1}) - \alpha_1(t_j)) + \mathbf{R}_v(\boldsymbol{\alpha}(t_j))(\alpha_2(t_{j+1}) - \alpha_2(t_j)))$$

$$= (\mathbf{F}(\mathbf{R}(\boldsymbol{\alpha}(t_j))) \cdot \mathbf{R}_u(\boldsymbol{\alpha}(t_j)), \mathbf{F}(\mathbf{R}(\boldsymbol{\alpha}(t_j))) \cdot \mathbf{R}_v(\boldsymbol{\alpha}(t_j))) \cdot (\boldsymbol{\alpha}(t_{j+1}) - \boldsymbol{\alpha}(t_j))$$

By continuity of $\mathbf{F}$, $\mathbf{R}_u$ and $\mathbf{R}_v$, it follows that sum converges as $||\mathcal{P}|| \to 0$ to

$$\int_{\boldsymbol{\alpha}} ((\mathbf{F} \circ \mathbf{R}) \cdot \mathbf{R}_u, (\mathbf{F} \circ \mathbf{R}) \cdot \mathbf{R}_v) \cdot d\boldsymbol{\alpha}$$

Therefore, taking the limit as $||\mathcal{P}|| \to 0$ in 11.23

$$\left| \int_{\boldsymbol{\gamma}} \mathbf{F} \cdot d\boldsymbol{\gamma} - \int_{\boldsymbol{\alpha}} ((\mathbf{F} \circ \mathbf{R}) \cdot \mathbf{R}_u, (\mathbf{F} \circ \mathbf{R}) \cdot \mathbf{R}_v) \cdot d\boldsymbol{\alpha} \right| < \varepsilon V(\boldsymbol{\alpha}, [a,b]).$$

Since $\varepsilon > 0$ is arbitrary, this proves the lemma.

The following is a little identity which will allow a proof of Stoke's theorem to follow from Green's theorem. First recall the following definition from calculus of the curl of a vector field and the cross product of two vectors from calculus.

Definition 11.4.2 *Let $\mathbf{u} \equiv (a,b,c)$ and $\mathbf{v} \equiv (d,e,f)$ be two vectors in $\mathbb{R}^3$. Then*

$$\mathbf{u} \times \mathbf{v} \equiv \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a & b & c \\ d & e & f \end{vmatrix}$$

*where the determinant is expanded formally along the top row. Let $\mathbf{f} : U \to \mathbb{R}^3$ for $U \subseteq \mathbb{R}^3$ denote a vector field. The **curl** of the vector field yields another vector field and it is defined as follows.*

$$(\operatorname{curl}(\mathbf{f})(\mathbf{x}))_i \equiv (\nabla \times \mathbf{f}(\mathbf{x}))_i$$

where here ∂_j means the partial derivative with respect to x_j and the subscript of i in $(\operatorname{curl}(\mathbf{f})(\mathbf{x}))_i$ means the i^{th} Cartesian component of the vector, $\operatorname{curl}(\mathbf{f})(\mathbf{x})$. Thus the curl is evaluated by expanding the following determinant along the top row.

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f_1(x,y,z) & f_2(x,y,z) & f_3(x,y,z) \end{vmatrix}.$$

Note the similarity with the cross product. More precisely and less evocatively,

$$\nabla \times \mathbf{f}(x,y,z) =\equiv \left(\frac{\partial F_3}{\partial y}-\frac{\partial F_2}{\partial z}\right)\mathbf{i}+\left(\frac{\partial F_1}{\partial z}-\frac{\partial F_3}{\partial x}\right)\mathbf{j}+\left(\frac{\partial F_2}{\partial x}-\frac{\partial F_1}{\partial y}\right)\mathbf{k}.$$

In the above, $\mathbf{i}=\mathbf{e}_1, \mathbf{j}=\mathbf{e}_2$, *and* $\mathbf{k}=\mathbf{e}_3$ *the standard unit basis vectors for* $\mathbb{R}^3$.

With this definition, here is the identity.

Lemma 11.4.3 *Let* $\mathbf{R}: U \to V \subseteq \mathbb{R}^3$ *where* U *is an open subset of* $\mathbb{R}^2$ *and* V *is an open subset of* $\mathbb{R}^3$. *Suppose* $\mathbf{R}$ *is* C^2 *and let* $\mathbf{F}$ *be a* C^1 *vector field defined in* V.

$$(\mathbf{R}_u \times \mathbf{R}_v)\cdot(\nabla\times\mathbf{F})(\mathbf{R}(u,v)) = ((\mathbf{F}\circ\mathbf{R})_u\cdot\mathbf{R}_v - (\mathbf{F}\circ\mathbf{R})_v\cdot\mathbf{R}_u)(u,v). \tag{11.24}$$

Proof: Letting x,y,z denote the components of $\mathbf{R}(\mathbf{u})$ and f_1,f_2,f_3 denote the components of $\mathbf{F}$, and letting a subscripted variable denote the partial derivative with respect to that variable, the left side of 11.24 equals

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ x_u & y_u & z_u \\ x_v & y_v & z_v \end{vmatrix} \cdot \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial_x & \partial_y & \partial_z \\ f_1 & f_2 & f_3 \end{vmatrix}$$

$$= (f_{3y}-f_{2z})(y_u z_v - z_u y_v) + (f_{1z}-f_{3x})(z_u x_v - x_u z_v) + (f_{2x}-f_{1y})(x_u y_v - y_u x_v)$$

$$\begin{aligned} &= f_{3y}y_u z_v + f_{2z}z_u y_v + f_{1z}z_u x_v + f_{3x}x_u z_v + f_{2x}x_u y_v + f_{1y}y_u x_v \\ &- (f_{2z}y_u z_v + f_{3y}z_u y_v + f_{1z}x_u z_v + f_{3x}z_u x_v + f_{2x}y_u x_v + f_{1y}x_u y_v) \end{aligned}$$

$$\begin{aligned} &= f_{1y}y_u x_v + f_{1z}z_u x_v + f_{2x}x_u y_v + f_{2z}z_u y_v + f_{3x}x_u z_v + f_{3y}y_u z_v \\ &- (f_{1y}y_v x_u + f_{1z}z_v x_u + f_{2x}x_v y_u + f_{2z}z_v y_u + f_{3x}x_v z_u + f_{3y}y_v z_u) \end{aligned}$$

At this point add in and subtract off certain terms. Then the above equals

$$\begin{aligned} &= f_{1x}x_u x_v + f_{1y}y_u x_v + f_{1z}z_u x_v + f_{2x}x_u y_v + f_{2x}y_u y_v \\ &\quad +f_{2z}z_u y_v + f_{3x}x_u z_v + f_{3y}y_u z_v + f_{3z}z_u z_v \\ &- \left(\begin{array}{c} f_{1x}x_v x_u + f_{1y}y_v x_u + f_{1z}z_v x_u + f_{2x}x_v y_u + f_{2x}y_v y_u \\ +f_{2z}z_v y_u + f_{3x}x_v z_u + f_{3y}y_v z_u + f_{3z}z_v z_u \end{array}\right) \end{aligned}$$

$$\begin{aligned} &= \frac{\partial f_1\circ\mathbf{R}(u,v)}{\partial u}x_v + \frac{\partial f_2\circ\mathbf{R}(u,v)}{\partial u}y_v + \frac{\partial f_3\circ\mathbf{R}(u,v)}{\partial u}z_v \\ &- \left(\frac{\partial f_1\circ\mathbf{R}(u,v)}{\partial v}x_u + \frac{\partial f_2\circ\mathbf{R}(u,v)}{\partial v}y_u + \frac{\partial f_3\circ\mathbf{R}(u,v)}{\partial v}z_u\right) \end{aligned}$$

$$= ((\mathbf{F}\circ\mathbf{R})_u\cdot\mathbf{R}_v - (\mathbf{F}\circ\mathbf{R})_v\cdot\mathbf{R}_u)(u,v).$$

This proves the lemma.

Let U be a region in $\mathbb{R}^2$ for which Green's theorem holds. Thus Green's theorem says that for P,Q continuous on $U_i\cup\Gamma, P_v, Q_u \in L^1(U_i\cup\Gamma), P,Q$ being C^1 on U_i,

$$\int_U (Q_u - P_v)\,dm_2 = \int_{\partial U}\mathbf{f}\cdot d\boldsymbol{\alpha}$$

where ∂U consists of some simple closed rectifiable oriented curves as explained above. Here the u and v axes are in the same relation as the x and y axes.

Theorem 11.4.4 *(Stoke's Theorem) Let U be any region in $\mathbb{R}^2$ for which the conclusion of Green's theorem holds. Let $\mathbf{R} \in C^2\left(\overline{U}, \mathbb{R}^3\right)$ be a one to one function. Let*

$$\boldsymbol{\gamma}_j = \mathbf{R} \circ \boldsymbol{\alpha}_j,$$

where the $\boldsymbol{\alpha}_j$ are parameterizations for the oriented curves making up the boundary of U such that the conclusion of Green's theorem holds. Let S denote the surface,

$$S \equiv \{\mathbf{R}(u,v) : (u,v) \in U\},$$

Then for $\mathbf{F}$ a C^1 vector field defined near S,

$$\sum_{i=1}^{n} \int_{\boldsymbol{\gamma}_i} \mathbf{F} \cdot d\boldsymbol{\gamma}_i = \int_U (\mathbf{R}_u(u,v) \times \mathbf{R}_v(u,v)) \cdot (\nabla \times \mathbf{F}(\mathbf{R}(u,v)))\, dm_2$$

Proof: By Lemma 11.4.1,

$$\sum_{j=1}^{n} \int_{\boldsymbol{\gamma}_j} \mathbf{F} \cdot d\boldsymbol{\gamma}_j =$$

$$\sum_{j=1}^{n} \int_{\boldsymbol{\alpha}_j} ((\mathbf{F} \circ \mathbf{R}) \cdot \mathbf{R}_u, (\mathbf{F} \circ \mathbf{R}) \cdot \mathbf{R}_v) \cdot d\boldsymbol{\alpha}_j$$

By the assumption that the conclusion of Green's theorem holds for U, this equals

$$\begin{aligned} & \int_U \left[((\mathbf{F} \circ \mathbf{R}) \cdot \mathbf{R}_v)_u - ((\mathbf{F} \circ \mathbf{R}) \cdot \mathbf{R}_u)_v\right] dm_2 \\ = & \int_U \left[(\mathbf{F} \circ \mathbf{R})_u \cdot \mathbf{R}_v + (\mathbf{F} \circ \mathbf{R}) \cdot \mathbf{R}_{vu} - (\mathbf{F} \circ \mathbf{R}) \cdot \mathbf{R}_{uv} - (\mathbf{F} \circ \mathbf{R})_v \cdot \mathbf{R}_u\right] dm_2 \\ = & \int_U \left[(\mathbf{F} \circ \mathbf{R})_u \cdot \mathbf{R}_v - (\mathbf{F} \circ \mathbf{R})_v \cdot \mathbf{R}_u\right] dm_2 \end{aligned}$$

the last step holding by equality of mixed partial derivatives, a result of the assumption that $\mathbf{R}$ is C^2. Now by Lemma 11.4.3, this equals

$$\int_U (\mathbf{R}_u(u,v) \times \mathbf{R}_v(u,v)) \cdot (\nabla \times \mathbf{F}(\mathbf{R}(u,v)))\, dm_2$$

This proves Stoke's theorem.

With approximation arguments one can remove the assumption that $\mathbf{R}$ is C^2 and replace this condition with weaker conditions. This is not surprising because in the final result, only first derivatives of $\mathbf{R}$ occur.

11.5 Interpretation And Review

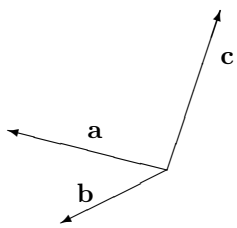
To understand the interpretation of Stoke's theorem in terms of an integral over the surface S, it is necessary to either do more theoretical development or to review some beginning calculus. I will do the latter here. First of all, it is important to understand the geometrical properties of the cross product. Those who have had a typical calculus course will probably not have seen this so I will present it here. It is elementary material which is a little out of place in an advanced calculus book but it is nevertheless useful and important and if you have not seen it, you should.

11.5.1 The Geometric Description Of The Cross Product

The cross product is a way of multiplying two vectors in $\mathbb{R}^3$. It is very different from the dot product in many ways. First the geometric meaning is discussed and then a description in terms of coordinates is given. Both descriptions of the cross product are important. The geometric description is essential in order to understand the applications to physics and geometry while the coordinate description is the only way to practically compute the cross product. In this presentation a vector is something which is characterized by direction and magnitude.

Definition 11.5.1 *Three vectors,* $\mathbf{a}, \mathbf{b}, \mathbf{c}$ *form a right handed system if when you extend the fingers of your right hand along the vector,* $\mathbf{a}$ *and close them in the direction of* $\mathbf{b}$*, the thumb points roughly in the direction of* $\mathbf{c}$*.*

For an example of a right handed system of vectors, see the following picture.



In this picture the vector $\mathbf{c}$ points upwards from the plane determined by the other two vectors. You should consider how a right hand system would differ from a left hand system. Try using your left hand and you will see that the vector, $\mathbf{c}$ would need to point in the opposite direction as it would for a right hand system.

From now on, the vectors, $\mathbf{i}, \mathbf{j}, \mathbf{k}$ will **always** form a right handed system. To repeat, if you extend the fingers of your right hand along $\mathbf{i}$ and close them in the direction $\mathbf{j}$, the thumb points in the direction of $\mathbf{k}$. Recall these are the basis vectors $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$.

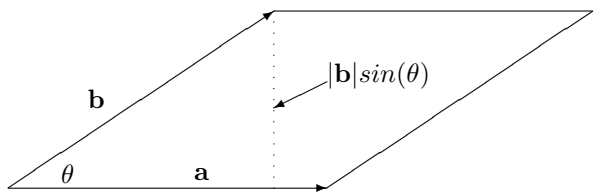
The following is the geometric description of the cross product. It gives both the direction and the magnitude and therefore specifies the vector.

Definition 11.5.2 *Let* $\mathbf{a}$ *and* $\mathbf{b}$ *be two vectors in* $\mathbb{R}^3$*. Then* $\mathbf{a} \times \mathbf{b}$ *is defined by the following two rules.*

1. $|\mathbf{a} \times \mathbf{b}| = |\mathbf{a}|\,|\mathbf{b}| \sin\theta$ *where* θ *is the included angle.*

2. $\mathbf{a} \times \mathbf{b} \cdot \mathbf{a} = 0$, $\mathbf{a} \times \mathbf{b} \cdot \mathbf{b} = 0$, *and* $\mathbf{a}, \mathbf{b}, \mathbf{a} \times \mathbf{b}$ *forms a right hand system.*

Note that $|\mathbf{a} \times \mathbf{b}|$ is the **area of the parallelogram** spanned by $\mathbf{a}$ and $\mathbf{b}$.



The cross product satisfies the following properties.

$$\mathbf{a}\times\mathbf{b} = -(\mathbf{b}\times\mathbf{a}) \ , \ \mathbf{a}\times\mathbf{a} = \mathbf{0}, \tag{11.25}$$

For α a scalar,

$$(\alpha\mathbf{a})\times\mathbf{b} = \alpha(\mathbf{a}\times\mathbf{b}) = \mathbf{a}\times(\alpha\mathbf{b}), \tag{11.26}$$

For $\mathbf{a},\mathbf{b}$, and $\mathbf{c}$ vectors, one obtains the distributive laws,

$$\mathbf{a}\times(\mathbf{b}+\mathbf{c}) = \mathbf{a}\times\mathbf{b}+\mathbf{a}\times\mathbf{c}, \tag{11.27}$$

$$(\mathbf{b}+\mathbf{c})\times\mathbf{a} = \mathbf{b}\times\mathbf{a}+\mathbf{c}\times\mathbf{a}. \tag{11.28}$$

Formula 11.25 follows immediately from the definition. The vectors $\mathbf{a}\times\mathbf{b}$ and $\mathbf{b}\times\mathbf{a}$ have the same magnitude, $|\mathbf{a}||\mathbf{b}|\sin\theta$, and an application of the right hand rule shows they have opposite direction. Formula 11.26 is also fairly clear. If α is a nonnegative scalar, the direction of $(\alpha\mathbf{a})\times\mathbf{b}$ is the same as the direction of $\mathbf{a}\times\mathbf{b}$,$\alpha(\mathbf{a}\times\mathbf{b})$ and $\mathbf{a}\times(\alpha\mathbf{b})$ while the magnitude is just α times the magnitude of $\mathbf{a}\times\mathbf{b}$ which is the same as the magnitude of $\alpha(\mathbf{a}\times\mathbf{b})$ and $\mathbf{a}\times(\alpha\mathbf{b})$. Using this yields equality in 11.26. In the case where $\alpha<0$, everything works the same way except the vectors are all pointing in the opposite direction and you must multiply by $|\alpha|$ when comparing their magnitudes. The distributive laws are much harder to establish but the second follows from the first quite easily. Thus, assuming the first, and using 11.25,

$$\begin{aligned}(\mathbf{b}+\mathbf{c})\times\mathbf{a} &= -\mathbf{a}\times(\mathbf{b}+\mathbf{c})\\ &= -(\mathbf{a}\times\mathbf{b}+\mathbf{a}\times\mathbf{c})\\ &= \mathbf{b}\times\mathbf{a}+\mathbf{c}\times\mathbf{a}.\end{aligned}$$

To verify the distributive law one can consider something called the box product.

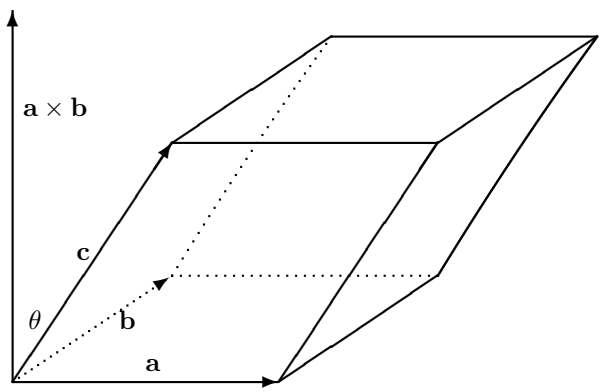
11.5.2 The Box Product, Triple Product

Definition 11.5.3 *A parallelepiped determined by the three vectors,* $\mathbf{a},\mathbf{b}$, *and* $\mathbf{c}$ *consists of*

$$\{r\mathbf{a}+s\mathbf{b}+t\mathbf{c} : r,s,t\in[0,1]\}.$$

That is, if you pick three numbers, r,s, *and* t *each in* $[0,1]$ *and form* $r\mathbf{a}+s\mathbf{b}+t\mathbf{c}$, *then the collection of all such points is what is meant by the parallelepiped determined by these three vectors.*

The following is a picture of such a thing.



You notice the area of the base of the parallelepiped, the parallelogram determined by the vectors, $\mathbf{a}$ and $\mathbf{b}$ has area equal to $|\mathbf{a}\times\mathbf{b}|$ while the altitude of the parallelepiped is $|\mathbf{c}|\cos\theta$ where θ is the angle shown in the picture between $\mathbf{c}$ and $\mathbf{a}\times\mathbf{b}$. Therefore, the volume of this parallelepiped is the area of the base times the altitude which is just

$$|\mathbf{a}\times\mathbf{b}|\,|\mathbf{c}|\cos\theta = \mathbf{a}\times\mathbf{b}\cdot\mathbf{c}.$$

This expression is known as the box product and is sometimes written as $[\mathbf{a},\mathbf{b},\mathbf{c}]$. You should consider what happens if you interchange the $\mathbf{b}$ with the $\mathbf{c}$ or the $\mathbf{a}$ with the $\mathbf{c}$. You can see geometrically from drawing pictures that this merely introduces a minus sign. In any case the box product of three vectors always equals either the volume of the parallelepiped determined by the three vectors or else minus this volume. From geometric reasoning like this you see that

$$\mathbf{a}\cdot\mathbf{b}\times\mathbf{c} = \mathbf{a}\times\mathbf{b}\cdot\mathbf{c}.$$

In other words, you can switch the $\times$ and the $\cdot$.

11.5.3 A Proof Of The Distributive Law For The Cross Product

Here is a proof of the distributive law for the cross product. Let $\mathbf{x}$ be a vector. From the above observation,

$$\begin{aligned}\mathbf{x}\cdot\mathbf{a}\times(\mathbf{b}+\mathbf{c}) &= (\mathbf{x}\times\mathbf{a})\cdot(\mathbf{b}+\mathbf{c})\\ &= (\mathbf{x}\times\mathbf{a})\cdot\mathbf{b}+(\mathbf{x}\times\mathbf{a})\cdot\mathbf{c}\\ &= \mathbf{x}\cdot\mathbf{a}\times\mathbf{b}+\mathbf{x}\cdot\mathbf{a}\times\mathbf{c}\\ &= \mathbf{x}\cdot(\mathbf{a}\times\mathbf{b}+\mathbf{a}\times\mathbf{c}).\end{aligned}$$

Therefore,

$$\mathbf{x}\cdot[\mathbf{a}\times(\mathbf{b}+\mathbf{c})-(\mathbf{a}\times\mathbf{b}+\mathbf{a}\times\mathbf{c})] = 0$$

for all $\mathbf{x}$. In particular, this holds for $\mathbf{x} = \mathbf{a}\times(\mathbf{b}+\mathbf{c})-(\mathbf{a}\times\mathbf{b}+\mathbf{a}\times\mathbf{c})$ showing that $\mathbf{a}\times(\mathbf{b}+\mathbf{c}) = \mathbf{a}\times\mathbf{b}+\mathbf{a}\times\mathbf{c}$ and this proves the distributive law for the cross product.

11.5.4 The Coordinate Description Of The Cross Product

Now from the properties of the cross product and its definition,

$$\begin{array}{ll}\mathbf{i}\times\mathbf{j}=\mathbf{k} & \mathbf{j}\times\mathbf{i}=-\mathbf{k}\\ \mathbf{k}\times\mathbf{i}=\mathbf{j} & \mathbf{i}\times\mathbf{k}=-\mathbf{j}\\ \mathbf{j}\times\mathbf{k}=\mathbf{i} & \mathbf{k}\times\mathbf{j}=-\mathbf{i}\end{array}$$

With this information, the following gives the coordinate description of the cross product.

Proposition 11.5.4 *Let* $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$ *and* $\mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$ *be two vectors. Then*

$$\begin{aligned}\mathbf{a}\times\mathbf{b} = (a_2b_3 - a_3b_2)\,\mathbf{i} + (a_3b_1 - a_1b_3)\,\mathbf{j} + \\ + (a_1b_2 - a_2b_1)\,\mathbf{k}.\end{aligned} \tag{11.29}$$

Proof: From the above table and the properties of the cross product listed,

$$\begin{aligned}&(a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}) \times (b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}) = \\ &a_1b_2\mathbf{i}\times\mathbf{j} + a_1b_3\mathbf{i}\times\mathbf{k} + a_2b_1\mathbf{j}\times\mathbf{i} + a_2b_3\mathbf{j}\times\mathbf{k} + \\ &+a_3b_1\mathbf{k}\times\mathbf{i} + a_3b_2\mathbf{k}\times\mathbf{j} \\ &= a_1b_2\mathbf{k} - a_1b_3\mathbf{j} - a_2b_1\mathbf{k} + a_2b_3\mathbf{i} + a_3b_1\mathbf{j} - a_3b_2\mathbf{i} \\ &= (a_2b_3 - a_3b_2)\,\mathbf{i} + (a_3b_1 - a_1b_3)\,\mathbf{j} + (a_1b_2 - a_2b_1)\,\mathbf{k}\end{aligned} \tag{11.30}$$

This proves the proposition.

The easy way to remember the above formula is to write it as follows.

$$\mathbf{a}\times\mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} \tag{11.31}$$

where you expand the determinant along the top row. This yields

$$(a_2b_3 - a_3b_2)\,\mathbf{i} - (a_1b_3 - a_3b_1)\,\mathbf{j} + (a_1b_2 - a_2b_1)\,\mathbf{k} \tag{11.32}$$

which is the same as 11.30.

11.5.5 The Integral Over A Two Dimensional Surface

First it is good to define what is meant by a smooth surface.

Definition 11.5.5 *Let* S *be a subset of* $\mathbb{R}^3$. *Then* S *is a* ***smooth surface*** *if there exists an open set,* $U \subseteq \mathbb{R}^2$ *and a* C^1 *function,* $\mathbf{R}$ *defined on* U *such that* $\mathbf{R}(U) = S, \mathbf{R}$ *is one to one, and for all* $(u, v) \in U$,

$$\mathbf{R}_u \times \mathbf{R}_v \neq \mathbf{0}. \tag{11.33}$$

This last condition ensures that there is always a well defined normal on S. *This function,* $\mathbf{R}$ *is called a parameterization of the surface. It is just like a parameterization of a curve but here there are two parameters,* u, v.

One way to think of this is that there is a piece of rubber occupying U in the plane and then it is taken and stretched in three dimensions. This gives S.

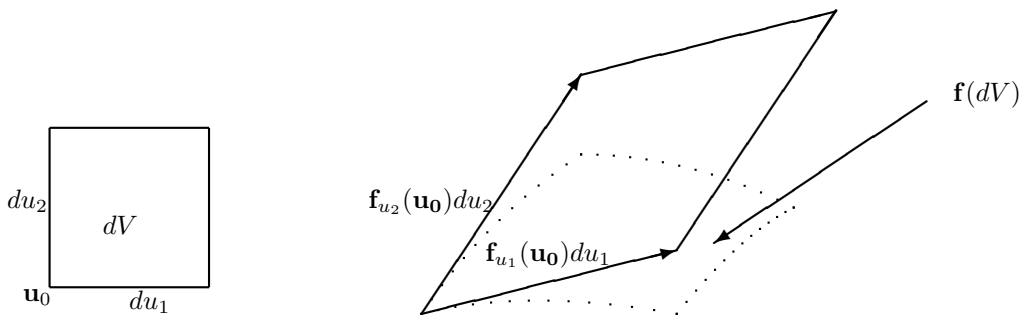
Definition 11.5.6 *Let* $\mathbf{u}_1, \mathbf{u}_2$ *be vectors in* $\mathbb{R}^3$. *The* 2 *dimensional parallelogram determined by these vectors will be denoted by* $P(\mathbf{u}_1, \mathbf{u}_2)$ *and it is defined as*

$$P(\mathbf{u}_1, \mathbf{u}_2) \equiv \left\{ \sum_{j=1}^{2} s_j\mathbf{u}_j : s_j \in [0,1] \right\}.$$

Then the area of this parallelogram is

$$\text{area } P(\mathbf{u}_1, \mathbf{u}_2) \equiv |\mathbf{u}_1 \times \mathbf{u}_2|.$$

Suppose then that $\mathbf{x} = \mathbf{R}(\mathbf{u})$ where $\mathbf{u} \in U$, a subset of $\mathbb{R}^2$ and $\mathbf{x}$ is a point in V, a subset of 3 dimensional space. Thus, letting the Cartesian coordinates of $\mathbf{x}$ be given by $\mathbf{x} = (x_1, x_2, x_3)^T$, each x_i being a function of $\mathbf{u}$, an infinitesimal rectangle located at $\mathbf{u}_0$ corresponds to an infinitesimal parallelogram located at $\mathbf{R}(\mathbf{u}_0)$ which is determined by the 2 vectors $\left\{\frac{\partial \mathbf{R}(\mathbf{u}_0)}{\partial u}\, du, \frac{\partial \mathbf{R}(\mathbf{u}_0)}{\partial v}\, dv\right\}$, each of which is tangent to the surface defined by $\mathbf{x} = \mathbf{R}(\mathbf{u})$. This is a very vague and unacceptable description. What exactly is an infinitesimal rectangle? However, it can all be made precise later and this is good motivation for the real thing.



From Definition 11.5.6, the volume of this infinitesimal parallelepiped located at $\mathbf{R}(\mathbf{u}_0)$ is given by

$$\left|\frac{\partial \mathbf{R}(\mathbf{u}_0)}{\partial u}\, du \times \frac{\partial \mathbf{R}(\mathbf{u}_0)}{\partial v}\, dv\right| = \left|\frac{\partial \mathbf{R}(\mathbf{u}_0)}{\partial u} \times \frac{\partial \mathbf{R}(\mathbf{u}_0)}{\partial v}\right| du dv \tag{11.34}$$

$$= |\mathbf{R}_u \times \mathbf{R}_v|\, du dv \tag{11.35}$$

This motivates the following definition of what is meant by the integral over a parametrically defined surface in $\mathbb{R}^3$.

Definition 11.5.7 *Suppose U is a subset of $\mathbb{R}^2$ and suppose $\mathbf{R} : U \to \mathbf{R}(U) = S \subseteq \mathbb{R}^3$ is a one to one and C^1 function. Then if $h : \mathbf{R}(U) \to \mathbb{R}$, define the 2 dimensional surface integral, $\int_{\mathbf{R}(U)} h(\mathbf{x})\, dS$ according to the following formula.*

$$\int_{\mathbf{S}} h(\mathbf{x})\, dS \equiv \int_U h(\mathbf{R}(\mathbf{u}))\, |\mathbf{R}_u(\mathbf{u}) \times \mathbf{R}_v(\mathbf{u})|\, du dv.$$

With this understanding, it becomes possible to interpret the meaning of Stoke's theorem. This is stated in the following theorem. Note that slightly more is assumed here than earlier. In particular, it is assumed that $\mathbf{R}_u \times \mathbf{R}_v \neq \mathbf{0}$. This allows the definition of a well defined normal vector which varies continuously over the surface, S.

Theorem 11.5.8 *(Stoke's Theorem) Let U be any region in $\mathbb{R}^2$ for which the conclusion of Green's theorem holds. Let $\mathbf{R} \in C^2\left(\overline{U_i}, \mathbb{R}^3\right)$ be a one to one function such that $\mathbf{R}_u \times \mathbf{R}_v \neq \mathbf{0}$ on U. Let*

$$\boldsymbol{\gamma}_j = \mathbf{R} \circ \boldsymbol{\alpha}_j,$$

where the $\boldsymbol{\alpha}_j$ are parameterizations for the oriented bounded variation curves bounding the region U oriented such that the conclusion of Green's theorem holds. Let S denote the surface,

$$S \equiv \{\mathbf{R}(u, v) : (u, v) \in U\},$$

Then for $\mathbf{F}$ *a* C^1 *vector field defined near* S,

$$\sum_{j=1}^{n}\int_{\gamma_j}\mathbf{F}\cdot d\boldsymbol{\gamma}_i = \int_U (\mathbf{R}_u\times\mathbf{R}_v)\cdot(\nabla\times\mathbf{F})(\mathbf{R}(u,v))\,dm_2 \tag{11.36}$$

$$= \int_S (\nabla\times\mathbf{F})\cdot\mathbf{n}dS \tag{11.37}$$

Proof: Formula 11.36 was established in Theorem 11.4.4. The unit normal of the point $\mathbf{R}(u,v)$ of S is $(\mathbf{R}_u\times\mathbf{R}_v)/|\mathbf{R}_u\times\mathbf{R}_v|$ and from the definition of the integral over the surface, Definition 11.5.7, Formula 11.37 follows.

11.6 Introduction To Complex Analysis

11.6.1 Basic Theorems, The Cauchy Riemann Equations

With Green's theorem and the technique of proof used in proving it, it is possible to present the most important parts of complex analysis almost effortlessly. I will do this here and leave some of the other parts for the exercises. Recall the complex numbers should be considered as points in the plane. Thus a complex number is of the form $x+iy$ where $i^2=-1$. The complex conjugate is defined by

$$\overline{x+iy}\equiv x-iy$$

and for z a complex number,

$$|z|\equiv(z\overline{z})^{1/2}=\sqrt{x^2+y^2}.$$

Thus when $x+iy$ is considered an ordered pair $(x,y)\in\mathbb{R}^2$ the magnitude of a complex number is nothing more than the usual norm of the ordered pair. Also for $z=x+iy, w=u+iv$,

$$|z-w|=\sqrt{(x-u)^2+(y-v)^2}$$

so in terms of all topological considerations, $\mathbb{R}^2$ is the same as $\mathbb{C}$. Thus to say $z\to f(z)$ is continuous, is the same as saying

$$(x,y)\to u(x,y),\ (x,y)\to v(x,y)$$

are continuous where $f(z)\equiv u(x,y)+iv(x,y)$ with u and v being called the real and imaginary parts of f. The only new thing is that writing an ordered pair (x,y) as $x+iy$ with the convention $i^2=-1$ makes $\mathbb{C}$ into a field. Now here is the definition of what it means for a function to be analytic.

Definition 11.6.1 *Let* U *be an open subset of* $\mathbb{C}$ $(\mathbb{R}^2)$ *and let* $f:U\to\mathbb{C}$ *be a function. Then* f *is said to be analytic on* U *if for every* $z\in U$,

$$\lim_{\Delta z\to 0}\frac{f(z+\Delta z)-f(z)}{\Delta z}\equiv f'(z)$$

exists and is a continuous function of $z\in U$. *For a function having values in* $\mathbb{C}$ *denote by* $u(x,y)$ *the real part of* f *and* $v(x,y)$ *the imaginary part. Both* u *and* v *have real values and*

$$f(x+iy)\equiv f(z)\equiv u(x,y)+iv(x,y)$$

Proposition 11.6.2 *Let U be an open subset of $\mathbb{C}$. Then $f : U \to \mathbb{C}$ is analytic if and only if for*

$$f(x+iy) \equiv u(x,y) + iv(x,y)$$

$u(x,y), v(x,y)$ being the real and imaginary parts of f, it follows

$$u_x(x,y) = v_y(x,y),\ u_y(x,y) = -v_x(x,y)$$

and all these partial derivatives, u_x, u_y, v_x, v_y are continuous on U. (The above equations are called the Cauchy Riemann equations.)

Proof: First suppose f is analytic. First let $\Delta z = ih$ and take the limit of the difference quotient as $h \to 0$ in the definition. Thus from the definition,

$$\begin{aligned} f'(z) &\equiv \lim_{h\to 0} \frac{f(z+ih) - f(z)}{ih} \\ &= \lim_{h\to 0} \frac{u(x,y+h) + iv(x,y+h) - (u(x,y) + iv(x,y))}{ih} \\ &= \lim_{h\to 0} \frac{1}{i}(u_y(x,y) + iv_y(x,y)) = -iu_y(x,y) + v_y(x,y) \end{aligned}$$

Next let $\Delta z = h$ and take the limit of the difference quotient as $h \to 0$.

$$\begin{aligned} f'(z) &\equiv \lim_{h\to 0} \frac{f(z+h) - f(z)}{h} \\ &= \lim_{h\to 0} \frac{u(x+h,y) + iv(x+h,y) - (u(x,y) + iv(x,y))}{h} \\ &= u_x(x,y) + iv_x(x,y). \end{aligned}$$

Therefore, equating real and imaginary parts,

$$u_x = v_y,\ v_x = -u_y$$

and this yields the Cauchy Riemann equations. Since $z \to f'(z)$ is continuous, it follows the real and imaginary parts of this function must also be continuous. Thus from the above formulas for $f'(z)$, it follows from the continuity of $z \to f'(z)$ all the partial derivatives of the real and imaginary parts are continuous.

Next suppose the Cauchy Riemann equations hold and these partial derivatives are all continuous. For $\Delta z = h + ik$,

$$\begin{aligned} f(z+\Delta z) - f(z) &= u(x+h,y+k) + iv(x+h,y+k) - (u(x,y) + iv(x,y)) \\ &= u_x(x,y)h + u_y(x,y)k + i(v_x(x,y)h + v_y(x,y)k) + o((h,k)) \\ &= u_x(x,y)h + u_y(x,y)k + i(v_x(x,y)h + v_y(x,y)k) + o(\Delta z) \end{aligned}$$

This follows from Theorem 6.5.1 which says that C^1 implies differentiable along with the definition of the norm (absolute value) in $\mathbb{C}$. By the Cauchy Riemann equations this equals

$$\begin{aligned} &= u_x(x,y)h - v_x(x,y)k + i(v_x(x,y)h + u_x(x,y)k) + o(\Delta z) \\ &= u_x(x,y)(h+ik) + iv_x(x,y)(h+ik) + o(\Delta z) \\ &= u_x(x,y)\Delta z + iv_x(x,y)\Delta z + o(\Delta z) \end{aligned}$$

Dividing by Δz and taking a limit yields $f'(z)$ exists and equals $u_x(x,y) + iv_x(x,y)$ which are assumed to be continuous. This proves the proposition.

11.6.2 Contour Integrals

The most important tools in complex analysis are Cauchy's theorem in some form and Cauchy's formula for an analytic function. I will give one of the very best versions of these theorems. They all involve something called a contour integral. Now a contour integral is just a sort of line integral. Here is the definition.

Definition 11.6.3 *Let $\gamma : [a,b] \to \mathbb{C}$ be of bounded variation and let $f : \gamma^* \to \mathbb{C}$. Letting $\mathcal{P} \equiv \{t_0, \cdots, t_n\}$ where $a = t_0 < t_1 < \cdots < t_n = b$, define*

$$||\mathcal{P}|| \equiv \max\{|t_j - t_{j-1}| : j = 1, \cdots, n\}$$

and the Riemann Stieltjes sum by

$$S(\mathcal{P}) \equiv \sum_{j=1}^{n} f(\gamma(\tau_j))(\gamma(t_j) - \gamma(t_{j-1}))$$

where $\tau_j \in [t_{j-1}, t_j]$. (Note this notation is a little sloppy because it does not identify the specific point, τ_j used. It is understood that this point is arbitrary.) Define $\int_\gamma f(z)\,dz$ as the unique number which satisfies the following condition. For all $\varepsilon > 0$ there exists a $\delta > 0$ such that if $||\mathcal{P}|| \le \delta$, then

$$\left| \int_\gamma f(z)\,dz - S(\mathcal{P}) \right| < \varepsilon.$$

Sometimes this is written as

$$\int_\gamma f(z)\,dz \equiv \lim_{||\mathcal{P}|| \to 0} S(\mathcal{P}).$$

You note that this is essentially the same definition given earlier for the line integral only this time the function has values in $\mathbb{C}$ rather than $\mathbb{R}^n$ and there is no dot product involved. Instead, you multiply by the complex number $\gamma(t_j) - \gamma(t_{j-1})$ in the Riemann Stieltjes sum. To tie this in with the line integral even more, consider a typical term in the sum for $S(\mathcal{P})$. Let $\gamma(t) = \gamma_1(t) + i\gamma_2(t)$. Then letting u be the real part of f and v the imaginary part, $S(\mathcal{P})$ equals

$$\begin{aligned}
&\sum_{j=1}^{n} (u(\gamma_1(\tau_j), \gamma_2(\tau_j)) + iv(\gamma_1(\tau_j), \gamma_2(\tau_j))) \\
&(\gamma_1(t_j) - \gamma_1(t_{j-1}) + i(\gamma_2(t_j) - \gamma_2(t_{j-1})))
\end{aligned}$$

$$\begin{aligned}
= \quad & \sum_{j=1}^{n} u(\gamma_1(\tau_j), \gamma_2(\tau_j))(\gamma_1(t_j) - \gamma_1(t_{j-1})) \\
& - \sum_{j=1}^{n} v(\gamma_1(\tau_j), \gamma_2(\tau_j))(\gamma_2(t_j) - \gamma_2(t_{j-1})) \\
& + i \sum_{j=1}^{n} v(\gamma_1(\tau_j), \gamma_2(\tau_j))(\gamma_1(t_j) - \gamma_1(t_{j-1})) \\
& + i \sum_{j=1}^{n} u(\gamma_1(\tau_j), \gamma_2(\tau_j))(\gamma_2(t_j) - \gamma_2(t_{j-1}))
\end{aligned}$$

Combining these leads to

$$\sum_{j=1}^{n} \left(u\left(\gamma\left(\tau_j\right)\right), -v\left(\gamma\left(\tau_j\right)\right)\right) \cdot \left(\gamma\left(t_j\right) - \gamma\left(t_{j-1}\right)\right)$$
$$+i \sum_{j=1}^{n} \left(v\left(\gamma\left(\tau_j\right)\right), u\left(\gamma\left(\tau_j\right)\right)\right) \cdot \left(\gamma\left(t_j\right) - \gamma\left(t_{j-1}\right)\right) \tag{11.38}$$

Since the functions u and v are continuous, the limit as $||\mathcal{P}|| \to 0$ of the above equals

$$\int_\gamma (u, -v) \cdot d\gamma + i \int_\gamma (v, u) \cdot d\gamma$$

This proves most of the following lemma.

Lemma 11.6.4 *Let Γ be a rectifiable curve in $\mathbb{C}$ having parameterization γ which is continuous with bounded variation. Also let $f : \Gamma \to \mathbb{C}$ be continuous. Then the contour integral $\int_\gamma f(z)\, dz$ exists and is given by the sum of the following line integrals.*

$$\int_\gamma f(z)\, dz = \int_\gamma (u, -v) \cdot d\gamma + i \int_\gamma (v, u) \cdot d\gamma \tag{11.39}$$

Proof: The existence of the two line integrals as limits of $S(\mathcal{P})$ as $||\mathcal{P}|| \to 0$ follows from continuity of u, v and Theorem 11.2.3 along with the above discussion which decomposes the sum for the contour integral into the expression of 11.38 for which the two sums converge to the line integrals in the above formula. This proves the lemma.

The lemma implies all the algebraic properties for line integrals hold in the same way for contour integrals. In particular, if γ is C^1, then

$$\int_\gamma f(z)\, dz = \int_a^b f\left(\gamma(t)\right) \gamma'(t)\, dt.$$

Another important observation is the following.

Proposition 11.6.5 *Suppose $F'(z) = f(z)$ for all $z \in \Omega$, an open set containing γ^* where $\gamma : [a, b] \to \mathbb{C}$ is a continuous bounded variation curve. Then*

$$\int_\gamma f(z)\, dz = F\left(\gamma(b)\right) - F\left(\gamma(a)\right).$$

Proof: Letting u and v be real and imaginary parts of f, it follows from Lemma 11.6.4

$$\int_\gamma f(z)\, dz = \int_\gamma (u, -v) \cdot d\gamma + i \int_\gamma (v, u) \cdot d\gamma \tag{11.40}$$

Consider the real valued function

$$G(x, y) \equiv \frac{1}{2}\left(F(x + iy) + \overline{F(x + iy)}\right) \equiv \operatorname{Re} F(x + iy)$$

By assumption,

$$F'(x + iy) = f(x + iy) = u(x, y) + iv(x, y).$$

Thus it is routine to verify $\nabla G = (u, -v)$. Next let the real valued function H be defined by

$$H(x, y) \equiv \frac{1}{2i}\left(F(x + iy) - \overline{F(x + iy)}\right) \equiv \operatorname{Im} F(x + iy)$$

Then $\nabla H = (v, u)$ and so from 11.40 and Theorem 11.2.11

$$\begin{aligned}\int_\gamma f(z)\,dz &= G(\gamma(b)) - G(\gamma(a)) + i(H(\gamma(b)) - H(\gamma(a))) \\ &= F(\gamma(b)) - F(\gamma(a)).\end{aligned}$$

This proves the proposition.

A function F such that $F' = f$ is called a **primitive** of f. See how it acts a lot like a potential, the difference being that a primitive has complex, not real values. In calculus, in the context of a function of one real variable, this is often called an antiderivative and every continuous function has one thanks to the fundamental theorem of calculus. However, it will be shown below that the situation is not at all the same for functions of a complex variable.

11.6.3 The Cauchy Integral

The following is the first form of the Cauchy integral theorem.

Lemma 11.6.6 *Let U be an open set in $\mathbb{C}$ and let Γ be a simple closed rectifiable curve contained in U having parameterization γ. Also let f be analytic in U. Then*

$$\int_\gamma f(z)\,dz = 0.$$

Proof: This follows right away from the Cauchy Riemann equations and the formula 11.39. Assume without loss of generality the orientation of Γ is the positive orientation. If not, the argument is the same. Then from formula 11.39,

$$\int_\gamma f(z)\,dz = \int_\gamma (u, -v)\cdot d\gamma + i\int_\gamma (v, u)\cdot d\gamma$$

and by Green's theorem and U_i the inside of Γ this equals

$$\int_{U_i} (-v_x - u_y)\,dm_2 + i\int_\gamma (u_x - v_y)\,dm_2 = 0$$

by the Cauchy Riemann equations. This proves the lemma.

It is easy to improve on this result using the argument for proving Green's theorem. You only need continuity on the bounding curve. You also don't need to make any assumption about the functions u_x, etc. being in $L^1(U)$. The following is a very general version of the Cauchy integral theorem.

Theorem 11.6.7 *Let U_i be the inside of Γ a simple closed rectifiable curve having parameterization γ. Also let f be analytic in U_i and continuous on $U_i \cup \Gamma$. Then*

$$\int_\gamma f(z)\,dz = 0.$$

Proof: Let $\mathcal{B}_\delta, \mathcal{I}_\delta$ be those regions of Lemma 11.3.6 where as earlier $\mathcal{I}_\delta$ are those which have empty intersection with Γ and $\mathcal{B}_\delta$ are the border regions. Without loss of generality, assume Γ is positively oriented. As in the proof of Green's theorem you can apply the same argument to the line integrals on the right of 11.39 to obtain, just as in the proof of Green's theorem

$$\sum_{R\in\mathcal{I}_\delta}\int_{\partial R} f(z)\,dz + \sum_{R\in\mathcal{B}_\delta}\int_{\partial R} f(z)\,dz = \int_\gamma f(z)\,dz$$

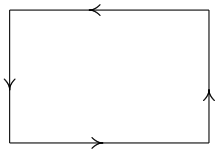
In this case the first sum on the left in the above formula equals 0 from Lemma 11.6.6 for any $\delta > 0$. Now just as in the proof of Green's theorem, you can choose δ small enough that

$$\sum_{R\in\mathcal{B}_\delta} \left| \int_{\partial R} f(z)\, dz \right| < \varepsilon.$$

Since ε is arbitrary this proves the theorem.

With this really marvelous theorem it is time to consider the Cauchy integral formula which represents the value of an analytic function at a point on the inside in terms of its values on the boundary. First here are some lemmas.

Lemma 11.6.8 *Let R be a rectangle such that ∂R is positively oriented. Recall this means the direction of motion is counter clockwise.*



Then if z is on the inside of ∂R,

$$\frac{1}{2\pi i}\int_{\partial R} \frac{1}{w-z} dw = 1$$

while if z is on the outside of ∂R, the above integral equals 0.

Proof: This follows from a routine computation and is left to you. In the case where z is on the outside of ∂R, the conclusion follows from the Cauchy integral formula Theorem 11.6.7 as you can verify by noting that $f(w) \equiv 1/(w-z)$ is analytic on an open set containing R and that in fact its derivative equals what you would think,

$$-1/(w-z)^2.$$

This proves the lemma.

Now with this little lemma, here is the Cauchy integral formula.

Theorem 11.6.9 *Let Γ be a positively oriented simple closed rectifiable curve having parameterization γ and let $z \in U_i$, the inside of Γ. Also let f be analytic on U_i, and continuous on $U_i \cup \Gamma$. Then*

$$f(z) = \frac{1}{2\pi i}\int_\gamma \frac{f(w)}{w-z} dw.$$

In particular, letting $f(z) \equiv 1$,

$$\frac{1}{2\pi i}\int_\gamma \frac{1}{w-z} dw = 1.$$

Proof: In constructing the special regions in the proof of Green's theorem, always choose δ such that the point z is not on any of the lines $m\delta = y$ and $x = k\delta$. This makes it possible to avoid thinking about the case where z is not on the interior of any of the rectangles of $\mathcal{I}_\delta$. Pick δ small enough that $\mathcal{I}_\delta \neq \emptyset$ and z is contained in some $R_0 \in \mathcal{I}_\delta$. From Lemma 11.6.8 it follows for each $R \in \mathcal{I}_\delta$

$$\frac{1}{2\pi i}\int_R \frac{f(w)}{w-z} dw - f(z) = \frac{1}{2\pi i}\int_R \frac{f(w)-f(z)}{w-z} dw$$

Then as in the proof of Theorem 11.6.7

$$\begin{aligned} & \frac{1}{2\pi i}\sum_{R\in\mathcal{I}_\delta}\int_{\partial R}\frac{f(w)-f(z)}{w-z}dw+\frac{1}{2\pi i}\sum_{R\in\mathcal{B}_\delta}\int_{\partial R}\frac{f(w)-f(z)}{w-z}dw \\ = & \frac{1}{2\pi i}\int_\gamma \frac{f(w)-f(z)}{w-z}dw \end{aligned}$$

By Theorem 11.6.7, all these integrals on the left equal 0 except for R_0, the one which contains z on its interior.

Thus the above reduces to

$$\frac{1}{2\pi i}\int_{\partial R_0}\frac{f(w)-f(z)}{w-z}dw=\frac{1}{2\pi i}\int_\gamma \frac{f(w)-f(z)}{w-z}dw$$

The integrand of the left converges to $f'(z)$ as $\delta\to 0$ and the length of R_0 also converges to 0 so it follows from Theorem 11.2.4 that the limit as $\delta\to 0$ in the above exists and yields

$$0=\frac{1}{2\pi i}\int_\gamma \frac{f(w)-f(z)}{w-z}dw=\frac{1}{2\pi i}\int_\gamma \frac{f(w)}{w-z}dw-f(z)\frac{1}{2\pi i}\int_\gamma \frac{1}{w-z}dw. \tag{11.41}$$

Consider the last integral above.

$$\sum_{R\in\mathcal{I}_\delta}\int_{\partial R}\frac{1}{w-z}dw+\sum_{R\in\mathcal{B}_\delta}\int_{\partial R}\frac{1}{w-z}dw=\int_\gamma \frac{1}{w-z}dw \tag{11.42}$$

As in the proof of Green's theorem, choosing δ small enough the second sum on the left in the above satisfies

$$\left|\sum_{R\in\mathcal{B}_\delta}\int_{\partial R}\frac{1}{w-z}dw\right|\le\sum_{R\in\mathcal{B}_\delta}\left|\int_{\partial R}\frac{1}{w-z}dw\right|<\varepsilon.$$

By Lemma 11.6.8, the first sum on the left in 11.42 equals

$$\int_{\partial R_0}\frac{1}{w-z}dw$$

where R_0 is the rectangle for which z is on the inside of ∂R_0. Then by this lemma again, this equals $2\pi i$. Therefore for such small δ, 11.42 reduces to

$$\left|2\pi i-\int_\gamma \frac{1}{w-z}dw\right|<\varepsilon$$

Since ε is arbitrary, this shows

$$\frac{1}{2\pi i}\int_\gamma \frac{1}{w-z}dw=1.$$

Now using this, 11.41 implies the claimed formula of the theorem. This proves the theorem.

Theorem 11.6.10 *Let $\gamma:[a,b]\to\mathbb{C}$ be of bounded variation. Let f be continuous on γ^*. For $z\notin\gamma^*$, define*

$$g(z)\equiv\int_\gamma \frac{f(w)}{w-z}dw$$

Then g is infinitely differentiable. Futhermore,

$$g^{(n)}(z)=n!\int_\gamma \frac{f(w)}{(w-z)^{n+1}}$$

Proof:

$$\left(g\left(z+h\right)-g\left(z\right)\right)/h=\frac{1}{h}\int_{\gamma}\left(\frac{f\left(w\right)}{\left(w-z-h\right)}-\frac{f\left(w\right)}{\left(w-z\right)}\right)dw$$

$$=\frac{1}{h}\int_{\gamma}f\left(w\right)\left(\frac{h}{\left(w-z-h\right)\left(w-z\right)}\right)dw=\int_{\gamma}f\left(w\right)\left(\frac{1}{\left(w-z-h\right)\left(w-z\right)}\right)dw$$

Consider only $h\in\mathbb{C}$ such that $2\left|h\right|<\operatorname{dist}\left(z,\gamma^{*}\right)$. The integrand converges to $1/\left(w-z\right)^{2}$. Then for these values of h,

$$\left|\frac{1}{\left(w-z-h\right)\left(w-z\right)}-\frac{1}{\left(w-z\right)^{2}}\right|=\left|\frac{h}{\left(w-z-h\right)\left(w-z\right)^{2}}\right|$$

$$\leq\frac{\left|h\right|}{\operatorname{dist}\left(z,\gamma^{*}\right)^{3}/2}=\frac{2\left|h\right|}{\operatorname{dist}\left(z,\gamma^{*}\right)^{3}}$$

and so the convergence of the integrand to

$$f\left(w\right)/\left(w-z\right)^{2}$$

is uniform for $\left|h\right|<\operatorname{dist}\left(z,\gamma^{*}\right)/2$. Using Theorem 11.2.4, it follows

$$\begin{aligned}g'\left(z\right) &= \lim_{h\to0}\frac{g\left(z+h\right)-g\left(z\right)}{h}\\ &= \lim_{h\to0}\frac{1}{h}\int_{\gamma}\left(\frac{f\left(w\right)}{\left(w-z-h\right)}-\frac{f\left(w\right)}{\left(w-z\right)}\right)dw\\ &= \int_{\gamma}\frac{f\left(w\right)}{\left(w-z\right)^{2}}dw.\end{aligned}$$

One can then differentiate the above expression using the same arguments. Continuing this way results in the following formula.

$$g^{(n)}\left(z\right)=n!\int_{\gamma}\frac{f\left(w\right)}{\left(w-z\right)^{n+1}}dw$$

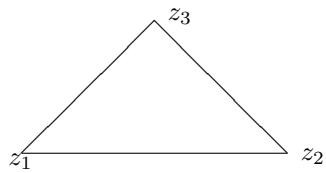
This proves the theorem.

It turns out that in the definition of what it means for a function defined on an open set, U to be analytic it is not necessary to say that $z\to f'\left(z\right)$ is continuous. In fact, this comes for free. The statement that $z\to f'\left(z\right)$ is continuous is REDUNDANT! The key to understanding this is the Cauchy Goursat theorem.

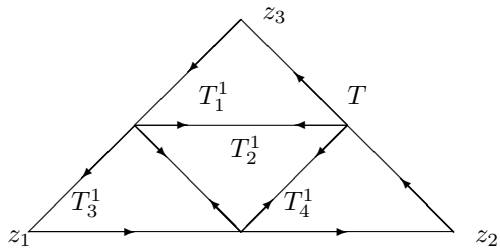
11.6.4 The Cauchy Goursat Theorem

If you have two points in $\mathbb{C}$, z_1 and z_2, you can consider $\gamma\left(t\right)\equiv z_1+t\left(z_2-z_1\right)$ for $t\in\left[0,1\right]$ to obtain a continuous bounded variation curve from z_1 to z_2. More generally, if $z_1,\cdots,z_m$ are points in $\mathbb{C}$ you can obtain a continuous bounded variation curve from z_1 to z_m which consists of first going from z_1 to z_2 and then from z_2 to z_3 and so on, till in the end one goes from z_{m-1} to z_m. Denote this piecewise linear curve as $\gamma\left(z_1,\cdots,z_m\right)$. Now let T be a

triangle with vertices z_1, z_2 and z_3 encountered in the counter clockwise direction as shown.



Denote by $\int_{\partial T} f(z)\,dz$, the expression, $\int_{\gamma(z_1,z_2,z_3,z_1)} f(z)\,dz$. Consider the following picture.



Thus

$$\int_{\partial T} f(z)\,dz = \sum_{k=1}^{4} \int_{\partial T_k^1} f(z)\,dz. \tag{11.43}$$

On the "inside lines" the integrals cancel because there are two integrals going in opposite directions for each of these inside lines.

Theorem 11.6.11 *(Cauchy Goursat) Let $f : \Omega \to \mathbb{C}$ have the property that $f'(z)$ exists for all $z \in \Omega$ and let T be a triangle contained in Ω. Then*

$$\int_{\partial T} f(w)\,dw = 0.$$

Proof: Suppose not. Then

$$\left| \int_{\partial T} f(w)\,dw \right| = \alpha \neq 0.$$

From 11.43 it follows

$$\alpha \leq \sum_{k=1}^{4} \left| \int_{\partial T_k^1} f(w)\,dw \right|$$

and so for at least one of these T_k^1, denoted from now on as T_1,

$$\left| \int_{\partial T_1} f(w)\,dw \right| \geq \frac{\alpha}{4}.$$

Now let T_1 play the same role as T. Subdivide as in the above picture, and obtain T_2 such that

$$\left| \int_{\partial T_2} f(w)\,dw \right| \geq \frac{\alpha}{4^2}.$$

Continue in this way, obtaining a sequence of triangles,

$$T_k \supseteq T_{k+1}, \operatorname{diam}(T_k) \leq \operatorname{diam}(T)\, 2^{-k},$$

and

$$\left|\int_{\partial T_k} f(w)\,dw\right| \geq \frac{\alpha}{4^k}.$$

Then let $z \in \cap_{k=1}^{\infty} T_k$ and note that by assumption, $f'(z)$ exists. Therefore, for all k large enough,

$$\int_{\partial T_k} f(w)\,dw = \int_{\partial T_k} (f(z) + f'(z)(w-z) + g(w))\,dw$$

where $|g(w)| < \varepsilon |w-z|$. Now observe that $w \to f(z) + f'(z)(w-z)$ has a primitive, namely,

$$F(w) = f(z)\,w + f'(z)(w-z)^2/2.$$

Therefore, by Proposition 11.6.5,

$$\int_{\partial T_k} f(w)\,dw = \int_{\partial T_k} g(w)\,dw.$$

From Theorem 11.2.4 applied to contour integrals or the definition of the contour integral,

$$\begin{aligned} \frac{\alpha}{4^k} &\leq \left|\int_{\partial T_k} g(w)\,dw\right| \leq \varepsilon \operatorname{diam}(T_k)\,(\text{length of } \partial T_k) \\ &\leq \varepsilon 2^{-k}\,(\text{length of } T)\operatorname{diam}(T)\,2^{-k}, \end{aligned}$$

and so

$$\alpha \leq \varepsilon\,(\text{length of } T)\operatorname{diam}(T).$$

Since ε is arbitrary, this shows $\alpha = 0$, a contradiction. Thus $\int_{\partial T} f(w)\,dw = 0$ as claimed.

This fundamental result yields the following important theorem.

Theorem 11.6.12 *(Morera[1]) Let Ω be an open set and let $f'(z)$ exist for all $z \in \Omega$. Let $D \equiv \overline{B(z_0, r)} \subseteq \Omega$. Then there exists $\varepsilon > 0$ such that f has a primitive on $B(z_0, r+\varepsilon)$. (Recall this is a function F such that $F'(z) = f(z)$.)*

Proof: Choose $\varepsilon > 0$ small enough that $B(z_0, r+\varepsilon) \subseteq \Omega$. Then for $w \in B(z_0, r+\varepsilon)$, define

$$F(w) \equiv \int_{\gamma(z_0, w)} f(u)\,du.$$

Then by the Cauchy Goursat theorem, Theorem 11.6.11, and $w \in B(z_0, r+\varepsilon)$, it follows that for $|h|$ small enough,

$$\frac{F(w+h) - F(w)}{h} = \frac{1}{h}\int_{\gamma(w, w+h)} f(u)\,du$$

$$= \frac{1}{h}\int_0^1 f(w+th)\,h\,dt = \int_0^1 f(w+th)\,dt$$

which converges to $f(w)$ due to the continuity of f at w. This proves the theorem.

The following is a slight generalization of the above theorem which is also referred to as Morera's theorem. It contains the proof that the condition of continuity of $z \to f'(z)$ is redundant.

[1] Giancinto Morera 1856-1909. This theorem or one like it dates from around 1886

Corollary 11.6.13 *Let Ω be an open set and suppose that whenever*

$$\gamma(z_1, z_2, z_3, z_1)$$

is a closed curve bounding a triangle T, which is contained in Ω, and f is a continuous function defined on Ω, it follows that

$$\int_{\gamma(z_1,z_2,z_3,z_1)} f(z)\,dz = 0,$$

then f is analytic on Ω. Also, if $f'(z)$ exists for $z \in \Omega$, then $z \to f'(z)$ is continuous.

Proof: As in the proof of Morera's theorem, let $\overline{B(z_0, r)} \subseteq \Omega$ and use the given condition to construct a primitive, F for f on $B(z_0, r)$. (As just shown in Theorem 11.6.12, the given condition is satisfied whenever $f'(z)$ exists for all $z \in \Omega$.) Then F is analytic and so by the Cauchy integral formula, for $z \in B(z_0, r)$

$$F(z) = \frac{1}{2\pi i}\int_{\partial B(z_0,r)} \frac{F(w)}{w - z} dw.$$

It follows from Theorem 11.6.10 that F and hence f have infinitely many derivatives, implying that f is analytic on $B(z_0, r)$. Since z_0 is arbitrary, this shows f is analytic on Ω. In particular $z \to f'(z)$ is continuous because actually this function is differentiable. This proves the corollary.

This shows that an equivalent definition of what it means for a function to be analytic is the following definition.

Definition 11.6.14 *Let U be an open set in $\mathbb{C}$ and suppose $f'(z)$ exists for all $z \in U$. Then f is called analytic.*

These theorems form the foundation for the study of functions of a complex variable. Some important theorems will be discussed in the exercises.

11.7 Exercises

1. Suppose $f : [a, b] \to [c, d]$ is continuous and one to one on (a, b). For $s \in (c, d)$, show

   $$d(f, (a, b), s) = \pm 1$$

   show it is 1 if f is increasing and -1 if f is decreasing. How can this be used to relate the degree to orientation?

2. In defining a simple curve the assumption was made that $\boldsymbol{\gamma}(t) \neq \boldsymbol{\gamma}(a)$ and $\boldsymbol{\gamma}(t) \neq \boldsymbol{\gamma}(b)$ if $t \in (a, b)$. Is this fussy condition really necessary? Which theorems and lemmas hold with simply assuming $\boldsymbol{\gamma}$ is one to one on (a, b)? Does the fussy condition follow from assuming $\boldsymbol{\gamma}$ is one to one on (a, b)?

3. Show that for many open sets in $\mathbb{R}^2$, Area of $U = \int_{\partial U} x dy$, and Area of $U = \int_{\partial U} -y dx$ and Area of $U = \frac{1}{2}\int_{\partial U} -y dx + x dy$. **Hint:** Use Green's theorem.

4. A closed polygon in the plane starts at (x_0, y_0), goes to (x_1, y_1), to (x_2, y_2) to $\cdots (x_n, y_n) = (x_0, y_0)$. Suppose the line segments never cross so that you have a simple closed curve. Using Green's theorem find a simple formula for the area of the parallelogram. You can use Problem 3. Get the area using a line integral obtained by adding the line integrals corresponding to the vertices of the polygon.

5. Let Γ be a simple C^1 oriented curve having parameterization $\boldsymbol{\gamma}$ where t is the time and suppose $\mathbf{f}$ is a force defined on Γ. Then the work done by $\mathbf{f}$ on an object of mass m as it moves over the curve is defined by

$$\int_{\gamma} \mathbf{f} \cdot d\boldsymbol{\gamma}$$

Newton's second law states that $\mathbf{f} = m\frac{d\mathbf{v}}{dt}$ where $\mathbf{v} \equiv \boldsymbol{\gamma}'(t)$. Let $\mathbf{v}_b = \boldsymbol{\gamma}'(b)$ with $\mathbf{v}_a$ defined similarly. Thus these are the final and initial velocities. Show the work equals

$$\frac{1}{2}m\left|\mathbf{v}_b\right|^2 - \frac{1}{2}m\left|\mathbf{v}_a\right|^2.$$

6. In the situation of the above problem, show that if $\mathbf{f}(\mathbf{x}) = \nabla F(\mathbf{x})$ where F is a potential, then if the motion is governed by the Newton's law it follows that for $\boldsymbol{\gamma}(t)$ the motion,

$$-F(\boldsymbol{\gamma}(t)) + \frac{1}{2}m\left|\boldsymbol{\gamma}'(t)\right|^2$$

is constant.

7. Generalize Stoke's theorem, Theorem 11.4.4 to the case where $\mathbf{R}$ is only assumed C^1.

8. Given an example of a simple closed rectifiable curve Γ and a horizontal line which intersects this curve in infinitely many points.

9. Let Γ be a simple closed rectifiable curve and let U_i be its inside. Show you can remove any finite number of circular disks from U_i and what remains will still be a region for which Green's theorem holds. **Hint:** You might get some ideas from looking at the proof of Lemma 11.3.6. This is much harder than it looks because you only know Γ is a simple closed rectifiable curve. Begin by punching one circular hole and go from there.

10. Let $\gamma : [a,b] \to \mathbb{R}$ be of bounded variation. Show there exist increasing functions $f(t)$ and $g(t)$ such that

$$\gamma(t) = f(t) - g(t).$$

Hint: You might let $f(t) = V(\gamma;[a,t])$. Show this is increasing and then consider $g(t) = f(t) - \gamma(t)$.

11. Using Problem 10 describe another way to obtain the integral $\int_\gamma f d\gamma$ for f a real valued function and γ a real valued curve of bounded variation as just described using the theory of Lebesgue integration. What exactly is this integral in this simple case? Next extend to the case where $\boldsymbol{\gamma}$ has values in $\mathbb{R}^n$ and $\mathbf{f} : \boldsymbol{\gamma}^* \to \mathbb{R}^n$. What are some advantages of using this other approach?

12. Suppose f is continuous but not analytic and a function of $z \in U \subseteq \mathbb{C}$. Show f has no primitive. When functions of real variables are considered, there are function spaces $C^m(U)$ which specify how many continuous derivatives the function has. Why are such function spaces irrelevant when considering functions of a complex variable?

13. Analytic functions are all just long polynomials. Prove this disappointing result. More precisely prove the following. If $f : U \to \mathbb{C}$ is analytic where U is an open set and if $B(z_0, r) \subseteq U$, then

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n \tag{11.44}$$

for all $|z - z_0| < r$. Furthermore,

$$a_n = \frac{f^{(n)}(z_0)}{n!}. \tag{11.45}$$

Hint: You use the Cauchy integral formula. For $z \in B(z_0, r)$ and C the positively oriented boundary,

$$\begin{aligned} f(z) &= \frac{1}{2\pi i}\int_C \frac{f(w)}{w-z} = \frac{1}{2\pi i}\int_C \frac{f(w)}{w-z_0}\frac{1}{1-\frac{z-z_0}{w-z_0}}dw \\ &= \frac{1}{2\pi i}\int_C \sum_{n=0}^{\infty}\frac{f(w)}{(w-z_0)^{n+1}}(z-z_0)^n\,dw \end{aligned}$$

Now explain why you can switch the sum and the integral. You will need to argue the sum converges uniformly which is what will justify this manipulation. Next use the result of Theorem 11.6.10.

14. Prove the following amazing result about the zeros of an analytic function. Let Ω be a connected open set (region) and let $f : \Omega \to X$ be analytic. Then the following are equivalent.

    (a) $f(z) = 0$ for all $z \in \Omega$

    (b) There exists $z_0 \in \Omega$ such that $f^{(n)}(z_0) = 0$ for all n.

    (c) There exists $z_0 \in \Omega$ which is a limit point of the set,

    $$Z \equiv \{z \in \Omega : f(z) = 0\}.$$

    Hint: From Problem 13, if *c.*) holds, then for z near z_0

    $$f(z) = \sum_{n=m}^{\infty}\frac{f^{(n)}(z_0)}{n!}(z-z_0)^n$$

    Say $f^{(n)}(z_0) \neq 0$. Then consider

    $$\frac{f(z)}{(z-z_0)^m} = \frac{f^{(m)}(z_0)}{m!} + \sum_{n=m+1}^{\infty}\frac{f^{(n)}(z_0)}{n!}(z-z_0)^{n-m}$$

    Now let $z_n \to z_0, z_n \neq z_0$ but $f(z_n) = 0$. What does this say about $f^{(m)}(z_0)$? Clearly the first two conditions are equivalent and they imply the third.

15. You want to define e^z for z complex such that it is analytic on $\mathbb{C}$. Using Problem 14 explain why there is at most one way to do it and still have it coincide with e^x when $z = x + i0$. Then show using the Cauchy Riemann equations that

    $$e^z \equiv e^x(\cos(y) + i\sin(y))$$

    is analytic and agrees with e^x when $z = x + i0$. Also show

    $$\frac{d}{dz}e^z = e^z.$$

    Hint: For the first part, suppose two functions, f, g work. Then consider $f - g$. this is analytic and has a zero set, $\mathbb{R}$.

16. Do the same thing as Problem 15 for $\sin(z), \cos(z)$. Also explain with a very short argument why all identities for these functions continue to hold for the extended functions. This argument shouldn't require any computations at all. Why is $\sin(z)$ no longer bounded if z is allowed to be complex? **Hint:** You might try something involving the above formula for e^z to get the definition.

17. Show that if f is analytic on $\mathbb{C}$ and $f'(z) = 0$ for all z, then $f(z) \equiv c$ for some constant $c \in \mathbb{C}$. You might want to use Problem 14 to do this really quickly. Now using Theorem 11.6.10 prove Liouville's theorem which states that a function which is analytic on all of $\mathbb{C}$ which is also bounded is constant. **Hint:** By that theorem,

$$f'(z) = \frac{1}{2\pi i}\int_{C_r} \frac{f(w)}{(w-z)^2} dw$$

where C_r is the positively oriented circle of radius r which is centered at z. Now consider what happens as $r \to \infty$. You might use the corresponding version of Theorem 11.2.4 applied to contour integrals and note the total length of C_r is $2\pi r$.

18. Using Problem 15 prove the fundamental theorem of algebra which says every non-constant polynomial having complex coefficients has at least one zero in $\mathbb{C}$. (This is the very best way to prove the fundamental theorem of algebra.) **Hint:** If $p(z)$ has no zeros, consider $1/p(z)$ and prove it must then be bounded and analytic on all of $\mathbb{C}$.

19. Let f be analytic on U_i, the inside of Γ, a rectifiable simple closed curve positively oriented with parameterization γ. Suppose also there are no zeros of f on Γ. Show then that the number of zeros, of f contained in U_i counted according to multiplicity is given by the formula

$$\frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)} dz$$

Hint: You ought to first show $f(z) = \prod_{k=1}^m (z - z_k)\, g(z)$ where the z_k are the zeros of f in U_i and $g(z)$ is an analytic function which never vanishes in $U_i \cup \Gamma$. In the above product there might be some repeats corresponding to repeated zeros.

20. An open connected set U is said to be star shaped if there exists a point $z_0 \in U$ called a star center such that the for all $z \in U, \gamma(z_0, z)^*$ as described in before the proof of the Cauchy Goursat theorem is contained in U. For example, pick any complex number α and consider everything left after leaving out the ray $\{t\alpha : t \geq 0\}$. Show this is star shaped with a star center $t\alpha$ for $t < 0$. Now for U a star shaped open connected set, suppose g is analytic on U and $g(z) \neq 0$ for all $z \in U$. Show there exists an analytic function h defined on U such that

$$e^{h(z)} = g(z).$$

This function $h(z)$ is like $\log(g(z))$. **Hint:** Use an argument like that used to prove Morera's theorem and the Cauchy Goursat theorem to obtain a primitive for g'/g, h_1. Next consider the function

$$ge^{-h_1}$$

Using the chain rule and the product rule, show $\frac{d}{dz}\left(ge^{-h_1}\right) = 0$. Using one of the results of Problem 17 show

$$g = ce^{h_1}$$

for some constant c. Tell why c can be written as e^{a+ib}. Then let $h = h_1 + a + ib$.

21. One of the most amazing theorems is the open mapping theorem. Let U be an open connected set in $\mathbb{C}$ and suppose $f : U \to \mathbb{C}$ is analytic. Then $f(U)$ is either a point or an open connected set. In the case where $f(U)$ is an open connected set, it follows that for each $z_0 \in U$, there exists an open set, V containing z_0 and $m \in \mathbb{N}$ such that for all $z \in V$,

$$f(z) = f(z_0) + \phi(z)^m \tag{11.46}$$

where $\phi : V \to B(0,\delta)$ is one to one, analytic and onto, $\phi(z_0) = 0$, $\phi'(z) \neq 0$ on V and ϕ^{-1} analytic on $B(0,\delta)$. If f is one to one then $m = 1$ for each z_0 and $f^{-1} : f(U) \to U$ is analytic. Consider the real valued function $f(x) = x^2$. $f(\mathbb{R})$ is neither a point nor an open connected set. This is a strictly complex analysis phenomenon. **Hint:**Work out the details of the following outline. Suppose $f(U)$ is not a point. Then using Problem 14 about the zeros of an analytic function there exists $r > 0$ such that for $z \in B(z_0, r) \setminus \{z_0\}$,

$$f(z) - f(z_0) \neq 0.$$

Explain why there exists $g(z)$ analytic and nonzero on $B(z_0, r)$ such that for some positive integer m,

$$f(z) - f(z_0) = (z - z_0)^m g(z)$$

Next one tries to take the m^{th} root of $g(z)$. Using Problem 20 there exists h analytic such that

$$g(z) = e^{h(z)},\ g(z) = \left(e^{h(z)/m}\right)^m$$

Now let $\phi(z) = (z - z_0)\, e^{h(z)/m}$. This yields the formula 11.46. Also $\phi'(z_0) = e^{h(z_0)/m} \neq 0$. Now consider

$$\phi(x + iy) = u(x,y) + iv(x,y)$$

and the map

$$\begin{pmatrix} x \\ y \end{pmatrix} \to \begin{pmatrix} u(x,y) \\ v(x,y) \end{pmatrix}$$

Here u, v are C^1 because ϕ is analytic. Use the Cauchy Riemann equations to verify the Jacobian of this transformation at (x_0, y_0) is nonzero. This is where you use $\phi'(z_0) \neq 0$. Use inverse function theorem to verify ϕ maps some open set V containing z_0 one to one and onto $B(0,\delta)$. Thus also ϕ^m maps V onto $B(0,\delta^m)$. Explain why it follows from 11.46 and the fact that z_0 is arbitrary that f is an open map. Since f is continuous and U is connected, so is $f(U)$. However, if $m > 1$, this mapping, f can't be one to one. To verify this,

$$e^{i2\pi/m}\phi(z_1) \neq \phi(z_1)$$

but both are in $B(0,\delta)$. Hence there exists $z_2 \neq z_1$ such that $\phi(z_2) = e^{i2\pi/m}\phi(z_1)$ (ϕ is one to one) but $f(z_2) = f(z_1)$. If f is one to one, then the above shows that f^{-1} is continuous and for each z, the m in the above is always 1 so $f'(z) = e^{h(z)/1} \neq 0$. Hence

$$\begin{aligned} \left(f^{-1}\right)'(f(z)) &= \lim_{f(z_1)\to f(z)} \frac{f^{-1}(f(z_1)) - f^{-1}(f(z))}{f(z_1) - f(z)} \\ &= \lim_{z_1 \to z} \frac{z_1 - z}{f(z_1) - f(z)} = \frac{1}{f'(z)} \end{aligned}$$

22. Let U be what is left when you leave out the ray $t\alpha$ for $t \geq 0$. This is a star shaped open set and $g(z) = z$ is nonzero on this set. Therefore, there exists $h(z)$ such that $z = e^{h(z)}$ by Problem 20. Explain why $h(z)$ is analytic on U. When $\alpha = -1$ this is called the principle branch of the logarithm. In this case define $Arg(z) \equiv \theta \in (-\pi, \pi)$ such that the given z equals $|z| e^{i\theta}$. Explain why this principle branch of the logarithm is
$$\log(z) = \ln(|z|) + iArg(z)$$
Note it follows from the open mapping theorem this is an analytic function on U. You don't have to fuss with any tedium in order to show this.

23. Suppose Γ is a simple closed curve and let U_i be the inside. Suppose f is analytic on U_i and continuous on $U_i \cup \Gamma$. Consider the function $z \to |f(z)|$. This is a continuous function. Show that if it achieves its maximum at any point of U_i then f must be a constant. **Hint:** You might use the open mapping theorem.

24. Let f, g be analytic on U_i, the inside of Γ, a rectifiable simple closed curve positively oriented with parameterization γ. Suppose either
$$|f(z) + g(z)| < |f(z)| + |g(z)| \text{ on } \Gamma$$
or
$$|f(z) - g(z)| < |f(z)| \text{ on } \Gamma$$
Let Z_f denote the number of zeros in U_i and let Z_g denote the number of zeros of g in U_i. Then neither f, g, nor f/g can equal zero anywhere on Γ and $Z_f = Z_g$. **Hint:** The first condition implies for all $z \in \Gamma$,
$$\frac{f(z)}{g(z)} \in \mathbb{C} \setminus [0, \infty)$$
Show there exists a primitive F for
$$\frac{(f/g)'}{f/g}.$$
and argue
$$0 = \int_\gamma \frac{(f/g)'}{f/g} dz = \int_\gamma \frac{f'}{g'} dz - \int_\gamma \frac{g'}{g} dz = Z_f - Z_g.$$
You could consider $F = L(f/g)$ where L is the analytic function defined on $\mathbb{C} \setminus [0, \infty)$ with the property that
$$e^{L(z)} = z.$$
Thus
$$e^{L(z)} L'(z) = 1,\ L'(z) = 1/z.$$
In the second case, show $g/f \notin (-\infty, 0]$ and so a similar thing can be done. This problem is a case of Rouche's theorem.

25. Use the result of Problem 24 to give another proof of the fundamental theorem of algebra as follows. Let $g(z)$ be a polynomial of degree n, $a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0$ where $a_n \neq 0$. Now let $f(z) = a_n z^n$. Let Γ be a big circle, large enough that $|f(z) - g(z)| < |f(z)|$ on this circle. Then tell why g and f have the same number of zeros where they are counted according to multiplicity.

26. Let $p(z) = z^7 + 11z^3 - 5z^2 + 5$. Identify a ball $B(0,r)$ which must contain all the zeros of $p(z)$. Try to make r reasonably small. Use Problem 24.

27. Here is another approach to the open mapping theorem which I think might be a little easier and shorter which is based on Rouche's theorem and makes no reference to real variable techniques. Let $f : U \to \mathbb{C}$ where U is an open connected set. Then $f(U)$ is either an open connected set or a point. **Hint:** Suppose $f(U)$ is not a point. Then explain why for any $z_0 \in U$ there exists $r > 0$ such that

$$f(z) - f(z_0) = g(z)(z - z_0)^m$$

where g is analytic and nonzero on $\overline{B(z_0, r)}$. Now consider the function $z \to f(z) - w$. I would like to use Rouche's theorem to claim this function has the same number of zeros, namely m as the function $z \to f(z) - f(z_0)$. Let

$$\delta = \min\{|f(z) - f(z_0)| : |z - z_0| = r\}$$

Then if $|w - f(z_0)| < \delta$,

$$|w - f(z_0)| = |f(z) - f(z_0) - (f(z) - w)| < \delta \leq |f(z) - f(z_0)|$$

for each $z \in \partial B(z_0, r)$ and so you can apply Rouche's theorem. What does this say about when f is one to one? Why is $f(U)$ open? Why is $f(U)$ connected?

28. Let $\gamma : [a,b] \to \mathbb{C}$ be of bounded variation, $\gamma(a) = \gamma(b)$ and suppose $z \notin \gamma^*$. Define

$$n(\gamma, z) \equiv \frac{1}{2\pi i} \int_\gamma \frac{dw}{w - z}.$$

This is called the winding number. When γ^* is positively oriented and a simple closed curve, this number equals 1 by the Cauchy integral formula. However, it is always an integer. Furthermore, $z \to n(\gamma, z)$ is continuous and so is constant on every component of $\mathbb{C} \setminus \gamma^*$. For z in the unbounded component, $n(\gamma, z) = 0$. Most modern treatments of complex analysis feature the winding number extensively in the statement of all the major theorems. This is because it makes possible the most general form of the theorems. Prove the above properties of the winding number. **Hint:** The continuity is easy. It follows right away from a simple estimate and Theorem 11.2.4 applied to contour integrals. The tricky part is in showing it is an integer. This is where it is convenient to use Theorem 11.2.6 applied to contour integrals. There exists $\eta : [a,b] \to \mathbb{C}$ which is C^1 on $[a,b]$ and

$$\max\{|\eta(t) - \gamma(t)| : t \in [a,b]\} < \varepsilon,$$

$$\eta(a) = \eta(b) = \gamma(a) = \gamma(b)$$

$$\left| \frac{1}{2\pi i} \int_\gamma \frac{dw}{w - z} - \frac{1}{2\pi i} \int_\eta \frac{dw}{w - z} \right| < \varepsilon$$

where $\varepsilon < \operatorname{dist}(z, \gamma^*)$. Thus $z \notin \eta^*$. Consider the contour integral which involves η and show it is an integer. Then there exists a sequence of these C^1 contours $\{\eta_k\}$ such that

$$\left| \frac{1}{2\pi i} \int_\gamma \frac{dw}{w - z} - \frac{1}{2\pi i} \int_{\eta_k} \frac{dw}{w - z} \right| \to 0.$$

Consequently, for all k large enough there can be no change in

$$\frac{1}{2\pi i}\int_{\eta_k}\frac{dw}{w-z}$$

which shows

$$\frac{1}{2\pi i}\int_{\gamma}\frac{dw}{w-z}$$

is an integer as claimed. So how do you show the contour integral involving η yields an integer? As mentioned above,

$$\frac{1}{2\pi i}\int_{\eta_k}\frac{dw}{w-z}=\frac{1}{2\pi i}\int_a^b\frac{\eta'(t)}{\eta(t)-z}dt$$

Let

$$g(t)\equiv\int_a^t\frac{\eta'(s)}{\eta(s)-z}ds$$

Formally this is a lot like some sort of $\log(\eta(s)-z)$ (recall beginning calculus) so it is reasonable to consider

$$\left(\frac{e^{g(t)}}{\eta(t)-z}\right)'.$$

Show this equals 0. Explain why this requires the function which is differentiated must be constant. Thus

$$\frac{e^{g(a)}}{\eta(a)-z}=\frac{e^{g(b)}}{\eta(b)-z}$$

Now $\eta(a)=\eta(b), g(a)=0$, and so $e^{g(a)}=1=e^{g(b)}$. Explain why this requires $g(b)=2m\pi i$ for m an integer. Now this gives the desired result.

29. Let

$$B'(a,r)\equiv\{z\in\mathbb{C}\text{ such that }0<|z-a|<r\}.$$

Thus this is the usual ball without the center. A function is said to have an isolated singularity at the point $a\in\mathbb{C}$ if f is analytic on $B'(a,r)$ for some $r>0$.

An isolated singularity of f is said to be removable if there exists an analytic function, g analytic at a and near a such that $f=g$ at all points near a. A major theorem is the following.

Theorem 11.7.1 *Let $f:B'(a,r)\to X$ be analytic. Thus f has an isolated singularity at a. Suppose also that*

$$\lim_{z\to a}f(z)(z-a)=0.$$

Then there exists a unique analytic function, $g:B(a,r)\to X$ such that $g=f$ on $B'(a,r)$. Thus the singularity at a is removable.

Prove this theorem. **Hint:** Let $h(z)=f(z)(z-a)^2$. Then $h(a)=0$ and $h'(a)$ exists and equals 0. Show this. Also h is analytic near a. Therefore,

$$h(z)=\sum_{k=2}^{\infty}a_k(z-a)^k$$

Maybe consider $g(z)=h(z)/(z-a)^2$. Argue g is analytic and equals f for z near a.

30. Another really amazing theorem in complex analysis is the Casorati Weierstrass theorem.

Theorem 11.7.2 *Let a be an isolated singularity and suppose for some $r > 0$, $f(B'(a,r))$ is not dense in $\mathbb{C}$. Then either a is a removable singularity or there exist finitely many $b_1, \cdots, b_M$ for some finite number, M such that for z near a,*

$$f(z) = g(z) + \sum_{k=1}^{M} \frac{b_{-k}}{(z-a)^k} \tag{11.47}$$

where $g(z)$ is analytic near a. When the above formula holds, f is said to have a pole of order M at a.

Prove this theorem. **Hint:** Suppose a is not removable and $B(z_0, \delta)$ has no points of $f(B'(a,r))$. Such a ball must exist if $f(B'(a,r))$ is not dense in the plane. this means that for all $0 < |z-a| < r$,

$$|f(z) - z_0| \geq \delta > 0$$

Hence

$$\lim_{z \to a} \frac{1}{f(z) - z_0} (z-a) = 0$$

and so $1/(f(z) - z_0)$ has a removable sincularity at a. See Problem 29. Let $g(z)$ be analytic at and near a and agree with this function. Thus

$$g(z) = \sum_{n=0}^{\infty} a_n (z-a)^n .$$

There are two cases, $g(a) = 0$ and $g(a) \neq 0$. First suppose $g(a) = 0$. Then explain why

$$g(z) = h(z)(z-a)^m$$

where $h(z)$ is analytic and non zero near a. Then

$$f(z) - z_0 = \frac{1}{h(z)} \frac{1}{(z-a)^m}$$

Show this yields the desired conclusion. Next suppose $g(a) \neq 0$. Then explain why $g(z) \neq 0$ near a and this would contradict the assertion that a is not removable.

31. One of the very important techniques in complex analysis is the method of residues. When a is a pole the residue of f at a denoted by $\operatorname{res}(f,a)$, is defined as b_{-1} in 11.47. Suppose a is a pole and Γ is a simple closed rectifiable curve containing a on the inside with no other singular points on Γ or anywhere else inside Γ. Show that under these conditions,

$$\int_{\Gamma} f(z)\, dz = 2\pi i (\operatorname{res}(f,a))$$

Also describe a way to find $\operatorname{res}(f,a)$ by multiplying by $(z-a)^m$ and differentiating. **Hint:** You should show $\int_{\Gamma} \frac{1}{(z-a)^m} dz = 0$ whenever $m > 1$. This is because the function has a primitive.

32. Using Problem 9 give a holy version of the Cauchy integral theorem. This is it. Let Γ be a positively oriented rectifiable simple closed curve with inside U_i and remove finitely many open discs $B(z_j, r_j)$ from U_i. Thus the result is a holy region. Suppose f is analytic on some open set containing $\overline{U_i} \setminus \cup_{j=1}^{n} B(z_j, r_j)$. Then letting Γ_j denote the negatively oriented boundary of $B(z_j, r_j)$, show

$$0 = \int_\gamma f(z)\, dz + \sum_{j=1}^{n} \int_{\gamma_j} f(z)\, dz$$

where γ_j is a parameterization for Γ_j. **Hint:** The proof is the same as given earlier. You just use Green's theorem.

33. Let Γ be a simple closed curve and suppose on its inside there are finitely many poles for a function f which is analytic near Γ. Call these poles $\{z_k\}_{k=1}^{n}$. Then

$$\int_\gamma f(z)\, dz = 2\pi i \sum_{j=1}^{n} \operatorname{res}(f, z_j)$$

This is the very important residue theorem for computing line integrals. **Hint:** You should use Problem 32 and Problem 30, the Casorati Weierstrass theorem.

Hausdorff Measures And Area Formula

12.1 Definition Of Hausdorff Measures

This chapter is on Hausdorff measures. First I will discuss some outer measures. In all that is done here, $\alpha(n)$ will be the volume of the ball in $\mathbb{R}^n$ which has radius 1. This volume is the usual Lebesgue measure and the balls will be determined by the usual norm on $\mathbb{R}^n$.

Definition 12.1.1 *For a set, E, denote by $r(E)$ the number which is half the diameter of E. Thus*

$$r(E) \equiv \frac{1}{2}\sup\{|\mathbf{x}-\mathbf{y}| : \mathbf{x},\mathbf{y} \in E\} \equiv \frac{1}{2}\operatorname{diam}(E)$$

Let $E \subseteq \mathbb{R}^n$.

$$\mathcal{H}^s_\delta(E) \equiv \inf\{\sum_{j=1}^{\infty} \beta(s)(r(C_j))^s : E \subseteq \cup_{j=1}^{\infty} C_j, r(C_j) \leq \delta\}$$

$$\mathcal{H}^s(E) \equiv \lim_{\delta \to 0} \mathcal{H}^s_\delta(E).$$

In the above definition, $\beta(s)$ is an appropriate positive constant depending on s. Later I will tell what this constant is but it is not important for now. It will be chosen in such a way that whenever n is a positive integer, $\mathcal{H}^n([0,1)^n) = 1 = m_n([0,1)^n)$. In fact, this is all you need to know about it.

Lemma 12.1.2 *$\mathcal{H}^s$ and $\mathcal{H}^s_\delta$ are outer measures.*

Proof: It is clear that $\mathcal{H}^s(\emptyset) = 0$ and if $A \subseteq B$, then $\mathcal{H}^s(A) \leq \mathcal{H}^s(B)$ with similar assertions valid for $\mathcal{H}^s_\delta$. Suppose $E = \cup_{i=1}^{\infty} E_i$ and $\mathcal{H}^s_\delta(E_i) < \infty$ for each i. Let $\{C^i_j\}_{j=1}^{\infty}$ be a covering of E_i with

$$\sum_{j=1}^{\infty} \beta(s)(r(C^i_j))^s - \varepsilon/2^i < \mathcal{H}^s_\delta(E_i)$$

and $\mathrm{diam}(C_j^i) \leq \delta$. Then

$$\begin{aligned}\mathcal{H}_\delta^s(E) &\leq \sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\beta(s)(r(C_j^i))^s \\ &\leq \sum_{i=1}^{\infty}\mathcal{H}_\delta^s(E_i)+\varepsilon/2^i \\ &\leq \varepsilon+\sum_{i=1}^{\infty}\mathcal{H}_\delta^s(E_i).\end{aligned}$$

It follows that since $\varepsilon > 0$ is arbitrary,

$$\mathcal{H}_\delta^s(E) \leq \sum_{i=1}^{\infty}\mathcal{H}_\delta^s(E_i)$$

which shows $\mathcal{H}_\delta^s$ is an outer measure. Now notice that $\mathcal{H}_\delta^s(E)$ is increasing as $\delta \to 0$. Picking a sequence δ_k decreasing to 0, the monotone convergence theorem implies

$$\mathcal{H}^s(E) \leq \sum_{i=1}^{\infty}\mathcal{H}^s(E_i).$$

This proves the lemma.

The outer measure $\mathcal{H}^s$ is called s dimensional Hausdorff measure when restricted to the σ algebra of $\mathcal{H}^s$ measurable sets. Recall these are the sets, E such that for all S,

$$\mathcal{H}^s(S) = \mathcal{H}^s(S\cap E)+\mathcal{H}^s(S\setminus E).$$

Next I will show the σ algebra of $\mathcal{H}^s$ measurable sets includes the Borel sets. This is done by the following very interesting condition known as Caratheodory's criterion.

12.1.1 Properties Of Hausdorff Measure

Definition 12.1.3 *For two sets, A, B in a metric space, define*

$$\mathrm{dist}(A,B) \equiv \inf\{||\mathbf{x}-\mathbf{y}|| : \mathbf{x}\in A, \mathbf{y}\in B\}.$$

Theorem 12.1.4 *Let μ be an outer measure on the subsets of X, a closed subset of a normed vector space and suppose*

$$\mu(A\cup B) = \mu(A)+\mu(B)$$

whenever $\mathrm{dist}(A,B) > 0$*, then the σ algebra of measurable sets contains the Borel sets.*

Proof: It suffices to show that closed sets are in $\mathcal{F}$, the σ-algebra of measurable sets, because then the open sets are also in $\mathcal{F}$ and consequently $\mathcal{F}$ contains the Borel sets. Let K be closed and let S be a subset of Ω. Is $\mu(S) \geq \mu(S\cap K)+\mu(S\setminus K)$? It suffices to assume $\mu(S) < \infty$. Let

$$K_n \equiv \{x : \mathrm{dist}(x,K) \leq \frac{1}{n}\}$$

By Lemma 7.4.4 on Page 152, $x \to \mathrm{dist}(x,K)$ is continuous and so K_n is closed. By the assumption of the theorem,

$$\mu(S) \geq \mu((S \cap K) \cup (S \setminus K_n)) = \mu(S \cap K) + \mu(S \setminus K_n) \tag{12.1}$$

since $S \cap K$ and $S \setminus K_n$ are a positive distance apart. Now

$$\mu(S \setminus K_n) \leq \mu(S \setminus K) \leq \mu(S \setminus K_n) + \mu((K_n \setminus K) \cap S). \tag{12.2}$$

If $\lim_{n\to\infty} \mu((K_n \setminus K) \cap S) = 0$ then the theorem will be proved because this limit along with 12.2 implies $\lim_{n\to\infty} \mu(S \setminus K_n) = \mu(S \setminus K)$ and then taking a limit in 12.1, $\mu(S) \geq \mu(S \cap K) + \mu(S \setminus K)$ as desired. Therefore, it suffices to establish this limit.

Since K is closed, a point, $x \notin K$ must be at a positive distance from K and so

$$K_n \setminus K = \cup_{k=n}^{\infty} K_k \setminus K_{k+1}.$$

Therefore

$$\mu(S \cap (K_n \setminus K)) \leq \sum_{k=n}^{\infty} \mu(S \cap (K_k \setminus K_{k+1})). \tag{12.3}$$

If

$$\sum_{k=1}^{\infty} \mu(S \cap (K_k \setminus K_{k+1})) < \infty, \tag{12.4}$$

then $\mu(S \cap (K_n \setminus K)) \to 0$ because it is dominated by the tail of a convergent series so it suffices to show 12.4.

$$\sum_{k=1}^{M} \mu(S \cap (K_k \setminus K_{k+1})) =$$

$$\sum_{k \text{ even}, k\leq M} \mu(S \cap (K_k \setminus K_{k+1})) + \sum_{k \text{ odd}, k\leq M} \mu(S \cap (K_k \setminus K_{k+1})). \tag{12.5}$$

By the construction, the distance between any pair of sets, $S \cap (K_k \setminus K_{k+1})$ for different even values of k is positive and the distance between any pair of sets, $S \cap (K_k \setminus K_{k+1})$ for different odd values of k is positive. Therefore,

$$\sum_{k \text{ even}, k\leq M} \mu(S \cap (K_k \setminus K_{k+1})) + \sum_{k \text{ odd}, k\leq M} \mu(S \cap (K_k \setminus K_{k+1})) \leq$$

$$\mu(\bigcup_{k \text{ even}} S \cap (K_k \setminus K_{k+1})) + \mu(\bigcup_{k \text{ odd}} S \cap (K_k \setminus K_{k+1})) \leq 2\mu(S) < \infty$$

and so for all M, $\sum_{k=1}^{M} \mu(S \cap (K_k \setminus K_{k+1})) \leq 2\mu(S)$ showing 12.4 and proving the theorem.

The next theorem applies the Caratheodory criterion above to $\mathcal{H}^s$.

Theorem 12.1.5 *The σ algebra of $\mathcal{H}^s$ measurable sets contains the Borel sets and $\mathcal{H}^s$ has the property that for all $E \subseteq \mathbb{R}^n$, there exists a Borel set $F \supseteq E$ such that $\mathcal{H}^s(F) = \mathcal{H}^s(E)$.*

Proof: Let $\text{dist}(A, B) = 2\delta_0 > 0$. Is it the case that

$$\mathcal{H}^s(A) + \mathcal{H}^s(B) = \mathcal{H}^s(A \cup B)?$$

This is what is needed to use Caratheodory's criterion.

Let $\{C_j\}_{j=1}^{\infty}$ be a covering of $A \cup B$ such that $r(C_j) \leq \delta < \delta_0/2$ for each j and

$$\mathcal{H}^s_\delta(A \cup B) + \varepsilon > \sum_{j=1}^{\infty} \beta(s)(r(C_j))^s.$$

Thus

$$\mathcal{H}^s_\delta(A \cup B)\dot{} + \varepsilon > \sum_{j\in J_1} \beta(s)(r(C_j))^s + \sum_{j\in J_2} \beta(s)(r(C_j))^s$$

where

$$J_1 = \{j : C_j \cap A \neq \emptyset\},\ J_2 = \{j : C_j \cap B \neq \emptyset\}.$$

Recall $\operatorname{dist}(A, B) = 2\delta_0$ and so $J_1 \cap J_2 = \emptyset$. It follows

$$\mathcal{H}^s_\delta(A \cup B) + \varepsilon > \mathcal{H}^s_\delta(A) + \mathcal{H}^s_\delta(B).$$

Letting $\delta \to 0$, and noting $\varepsilon > 0$ was arbitrary, yields

$$\mathcal{H}^s(A \cup B) \geq \mathcal{H}^s(A) + \mathcal{H}^s(B).$$

Equality holds because $\mathcal{H}^s$ is an outer measure. By Caratheodory's criterion, $\mathcal{H}^s$ is a Borel measure.

To verify the second assertion, note first there is no loss of generality in letting $\mathcal{H}^s(E) < \infty$. Let

$$E \subseteq \cup_{j=1}^\infty C_j,\ r(C_j) < \delta,$$

and

$$\mathcal{H}^s_\delta(E) + \delta > \sum_{j=1}^\infty \beta(s)(r(C_j))^s.$$

Let

$$F_\delta = \cup_{j=1}^\infty \overline{C_j}.$$

Thus $F_\delta \supseteq E$ and

$$\begin{aligned} \mathcal{H}^s_\delta(E) &\leq \mathcal{H}^s_\delta(F_\delta) \leq \sum_{j=1}^\infty \beta(s)(r(\overline{C_j}))^s \\ &= \sum_{j=1}^\infty \beta(s)(r(C_j))^s < \delta + \mathcal{H}^s_\delta(E). \end{aligned}$$

Let $\delta_k \to 0$ and let $F = \cap_{k=1}^\infty F_{\delta_k}$. Then $F \supseteq E$ and

$$\mathcal{H}^s_{\delta_k}(E) \leq \mathcal{H}^s_{\delta_k}(F) \leq \mathcal{H}^s_{\delta_k}(F_\delta) \leq \delta_k + \mathcal{H}^s_{\delta_k}(E).$$

Letting $k \to \infty$,

$$\mathcal{H}^s(E) \leq \mathcal{H}^s(F) \leq \mathcal{H}^s(E)$$

This proves the theorem.

A measure satisfying the first conclusion of Theorem 12.1.5 is sometimes called a Borel regular measure.

12.1.2 $\mathcal{H}^n$ And m_n

Next I will compare $\mathcal{H}^n$ and m_n. To do this, recall the following covering theorem which is a summary of Corollaries 9.7.5 and 9.7.4 found on Page 220.

Theorem 12.1.6 *Let $E \subseteq \mathbb{R}^n$ and let $\mathcal{F}$, be a collection of balls of bounded radii such that $\mathcal{F}$ covers E in the sense of Vitali. Then there exists a countable collection of disjoint balls from $\mathcal{F}$, $\{B_j\}_{j=1}^\infty$, such that $\overline{m_n}(E \setminus \cup_{j=1}^\infty B_j) = 0$.*

In the next lemma, the balls are the usual balls taken with respect to the usual distance in $\mathbb{R}^n$.

Lemma 12.1.7 *If $m_n(S) = 0$ then $\mathcal{H}^n(S) = \mathcal{H}^n_\delta(S) = 0$. Also, there exists a constant, k such that $\mathcal{H}^n(E) \leq k m_n(E)$ for all E Borel. Also, if $Q_0 \equiv [0,1)^n$, the unit cube, then $\mathcal{H}^n([0,1)^n) > 0$.*

Proof: Suppose first $m_n(S) = 0$. First suppose S is bounded. Then by outer regularity, there exists a bounded open V containing S and $m_n(V) < \varepsilon$. For each $\mathbf{x} \in S$, there exists a ball $B_{\mathbf{x}}$ such that $\widehat{B_{\mathbf{x}}} \subseteq V$ and $\delta > r\left(\widehat{B_{\mathbf{x}}}\right)$. By the Vitali covering theorem there is a sequence of disjoint balls $\{B_k\}$ such that $\left\{\widehat{B_k}\right\}$ covers S. Then letting $\alpha(n)$ be the Lebesgue measure of the unit ball in $\mathbb{R}^n$

$$\begin{aligned}\mathcal{H}^n_\delta(S) &\leq \sum_k \beta(n)\, r\left(\widehat{B_k}\right)^n = \frac{\beta(n)}{\alpha(n)} 5^n \sum_k \alpha(n)\, r(B_k)^n \\ &\leq \frac{\beta(n)}{\alpha(n)} 5^n m_n(V) < \frac{\beta(n)}{\alpha(n)} 5^n \varepsilon\end{aligned}$$

Since ε is arbitrary, this shows $\mathcal{H}^n_\delta(S) = 0$ and now it follows $\mathcal{H}^n(S) = 0$. In case S is not bounded, let $S_m = B(\mathbf{0},m) \cap S$. Then $\mathcal{H}^n_\delta(S_m) = 0$ and so letting $m \to \infty, \mathcal{H}^n_\delta(S) = 0$ also. Then as before, $\mathcal{H}^n(S) = 0$.

Letting U be an open set and $\delta > 0$, consider all balls, B contained in U which have diameters less than δ. This is a Vitali covering of U and therefore by Theorem 12.1.6, there exists $\{B_i\}$, a sequence of disjoint balls of radii less than δ contained in U such that $\cup_{i=1}^\infty B_i$ differs from U by a set of Lebesgue measure zero. Let $\alpha(n)$ be the Lebesgue measure of the unit ball in $\mathbb{R}^n$. Then from what was just shown,

$$\begin{aligned}\mathcal{H}^n_\delta(U) &= \mathcal{H}^n_\delta(\cup_i B_i) \leq \sum_{i=1}^\infty \beta(n)\, r(B_i)^n = \frac{\beta(n)}{\alpha(n)} \sum_{i=1}^\infty \alpha(n)\, r(B_i)^n \\ &= \frac{\beta(n)}{\alpha(n)} \sum_{i=1}^\infty m_n(B_i) = \frac{\beta(n)}{\alpha(n)} m_n(U) \equiv k m_n(U).\end{aligned}$$

Now letting E be Borel, it follows from the outer regularity of m_n there exists a decreasing sequence of open sets, $\{V_i\}$ containing E such such that $m_n(V_i) \to m_n(E)$. Then from the above,

$$\mathcal{H}^n_\delta(E) \leq \lim_{i\to\infty} \mathcal{H}^n_\delta(V_i) \leq \lim_{i\to\infty} k m_n(V_i) = k m_n(E).$$

Since $\delta > 0$ is arbitrary, it follows that also

$$\mathcal{H}^n(E) \leq k m_n(E).$$

This proves the first part of the lemma.

To verify the second part, note that it is obvious $\mathcal{H}^n_\delta$ and $\mathcal{H}^n$ are translation invariant because diameters of sets do not change when translated. Therefore, if $\mathcal{H}^n([0,1)^n) = 0$, it follows $\mathcal{H}^n(\mathbb{R}^n) = 0$ because $\mathbb{R}^n$ is the countable union of translates of $Q_0 \equiv [0,1)^n$. Since each $\mathcal{H}^n_\delta$ is no larger than $\mathcal{H}^n$, the same must hold for $\mathcal{H}^n_\delta$. Therefore, there exists a sequence of sets, $\{C_i\}$ each having diameter less than δ such that the union of these sets equals $\mathbb{R}^n$ but

$$1 > \sum_{i=1}^\infty \beta(n)\, r(C_i)^n.$$

Now let B_i be a ball having radius equal to $\text{diam}\,(C_i) = 2r\,(C_i)$ which contains C_i. It follows

$$m_n\,(B_i) = \alpha\,(n)\,2^n r\,(C_i)^n = \frac{\alpha\,(n)\,2^n}{\beta\,(n)}\beta\,(n)\,r\,(C_i)^n$$

which implies

$$1 > \sum_{i=1}^{\infty} \beta\,(n)\,r\,(C_i)^n = \sum_{i=1}^{\infty} \frac{\beta\,(n)}{\alpha\,(n)\,2^n} m_n\,(B_i) = \infty,$$

a contradiction. This proves the lemma.

Lemma 12.1.8 *Every open set in $\mathbb{R}^n$ is the countable disjoint union of half open boxes of the form*

$$\prod_{i=1}^{n} (a_i, a_i + 2^{-k}]$$

where $a_i = l2^{-k}$ for some integers, l, k. The sides of these boxes are of equal length. One could also have half open boxes of the form

$$\prod_{i=1}^{n} [a_i, a_i + 2^{-k})$$

and the conclusion would be unchanged.

Proof: Let

$$\mathcal{C}_k = \{\text{All half open boxes } \prod_{i=1}^{n} (a_i, a_i + 2^{-k}] \text{ where}$$

$$a_i = l2^{-k} \text{ for some integer } l.\}$$

Thus $\mathcal{C}_k$ consists of a countable disjoint collection of boxes whose union is $\mathbb{R}^n$. This is sometimes called a tiling of $\mathbb{R}^n$. Think of tiles on the floor of a bathroom and you will get the idea. Note that each box has diameter no larger than $2^{-k}\sqrt{n}$. This is because if

$$\mathbf{x}, \mathbf{y} \in \prod_{i=1}^{n} (a_i, a_i + 2^{-k}],$$

then $|x_i - y_i| \leq 2^{-k}$. Therefore,

$$|\mathbf{x} - \mathbf{y}| \leq \left(\sum_{i=1}^{n} \left(2^{-k}\right)^2 \right)^{1/2} = 2^{-k}\sqrt{n}.$$

Let U be open and let $\mathcal{B}_1 \equiv$ all sets of $\mathcal{C}_1$ which are contained in U. If $\mathcal{B}_1, \cdots, \mathcal{B}_k$ have been chosen, $\mathcal{B}_{k+1} \equiv$ all sets of $\mathcal{C}_{k+1}$ contained in

$$U \setminus \cup \left(\cup_{i=1}^{k} \mathcal{B}_i \right).$$

Let $\mathcal{B}_\infty = \cup_{i=1}^{\infty} \mathcal{B}_i$. In fact $\cup \mathcal{B}_\infty = U$. Clearly $\cup \mathcal{B}_\infty \subseteq U$ because every box of every $\mathcal{B}_i$ is contained in U. If $p \in U$, let k be the smallest integer such that p is contained in a box from $\mathcal{C}_k$ which is also a subset of U. Thus

$$p \in \cup \mathcal{B}_k \subseteq \cup \mathcal{B}_\infty.$$

Hence $\mathcal{B}_\infty$ is the desired countable disjoint collection of half open boxes whose union is U. The last assertion about the other type of half open rectangle is obvious. This proves the lemma.

Theorem 12.1.9 *By choosing $\beta(n)$ properly, one can obtain $\mathcal{H}^n = m_n$ on all Lebesgue measurable sets.*

Proof: I will show $\mathcal{H}^n$ is a positive multiple of m_n for any choice of $\beta(n)$. Define

$$k = \frac{m_n(Q_0)}{\mathcal{H}^n(Q_0)}$$

where $Q_0 = [0,1)^n$ is the half open unit cube in $\mathbb{R}^n$. I will show $k\mathcal{H}^n(E) = m_n(E)$ for any Lebesgue measurable set. When this is done, it will follow that by adjusting $\beta(n)$ the multiple can be taken to be 1.

Let $Q = \prod_{i=1}^n [a_i, a_i + 2^{-k})$ be a half open box where $a_i = l2^{-k}$. Thus Q_0 is the union of $\left(2^k\right)^n$ of these identical half open boxes. By translation invariance, of $\mathcal{H}^n$ and m_n

$$\left(2^k\right)^n \mathcal{H}^n(Q) = \mathcal{H}^n(Q_0) = \frac{1}{k} m_n(Q_0) = \frac{1}{k}\left(2^k\right)^n m_n(Q).$$

Therefore, $k\mathcal{H}^n(Q) = m_n(Q)$ for any such half open box and by translation invariance, for the translation of any such half open box. It follows from Lemma 12.1.8 that $k\mathcal{H}^n(U) = m_n(U)$ for all open sets. It follows immediately, since every compact set is the countable intersection of open sets that $k\mathcal{H}^n = m_n$ on compact sets. Therefore, they are also equal on all closed sets because every closed set is the countable union of compact sets. Now let F be an arbitrary Lebesgue measurable set. I will show that F is $\mathcal{H}^n$ measurable and that $k\mathcal{H}^n(F) = m_n(F)$. Let $F_l = B(\mathbf{0}, l) \cap F$. Then there exists H a countable union of compact sets and G a countable intersection of open sets such that

$$H \subseteq F_l \subseteq G \tag{12.6}$$

and $m_n(G \setminus H) = 0$ which implies by Lemma 12.1.7

$$m_n(G \setminus H) = k\mathcal{H}^n(G \setminus H) = 0. \tag{12.7}$$

To do this, let $\{G_i\}$ be a decreasing sequence of bounded open sets containing F_l and let $\{H_i\}$ be an increasing sequence of compact sets contained in F_l such that

$$k\mathcal{H}^n(G_i \setminus H_i) = m_n(G_i \setminus H_i) < 2^{-i}$$

Then letting $G = \cap_i G_i$ and $H = \cup_i H_i$ this establishes 12.6 and 12.7. Then by completeness of $\mathcal{H}^n$ it follows F_l is $\mathcal{H}^n$ measurable and

$$k\mathcal{H}^n(F_l) = k\mathcal{H}^n(H) = m_n(H) = m_n(F_l).$$

Now taking $l \to \infty$, it follows F is $\mathcal{H}^n$ measurable and $k\mathcal{H}^n(F) = m_n(F)$. Therefore, adjusting $\beta(n)$ it can be assumed the constant, k is 1. This proves the theorem.

The exact determination of $\beta(n)$ is more technical. You can skip it if you want. Just remember $\beta(n)$ is chosen such that $\mathcal{H}^n([0,1)^n) = 1$. It turns out this will require $\beta(n) = \alpha(n)$ where $\alpha(n)$ is the volume of the unit ball taken with respect to the usual norm. The optional sections are starred.

12.2 Technical Considerations*

Let $\alpha(n)$ be the volume of the unit ball in $\mathbb{R}^n$. Thus the volume of $B(\mathbf{0}, r)$ in $\mathbb{R}^n$ is $\alpha(n)r^n$ from the change of variables formula. There is a very important and interesting inequality known as the isodiametric inequality which says that if A is any set in $\mathbb{R}^n$, then

$$\overline{m}(A) \leq \alpha(n)(2^{-1}\mathrm{diam}(A))^n.$$

This inequality may seem obvious at first but it is not really. The reason it is not is that there are sets which are not subsets of any sphere having the same diameter as the set. For example, consider an equilateral triangle.

Lemma 12.2.1 *Let $f : \mathbb{R}^{n-1} \to [0,\infty)$ be Borel measurable and let*

$$S = \{(\mathbf{x},y) : |y| < f(\mathbf{x})\}.$$

Then S is a Borel set in $\mathbb{R}^n$.

Proof: Set s_k be an increasing sequence of Borel measurable functions converging pointwise to f.

$$s_k(\mathbf{x}) = \sum_{m=1}^{N_k} c_m^k \mathcal{X}_{E_m^k}(\mathbf{x}).$$

Let

$$S_k = \cup_{m=1}^{N_k} E_m^k \times (-c_m^k, c_m^k).$$

Then $(\mathbf{x},y) \in S_k$ if and only if $f(\mathbf{x}) > 0$ and $|y| < s_k(\mathbf{x}) \leq f(\mathbf{x})$. It follows that $S_k \subseteq S_{k+1}$ and

$$S = \cup_{k=1}^{\infty} S_k.$$

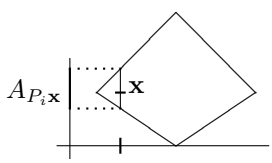
But each S_k is a Borel set and so S is also a Borel set. This proves the lemma.

Let P_i be the projection onto

$$\text{span}\,(\mathbf{e}_1, \cdots, \mathbf{e}_{i-1}, \mathbf{e}_{i+1}, \cdots, \mathbf{e}_n)$$

where the $\mathbf{e}_k$ are the standard basis vectors in $\mathbb{R}^n$, $\mathbf{e}_k$ being the vector having a 1 in the k^{th} slot and a 0 elsewhere. Thus $P_i\mathbf{x} \equiv \sum_{j\neq i} x_j \mathbf{e}_j$. Also let

$$A_{P_i\mathbf{x}} \equiv \{x_i : (x_1, \cdots, x_i, \cdots, x_n) \in A\}$$



$P_i\mathbf{x} \in \text{span}\{\mathbf{e}_1, \cdots, \mathbf{e}_{i-1}\mathbf{e}_{i+1}, \cdots, \mathbf{e}_n\}$.

Lemma 12.2.2 *Let $A \subseteq \mathbb{R}^n$ be a Borel set. Then $P_i\mathbf{x} \to m(A_{P_i\mathbf{x}})$ is a Borel measurable function defined on $P_i(\mathbb{R}^n)$.*

Proof: Let $\mathcal{K}$ be the π system consisting of sets of the form $\prod_{j=1}^n A_j$ where A_i is Borel. Also let $\mathcal{G}$ denote those Borel sets of $\mathbb{R}^n$ such that if $A \in \mathcal{G}$ then

$$P_i\mathbf{x} \to m((A \cap R_k)_{P_i\mathbf{x}}) \text{ is Borel measurable.}$$

where $R_k = (-k,k)^n$. Thus $\mathcal{K} \in \mathcal{G}$. If $A \in \mathcal{G}$

$$P_i\mathbf{x} \to m\left(\left(A^C \cap R_k\right)_{P_i\mathbf{x}}\right)$$

is Borel measurable because it is of the form

$$m\left((R_k)_{P_i\mathbf{x}}\right) - m\left((A \cap R_k)_{P_i\mathbf{x}}\right)$$

and these are Borel measurable functions of $P_i\mathbf{x}$. Also, if $\{A_i\}$ is a disjoint sequence of sets in $\mathcal{G}$ then

$$m\left(\left(\cup_i A_i \cap R_k\right)_{P_i\mathbf{x}}\right) = \sum_i m\left(\left(A_i \cap R_k\right)_{P_i\mathbf{x}}\right)$$

and each function of $P_i\mathbf{x}$ is Borel measurable. Thus by the lemma on π systems $\mathcal{G} = \mathcal{B}(\mathbb{R}^n)$ and this proves the lemma.

Now let $A \subseteq \mathbb{R}^n$ be Borel. Let P_i be the projection onto

$$\text{span}\left(\mathbf{e}_1, \cdots, \mathbf{e}_{i-1}, \mathbf{e}_{i+1}, \cdots, \mathbf{e}_n\right)$$

and as just described,

$$A_{P_i\mathbf{x}} = \{y \in \mathbb{R} \ : P_i\mathbf{x} + y\mathbf{e}_i \in A\}$$

Thus for $\mathbf{x} = (x_1, \cdots, x_n)$,

$$A_{P_i\mathbf{x}} = \{y \in \mathbb{R} : (x_1, \cdots, x_{i-1}, y, x_{i+1}, \cdots, x_n) \in A\}.$$

Since A is Borel, it follows from Lemma 12.2.1 that

$$P_i\mathbf{x} \to m(A_{P_i\mathbf{x}})$$

is a Borel measurable function on $P_i\mathbb{R}^n = \mathbb{R}^{n-1}$.

12.2.1 Steiner Symmetrization*

Define

$$S(A, \mathbf{e}_i) \equiv \{\mathbf{x} = P_i\mathbf{x} + y\mathbf{e}_i : |y| < 2^{-1}m(A_{P_i\mathbf{x}})\}$$

Lemma 12.2.3 *Let A be a Borel subset of $\mathbb{R}^n$. Then $S(A, \mathbf{e}_i)$ satisfies*

$$P_i\mathbf{x} + y\mathbf{e}_i \in S(A, \mathbf{e}_i) \text{ if and only if } P_i\mathbf{x} - y\mathbf{e}_i \in S(A, \mathbf{e}_i),$$

$$S(A, \mathbf{e}_i) \text{ is a Borel set in } \mathbb{R}^n,$$

$$m_n(S(A, \mathbf{e}_i)) = m_n(A), \tag{12.8}$$

$$diam(S(A, \mathbf{e}_i)) \leq \ diam(A). \tag{12.9}$$

Proof: The first assertion is obvious from the definition. The Borel measurability of $S(A, \mathbf{e}_i)$ follows from the definition and Lemmas 12.2.2 and 12.2.1. To show Formula 12.8,

$$\begin{aligned} m_n(S(A, \mathbf{e}_i)) &= \int_{P_i\mathbb{R}^n} \int_{-2^{-1}m(A_{P_i\mathbf{x}})}^{2^{-1}m(A_{P_i\mathbf{x}})} dx_i dx_1 \cdots dx_{i-1} dx_{i+1} \cdots dx_n \\ &= \int_{P_i\mathbb{R}^n} m(A_{P_i\mathbf{x}}) dx_1 \cdots dx_{i-1} dx_{i+1} \cdots dx_n \\ &= m(A). \end{aligned}$$

Now suppose $\mathbf{x}_1$ and $\mathbf{x}_2 \in S(A, \mathbf{e}_i)$

$$\mathbf{x}_1 = P_i\mathbf{x}_1 + y_1\mathbf{e}_i, \ \mathbf{x}_2 = P_i\mathbf{x}_2 + y_2\mathbf{e}_i.$$

For $\mathbf{x} \in A$ define

$$l(\mathbf{x}) = \sup\{y : P_i\mathbf{x} + y\mathbf{e}_i \in A\}.$$

$$g(\mathbf{x}) = \inf\{y : P_i\mathbf{x} + y\mathbf{e}_i \in A\}.$$

Then it is clear that

$$l(\mathbf{x}_1) - g(\mathbf{x}_1) \geq m(A_{P_i\mathbf{x}_1}) \geq 2|y_1|, \tag{12.10}$$

$$l(\mathbf{x}_2) - g(\mathbf{x}_2) \geq m(A_{P_i\mathbf{x}_2}) \geq 2|y_2|. \tag{12.11}$$

Claim: $|y_1 - y_2| \leq |l(\mathbf{x}_1) - g(\mathbf{x}_2)|$ or $|y_1 - y_2| \leq |l(\mathbf{x}_2) - g(\mathbf{x}_1)|$.

Proof of Claim: If not,

$$\begin{aligned} 2|y_1 - y_2| &> |l(\mathbf{x}_1) - g(\mathbf{x}_2)| + |l(\mathbf{x}_2) - g(\mathbf{x}_1)| \\ &\geq |l(\mathbf{x}_1) - g(\mathbf{x}_1) + l(\mathbf{x}_2) - g(\mathbf{x}_2)| \\ &= l(\mathbf{x}_1) - g(\mathbf{x}_1) + l(\mathbf{x}_2) - g(\mathbf{x}_2). \\ &\geq 2\,|y_1| + 2\,|y_2| \end{aligned}$$

by 12.10 and 12.11 contradicting the triangle inequality.

Now suppose $|y_1 - y_2| \leq |l(\mathbf{x}_1) - g(\mathbf{x}_2)|$. From the claim,

$$\begin{aligned} |\mathbf{x}_1 - \mathbf{x}_2| &= (|P_i\mathbf{x}_1 - P_i\mathbf{x}_2|^2 + |y_1 - y_2|^2)^{1/2} \\ &\leq (|P_i\mathbf{x}_1 - P_i\mathbf{x}_2|^2 + |l(\mathbf{x}_1) - g(\mathbf{x}_2)|^2)^{1/2} \\ &\leq (|P_i\mathbf{x}_1 - P_i\mathbf{x}_2|^2 + (|z_1 - z_2| + 2\varepsilon)^2)^{1/2} \\ &\leq \operatorname{diam}(A) + O(\sqrt{\varepsilon}) \end{aligned}$$

where z_1 and z_2 are such that $P_i\mathbf{x}_1 + z_1\mathbf{e}_i \in A$, $P_i\mathbf{x}_2 + z_2\mathbf{e}_i \in A$, and

$$|z_1 - l(\mathbf{x}_1)| < \varepsilon \text{ and } |z_2 - g(\mathbf{x}_2)| < \varepsilon.$$

If $|y_1 - y_2| \leq |l(\mathbf{x}_2) - g(\mathbf{x}_1)|$, then we use the same argument but let

$$|z_1 - g(\mathbf{x}_1)| < \varepsilon \text{ and } |z_2 - l(\mathbf{x}_2)| < \varepsilon,$$

Since $\mathbf{x}_1, \mathbf{x}_2$ are arbitrary elements of $S(A, \mathbf{e}_i)$ and ε is arbitrary, this proves 12.9.

The next lemma says that if A is already symmetric with respect to the j^{th} direction, then this symmetry is not destroyed by taking $S(A, \mathbf{e}_i)$.

Lemma 12.2.4 *Suppose A is a Borel set in $\mathbb{R}^n$ such that $P_j\mathbf{x} + \mathbf{e}_j x_j \in A$ if and only if $P_j\mathbf{x} + (-x_j)\mathbf{e}_j \in A$. Then if $i \neq j$, $P_j\mathbf{x} + \mathbf{e}_j x_j \in S(A, \mathbf{e}_i)$ if and only if $P_j\mathbf{x} + (-x_j)\mathbf{e}_j \in S(A, \mathbf{e}_i)$.*

Proof: By definition,

$$P_j\mathbf{x} + \mathbf{e}_j x_j \in S(A, \mathbf{e}_i)$$

if and only if

$$|x_i| < 2^{-1} m(A_{P_i(P_j\mathbf{x} + \mathbf{e}_j x_j)}).$$

Now

$$x_i \in A_{P_i(P_j\mathbf{x} + \mathbf{e}_j x_j)}$$

if and only if

$$x_i \in A_{P_i(P_j\mathbf{x} + (-x_j)\mathbf{e}_j)}$$

by the assumption on A which says that A is symmetric in the $\mathbf{e}_j$ direction. Hence

$$P_j\mathbf{x} + \mathbf{e}_j x_j \in S(A, \mathbf{e}_i)$$

if and only if

$$|x_i| < 2^{-1} m(A_{P_i(P_j\mathbf{x} + (-x_j)\mathbf{e}_j)})$$

if and only if

$$P_j\mathbf{x} + (-x_j)\mathbf{e}_j \in S(A, \mathbf{e}_i).$$

This proves the lemma.

12.2.2 The Isodiametric Inequality*

The next theorem is called the isodiametric inequality. It is the key result used to compare Lebesgue and Hausdorff measures.

Theorem 12.2.5 *Let A be any Lebesgue measurable set in $\mathbb{R}^n$. Then*

$$m_n(A) \leq \alpha(n)(r(A))^n.$$

Proof: Suppose first that A is Borel. Let $A_1 = S(A, \mathbf{e}_1)$ and let $A_k = S(A_{k-1}, \mathbf{e}_k)$. Then by the preceding lemmas, A_n is a Borel set, $\operatorname{diam}(A_n) \leq \operatorname{diam}(A)$, $m_n(A_n) = m_n(A)$, and A_n is symmetric. Thus $\mathbf{x} \in A_n$ if and only if $-\mathbf{x} \in A_n$. It follows that

$$A_n \subseteq \overline{B(\mathbf{0}, r(A_n))}.$$

(If $\mathbf{x} \in A_n \setminus \overline{B(\mathbf{0}, r(A_n))}$, then $-\mathbf{x} \in A_n \setminus \overline{B(\mathbf{0}, r(A_n))}$ and so $\operatorname{diam}(A_n) \geq 2|\mathbf{x}| > \operatorname{diam}(A_n)$.) Therefore,

$$m_n(A_n) \leq \alpha(n)(r(A_n))^n \leq \alpha(n)(r(A))^n.$$

It remains to establish this inequality for arbitrary measurable sets. Letting A be such a set, let $\{K_n\}$ be an increasing sequence of compact subsets of A such that

$$m(A) = \lim_{k\to\infty} m(K_k).$$

Then

$$\begin{aligned} m(A) &= \lim_{k\to\infty} m(K_k) \leq \limsup_{k\to\infty} \alpha(n)(r(K_k))^n \\ &\leq \alpha(n)(r(A))^n. \end{aligned}$$

This proves the theorem.

12.2.3 The Proper Value Of $\beta(n)$*

I will show that the proper determination of $\beta(n)$ is $\alpha(n)$, the volume of the unit ball. Since $\beta(n)$ has been adjusted such that $k = 1$, $m_n(B(\mathbf{0}, 1)) = \mathcal{H}^n(B(\mathbf{0}, 1))$. There exists a covering of $B(\mathbf{0},1)$ of sets of radii less than δ, $\{C_i\}_{i=1}^{\infty}$ such that

$$\mathcal{H}^n_\delta(B(\mathbf{0}, 1)) + \varepsilon > \sum_i \beta(n) r(C_i)^n$$

Then by Theorem 12.2.5, the isodiametric inequality,

$$\begin{aligned} \mathcal{H}^n_\delta(B(\mathbf{0}, 1)) + \varepsilon &> \sum_i \beta(n) r(C_i)^n = \frac{\beta(n)}{\alpha(n)} \sum_i \alpha(n) r\left(\overline{C}_i\right)^n \\ &\geq \frac{\beta(n)}{\alpha(n)} \sum_i m_n\left(\overline{C}_i\right) \geq \frac{\beta(n)}{\alpha(n)} m_n(B(\mathbf{0}, 1)) = \frac{\beta(n)}{\alpha(n)} \mathcal{H}^n(B(\mathbf{0}, 1)) \end{aligned}$$

Now taking the limit as $\delta \to 0$,

$$\mathcal{H}^n(B(\mathbf{0}, 1)) + \varepsilon \geq \frac{\beta(n)}{\alpha(n)} \mathcal{H}^n(B(\mathbf{0}, 1))$$

and since $\varepsilon > 0$ is arbitrary, this shows $\alpha(n) \geq \beta(n)$.

By the Vitali covering theorem, there exists a sequence of disjoint balls, $\{B_i\}$ such that $B(\mathbf{0},1) = (\cup_{i=1}^{\infty} B_i) \cup N$ where $m_n(N) = 0$. Then $\mathcal{H}_\delta^n(N) = 0$ can be concluded because $\mathcal{H}_\delta^n \leq \mathcal{H}^n$ and Lemma 12.1.7. Using $m_n(B(\mathbf{0},1)) = \mathcal{H}^n(B(\mathbf{0},1))$ again,

$$\begin{aligned}
\mathcal{H}_\delta^n(B(\mathbf{0},1)) &= \mathcal{H}_\delta^n(\cup_i B_i) \leq \sum_{i=1}^{\infty} \beta(n) r(B_i)^n \\
&= \frac{\beta(n)}{\alpha(n)} \sum_{i=1}^{\infty} \alpha(n) r(B_i)^n = \frac{\beta(n)}{\alpha(n)} \sum_{i=1}^{\infty} m_n(B_i) \\
&= \frac{\beta(n)}{\alpha(n)} m_n(\cup_i B_i) = \frac{\beta(n)}{\alpha(n)} m_n(B(\mathbf{0},1)) = \frac{\beta(n)}{\alpha(n)} \mathcal{H}^n(B(\mathbf{0},1))
\end{aligned}$$

which implies $\alpha(n) \leq \beta(n)$ and so the two are equal. This proves that if $\alpha(n) = \beta(n)$, then the $\mathcal{H}^n = m_n$ on the measurable sets of $\mathbb{R}^n$.

This gives another way to think of Lebesgue measure which is a particularly nice way because it is coordinate free, depending only on the notion of distance.

For $s < n$, note that $\mathcal{H}^s$ is not a Radon measure because it will not generally be finite on compact sets. For example, let $n = 2$ and consider $\mathcal{H}^1(L)$ where L is a line segment joining $(0,0)$ to $(1,0)$. Then $\mathcal{H}^1(L)$ is no smaller than $\mathcal{H}^1(L)$ when L is considered a subset of $\mathbb{R}^1, n = 1$. Thus by what was just shown, $\mathcal{H}^1(L) \geq 1$. Hence $\mathcal{H}^1([0,1] \times [0,1]) = \infty$. The situation is this: L is a one-dimensional object inside $\mathbb{R}^2$ and $\mathcal{H}^1$ is giving a one-dimensional measure of this object. In fact, Hausdorff measures can make such heuristic remarks as these precise. Define the Hausdorff dimension of a set, A, as

$$\dim(A) = \inf\{s : \mathcal{H}^s(A) = 0\}$$

12.2.4 A Formula For $\alpha(n)^*$

What is $\alpha(n)$? Recall the gamma function which makes sense for all $p > 0$.

$$\Gamma(p) \equiv \int_0^{\infty} e^{-t} t^{p-1} dt.$$

Lemma 12.2.6 *The following identities hold.*

$$p\Gamma(p) = \Gamma(p+1),$$

$$\Gamma(p)\Gamma(q) = \left(\int_0^1 x^{p-1}(1-x)^{q-1} dx \right) \Gamma(p+q),$$

$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$$

Proof: Using integration by parts,

$$\begin{aligned}
\Gamma(p+1) &= \int_0^{\infty} e^{-t} t^p dt = -e^{-t} t^p |_0^{\infty} + p \int_0^{\infty} e^{-t} t^{p-1} dt \\
&= p\Gamma(p)
\end{aligned}$$

Next

$$\begin{aligned}\Gamma(p)\Gamma(q) &= \int_0^\infty e^{-t}t^{p-1}dt\int_0^\infty e^{-s}s^{q-1}ds\\ &= \int_0^\infty\int_0^\infty e^{-(t+s)}t^{p-1}s^{q-1}dtds\\ &= \int_0^\infty\int_s^\infty e^{-u}(u-s)^{p-1}s^{q-1}duds\\ &= \int_0^\infty\int_0^u e^{-u}(u-s)^{p-1}s^{q-1}dsdu\\ &= \int_0^\infty\int_0^1 e^{-u}(u-ux)^{p-1}(ux)^{q-1}udxdu\\ &= \int_0^\infty\int_0^1 e^{-u}u^{p+q-1}(1-x)^{p-1}x^{q-1}dxdu\\ &= \Gamma(p+q)\left(\int_0^1 x^{p-1}(1-x)^{q-1}dx\right).\end{aligned}$$

It remains to find $\Gamma\left(\frac{1}{2}\right)$.

$$\Gamma\left(\frac{1}{2}\right)=\int_0^\infty e^{-t}t^{-1/2}dt=\int_0^\infty e^{-u^2}\frac{1}{u}2udu=2\int_0^\infty e^{-u^2}du$$

Now

$$\begin{aligned}\left(\int_0^\infty e^{-x^2}dx\right)^2 &= \int_0^\infty e^{-x^2}dx\int_0^\infty e^{-y^2}dy=\int_0^\infty\int_0^\infty e^{-\left(x^2+y^2\right)}dxdy\\ &= \int_0^\infty\int_0^{\pi/2}e^{-r^2}rd\theta dr=\frac{1}{4}\pi\end{aligned}$$

and so

$$\Gamma\left(\frac{1}{2}\right)=2\int_0^\infty e^{-u^2}du=\sqrt{\pi}$$

This proves the lemma.

Next let n be a positive integer.

Theorem 12.2.7 *$\alpha(n)=\pi^{n/2}(\Gamma(n/2+1))^{-1}$ where $\Gamma(s)$ is the gamma function*

$$\Gamma(s)=\int_0^\infty e^{-t}t^{s-1}dt.$$

Proof: First let $n=1$.

$$\Gamma(\frac{3}{2})=\frac{1}{2}\Gamma\left(\frac{1}{2}\right)=\frac{\sqrt{\pi}}{2}.$$

Thus

$$\pi^{1/2}(\Gamma(1/2+1))^{-1}=\frac{2}{\sqrt{\pi}}\sqrt{\pi}=2=\alpha(1).$$

and this shows the theorem is true if $n=1$.

Assume the theorem is true for n and let B_{n+1} be the unit ball in $\mathbb{R}^{n+1}$. Then by the result in $\mathbb{R}^n$,

$$m_{n+1}(B_{n+1})=\int_{-1}^1\alpha(n)(1-x_{n+1}^2)^{n/2}dx_{n+1}$$

$$= 2\alpha(n)\int_0^1 (1-t^2)^{n/2}dt.$$

Doing an integration by parts and using Lemma 12.2.6

$$\begin{aligned}
&= 2\alpha(n)n\int_0^1 t^2(1-t^2)^{(n-2)/2}dt\\
&= 2\alpha(n)n\frac{1}{2}\int_0^1 u^{1/2}(1-u)^{n/2-1}du\\
&= n\alpha(n)\int_0^1 u^{3/2-1}(1-u)^{n/2-1}du\\
&= n\alpha(n)\Gamma(3/2)\Gamma(n/2)(\Gamma((n+3)/2))^{-1}\\
&= n\pi^{n/2}(\Gamma(n/2+1))^{-1}(\Gamma((n+3)/2))^{-1}\Gamma(3/2)\Gamma(n/2)\\
&= n\pi^{n/2}(\Gamma(n/2)(n/2))^{-1}(\Gamma((n+1)/2+1))^{-1}\Gamma(3/2)\Gamma(n/2)\\
&= 2\pi^{n/2}\Gamma(3/2)(\Gamma((n+1)/2+1))^{-1}\\
&= \pi^{(n+1)/2}(\Gamma((n+1)/2+1))^{-1}.
\end{aligned}$$

This proves the theorem.

From now on, in the definition of Hausdorff measure, it will always be the case that $\beta(s) = \alpha(s)$. As shown above, this is the right thing to have $\beta(s)$ to equal if s is a positive integer because this yields the important result that Hausdorff measure is the same as Lebesgue measure. Note the formula, $\pi^{s/2}(\Gamma(s/2+1))^{-1}$ makes sense for any $s \geq 0$.

12.3 Hausdorff Measure And Linear Transformations

Hausdorff measure makes possible a unified development of n dimensional area including in one theory length and surface area. Imagine the boundary of an open set in $\mathbb{R}^3$. You would tend to think of this as something two dimensional. The way to measure it is with $\mathcal{H}^2$. Length can be measured by $\mathcal{H}^1$ and the boundary of an open set in $\mathbb{R}^4$ is measured in terms of $\mathcal{H}^3$ etc.

As in the case of Lebesgue measure, the first step in this is to understand basic considerations related to linear transformations. Recall that for $L \in \mathcal{L}(\mathbb{R}^k, \mathbb{R}^l)$, L^* is defined by

$$(L\mathbf{u}, \mathbf{v}) = (\mathbf{u}, L^*\mathbf{v}).$$

Also recall the right polar decomposition, Theorem 3.9.3 on Page 62. This theorem says you can write a linear transformation as the composition of two linear transformations, one which preserves length and the other which distorts, the right polar decomposition. The one which distorts is the one which will have a nontrivial interaction with Hausdorff measure while the one which preserves lengths does not change Hausdorff measure. These ideas are behind the following theorems and lemmas.

Lemma 12.3.1 *Let $R \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^m)$, $n \leq m$, and $R^*R = I$. Then if $A \subseteq \mathbb{R}^n$,*

$$\mathcal{H}^n(RA) = \mathcal{H}^n(A).$$

In fact, if $P : \mathbb{R}^n \to \mathbb{R}^m$ satisfies $|P\mathbf{x} - P\mathbf{y}| = |\mathbf{x} - \mathbf{y}|$, then

$$\mathcal{H}^n(PA) = \mathcal{H}^n(A).$$

Proof: Note that

$$|R(\mathbf{x}-\mathbf{y})|^2 = (R(\mathbf{x}-\mathbf{y}), R(\mathbf{x}-\mathbf{y})) = (R^*R(\mathbf{x}-\mathbf{y}), \mathbf{x}-\mathbf{y}) = |\mathbf{x}-\mathbf{y}|^2$$

Thus R preserves lengths.

Now let P be an arbitrary mapping which preserves lengths and let A be bounded, $P(A) \subseteq \cup_{j=1}^{\infty} C_j$, $r(C_j) < \delta$, and

$$\mathcal{H}^n_\delta(PA) + \varepsilon > \sum_{j=1}^{\infty} \alpha(n)(r(C_j))^n.$$

Since P preserves lengths, it follows P is one to one on $P(\mathbb{R}^n)$ and P^{-1} also preserves lengths on $P(\mathbb{R}^n)$. Replacing each C_j with $C_j \cap (PA)$,

$$\begin{aligned} \mathcal{H}^n_\delta(PA) + \varepsilon &> \sum_{j=1}^{\infty} \alpha(n) r(C_j \cap (PA))^n \\ &= \sum_{j=1}^{\infty} \alpha(n) r\left(P^{-1}(C_j \cap (PA))\right)^n \\ &\geq \mathcal{H}^n_\delta(A). \end{aligned}$$

Thus $\mathcal{H}^n_\delta(PA) \geq \mathcal{H}^n_\delta(A)$.

Now let $A \subseteq \cup_{j=1}^{\infty} C_j, \operatorname{diam}(C_j) \leq \delta$, and

$$\mathcal{H}^n_\delta(A) + \varepsilon \geq \sum_{j=1}^{\infty} \alpha(n) (r(C_j))^n$$

Then

$$\begin{aligned} \mathcal{H}^n_\delta(A) + \varepsilon &\geq \sum_{j=1}^{\infty} \alpha(n) (r(C_j))^n \\ &= \sum_{j=1}^{\infty} \alpha(n) (r(PC_j))^n \\ &\geq \mathcal{H}^n_\delta(PA). \end{aligned}$$

Hence $\mathcal{H}^n_\delta(PA) = \mathcal{H}^n_\delta(A)$. Letting $\delta \to 0$ yields the desired conclusion in the case where A is bounded. For the general case, let $A_r = A \cap B(\mathbf{0}, r)$. Then $\mathcal{H}^n(PA_r) = \mathcal{H}^n(A_r)$. Now let $r \to \infty$. This proves the lemma.

Lemma 12.3.2 *Let $F \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^m), n \leq m$, and let $F = RU$ where R and U are described in Theorem 3.9.3 on Page 62. Then if $A \subseteq \mathbb{R}^n$ is Lebesgue measurable,*

$$\mathcal{H}^n(FA) = \det(U) m_n(A).$$

Proof: Using Theorem 9.8.7 on Page 224 and Theorem 12.1.9,

$$\mathcal{H}^n(FA) = \mathcal{H}^n(RUA)$$

$$= \mathcal{H}^n(UA) = m_n(UA) = \det(U) m_n(A).$$

Definition 12.3.3 *Define J to equal* $\det(U)$. *Thus*

$$J = \det((F^*F)^{1/2}) = (\det(F^*F))^{1/2}.$$

12.4 The Area Formula

12.4.1 Preliminary Results

It was shown in Lemma 12.3.2 that for $F \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^m), m \geq n$

$$\mathcal{H}^n(FA) = \det(U) m_n(A)$$

where $F = RU$ with R preserving distances and $U \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^n)$ having all positive eigenvalues. The area formula gives a generalization of this simple relationship to the case where F is replaced by a nonlinear mapping, $\mathbf{h}$. It contains as a special case the earlier change of variables formula. The area formula has to do with n dimensional measure on a set in $\mathbb{R}^m$ where $m > n$. Thus it includes notions of length and area of curves or surfaces in higher dimensional space. For example, you can use these ideas to consider the two dimensional area of a surface in $\mathbb{R}^3$ or even in $\mathbb{R}^8$ and it can all be done in a unified and rational way. In addition, the area formula will not require integration over open sets. Measurable sets are good enough.

Assume $m \geq n$ and $\mathbf{h}$ maps an open set in $\mathbb{R}^n$ to $\mathbb{R}^m$. Also suppose

$$D\mathbf{h}(\mathbf{x}) \text{ exists for all } \mathbf{x} \in V, \tag{12.12}$$

Lemma 12.4.1 *If $T \subseteq V$ where V is an open set in $\mathbb{R}^n$ and $m_n(T) = 0$, then $\mathcal{H}^n(\mathbf{h}(T)) = 0$. If $\mathbf{h}$ is one to one, V and $\mathbf{h}(V)$ are both bounded, $N \subseteq \mathbf{h}(V), \mathcal{H}^n(N) = 0$, and for $D\mathbf{h}(\mathbf{x}) = R(\mathbf{x})U(\mathbf{x})$ the right polar decomposition, $U(\mathbf{x})^{-1}$ exists for all $\mathbf{x} \in V$, then $\overline{m_n}\left(\mathbf{h}^{-1}(N)\right) = 0$.*

Proof: Let

$$T_k \equiv \{\mathbf{x} \in T : ||D\mathbf{h}(\mathbf{x})|| < k\}.$$

Thus $T = \cup_k T_k$. I will show $\mathbf{h}(T_k)$ has $\mathcal{H}^n$ measure zero and then it will follow that

$$\mathbf{h}(T) = \cup_{k=1}^{\infty} \mathbf{h}(T_k)$$

must also have measure zero.

Let $\varepsilon > 0$ be given. By outer regularity, there exists an open set, W, containing T_k which is contained in V such that $m_n(W) < \frac{\varepsilon}{k^n 6^n}$. For $\mathbf{x} \in T_k$ it follows from differentiability,

$$\mathbf{h}(\mathbf{x} + \mathbf{v}) = \mathbf{h}(\mathbf{x}) + D\mathbf{h}(\mathbf{x})\mathbf{v} + o(\mathbf{v})$$

and so whenever $r_{\mathbf{x}}$ is small enough, $B(\mathbf{x}, 5r_{\mathbf{x}}) \subseteq W$ and whenever $|\mathbf{v}| < 5r_{\mathbf{x}}, |o(\mathbf{v})| < kr_{\mathbf{x}}$. Therefore, if $|\mathbf{v}| < 5r_{\mathbf{x}}$,

$$D\mathbf{h}(\mathbf{x})\mathbf{v} + o(\mathbf{v}) \in B(\mathbf{0}, 5kr_{\mathbf{x}}) + B(\mathbf{0}, kr_{\mathbf{x}}) \subseteq B(\mathbf{0}, 6kr_{\mathbf{x}})$$

and so

$$\mathbf{h}(B(\mathbf{x}, 5r_{\mathbf{x}})) \subseteq B(\mathbf{h}(\mathbf{x}), 6kr_{\mathbf{x}}).$$

Letting $\delta > 0$ be given, the Vitali covering theorem implies there exists a sequence of disjoint balls $\{B_i\}$, $B_i = B(\mathbf{x}_i, r_{\mathbf{x}_i})$, which are contained in W such that the sequence of enlarged balls, $\left\{\widehat{B}_i\right\}$, having the same center but 5 times the radius, covers T_k and $6kr_{\mathbf{x}_i} < \delta$. Then

$$\mathcal{H}^n_\delta(\mathbf{h}(T_k)) \leq \mathcal{H}^n_\delta\left(\mathbf{h}\left(\cup_{i=1}^{\infty} \widehat{B}_i\right)\right)$$

$$\leq \sum_{i=1}^{\infty} \mathcal{H}^n_\delta\left(\mathbf{h}\left(\widehat{B}_i\right)\right) \leq \sum_{i=1}^{\infty} \mathcal{H}^n_\delta(B(\mathbf{h}(\mathbf{x}_i), 6kr_{\mathbf{x}_i}))$$

$$\begin{aligned} &\leq \sum_{i=1}^{\infty} \alpha(n) (6kr_{\mathbf{x}_i})^n = (6k)^n \sum_{i=1}^{\infty} \alpha(n) r_{\mathbf{x}_i}^n \\ &= (6k)^n \sum_{i=1}^{\infty} m_n (B(\mathbf{x}_i, r_{\mathbf{x}_i})) \\ &\leq (6k)^n m_n(W) \leq (6k)^n \frac{\varepsilon}{k^n 6^n} = \varepsilon. \end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, this shows $\mathcal{H}_\delta^n(\mathbf{h}(T_k)) = 0$. Since δ is arbitrary, this implies $\mathcal{H}^n(\mathbf{h}(T_k)) = 0$. Now

$$\mathcal{H}^n(\mathbf{h}(T)) = \lim_{k\to\infty} \mathcal{H}^n(\mathbf{h}(T_k)) = 0.$$

It remains to verify the last claim. Recall

$$U(\mathbf{x}) = \sum_{k=1}^{n} a_k \mathbf{v}_k \mathbf{v}_k$$

where $\{\mathbf{v}_k\}$ is an orthonormal basis and since $U(\mathbf{x})^{-1}$ is given to exist, each $a_k > 0$. Therefore, $(U(\mathbf{x})\mathbf{w}, \mathbf{w}) \geq \delta(\mathbf{x})|\mathbf{w}|^2$ where $\delta(\mathbf{x}) > 0$.

Next I claim $\mathbf{h}^{-1}$ is continuous on $\mathbf{h}(V)$. Suppose then that $\mathbf{h}(\mathbf{x}_k) \to \mathbf{h}(\mathbf{x})$. Then let the sequentially compact set $\overline{B(\mathbf{x}, r)} \subseteq V$. Without loss of generality all $\mathbf{x}_k$ may be assumed to lie in $\overline{B(\mathbf{x}, r)}$. If $\{\mathbf{x}_k\}$ fails to converge to $\mathbf{x}$, then since $\overline{B(\mathbf{x}, r)}$ is sequentially compact, there exists a subsequence $\{\mathbf{x}_{k_l}\}$ converging to $\mathbf{z} \neq \mathbf{x}$. But then

$$\mathbf{h}(\mathbf{x}) = \lim_{l\to\infty} \mathbf{h}(\mathbf{x}_{k_l}) = \mathbf{h}(\mathbf{z})$$

a contradiction to $\mathbf{h}$ being one to one. Thus $\mathbf{x}_k \to \mathbf{x}$ and so $\mathbf{h}$ is continuous.

For $\eta > 0$ let

$$N_\eta \equiv \left\{\mathbf{y} \in N : \delta\left(\mathbf{h}^{-1}(\mathbf{y})\right) \geq \eta\right\}.$$

Then for $\mathbf{y} \in N_\eta$,

$$\begin{aligned} (\mathbf{y}+\mathbf{v}) - \mathbf{y} &= \mathbf{h}\left(\mathbf{h}^{-1}(\mathbf{y}+\mathbf{v})\right) - \mathbf{h}\left(\mathbf{h}^{-1}(\mathbf{y})\right) \\ &= R\left(\mathbf{h}^{-1}(\mathbf{y})\right) U\left(\mathbf{h}^{-1}(\mathbf{y})\right)\left(\mathbf{h}^{-1}(\mathbf{y}+\mathbf{v}) - \mathbf{h}^{-1}(\mathbf{y})\right) \\ &\quad + \mathbf{o}\left(\mathbf{h}^{-1}(\mathbf{y}+\mathbf{v}) - \mathbf{h}^{-1}(\mathbf{y})\right) \end{aligned}$$

therefore

$$R^*\left(\mathbf{h}^{-1}(\mathbf{y})\right)\mathbf{v} = U\left(\mathbf{h}^{-1}(\mathbf{y})\right)\left(\mathbf{h}^{-1}(\mathbf{y}+\mathbf{v}) - \mathbf{h}^{-1}(\mathbf{y})\right) + \mathbf{o}\left(\mathbf{h}^{-1}(\mathbf{y}+\mathbf{v}) - \mathbf{h}^{-1}(\mathbf{y})\right) \quad (12.13)$$

Using continuity of $\mathbf{h}^{-1}$, it follows that if $\mathbf{v}$ is small enough,

$$\left|\mathbf{o}\left(\mathbf{h}^{-1}(\mathbf{y}+\mathbf{v}) - \mathbf{h}^{-1}(\mathbf{y})\right)\right| \leq \frac{\eta}{2}\left|\mathbf{h}^{-1}(\mathbf{y}+\mathbf{v}) - \mathbf{h}^{-1}(\mathbf{y})\right|$$

Taking the inner product of both sides of 12.13 with $\mathbf{h}^{-1}(\mathbf{y}+\mathbf{v}) - \mathbf{h}^{-1}(\mathbf{y})$ yields

$$\begin{aligned} &\left|R^*\left(\mathbf{h}^{-1}(\mathbf{y})\right)\mathbf{v}\right|\left|\mathbf{h}^{-1}(\mathbf{y}+\mathbf{v}) - \mathbf{h}^{-1}(\mathbf{y})\right| \\ \geq\ &\eta\left|\mathbf{h}^{-1}(\mathbf{y}+\mathbf{v}) - \mathbf{h}^{-1}(\mathbf{y})\right|^2 - \frac{\eta}{2}\left|\mathbf{h}^{-1}(\mathbf{y}+\mathbf{v}) - \mathbf{h}^{-1}(\mathbf{y})\right|^2. \end{aligned}$$

Now since R preserves distances and $R^*R = I$

$$(R^*\mathbf{v}, R^*\mathbf{v}) = (\mathbf{v}, RR^*\mathbf{v}) \leq |\mathbf{v}|\,|RR^*\mathbf{v}| = |\mathbf{v}|\,|R^*\mathbf{v}|$$

and so

$$|R^*\mathbf{v}| \leq |\mathbf{v}|. \tag{12.14}$$

Thus the above formula implies

$$|\mathbf{v}| \geq \frac{\eta}{2}\left|\mathbf{h}^{-1}(\mathbf{y}+\mathbf{v}) - \mathbf{h}^{-1}(\mathbf{y})\right|. \tag{12.15}$$

Since N_η has $\mathcal{H}^n$ measure zero, there exist $\{C_k\}$ covering N_η such that $r(C_i) < \delta$ and

$$\varepsilon\eta^n/4^n > \sum_{k=1}^{\infty} \alpha(n)\, r(C_k)^n.$$

Without loss of generality each C_k has nonempty intersection with N_η, containing $\mathbf{y}_k$. Now $\left\{\mathbf{h}^{-1}(C_k)\right\}$ covers $\mathbf{h}^{-1}(N_\eta)$ and from 12.15

$$\operatorname{diam}\left(\mathbf{h}^{-1}(C_k)\right) \leq \frac{2}{\eta} \times 2 \times \operatorname{diam}(C_k)$$

and so

$$\begin{aligned}\mathcal{H}^n_{4\delta/\eta}\left(\mathbf{h}^{-1}(N_\eta)\right) &\leq \sum_k \alpha(n)\left(\frac{2}{\eta}\right)^n \left(\operatorname{diam}(C_k)^n\right) \\ &\leq \frac{4^n}{\eta^n}\sum_k \alpha(n)\, r(C_k)^n < \frac{\varepsilon\eta^n}{4^n}\frac{4^n}{\eta^n} = \varepsilon\end{aligned}$$

Since ε is arbitrary, $\mathcal{H}^n_{4\delta/\eta}\left(\mathbf{h}^{-1}(N_\eta)\right) = 0$ and letting $\delta \to 0$ yields

$$\mathcal{H}^n\left(\mathbf{h}^{-1}(N_\eta)\right) = m_n\left(\mathbf{h}^{-1}(N_\eta)\right) = 0$$

because by Theorem 12.1.9 Lebesgue and Hausdorff measure are the same on the Lebesgue measurable sets of $\mathbb{R}^n$. Now take $\eta_k \to 0$

$$m_n\left(\mathbf{h}^{-1}(N)\right) = \lim_{k\to\infty} m_n\left(\mathbf{h}^{-1}\left(N_{\eta_k}\right)\right) = 0.$$

This proves the lemma.

Lemma 12.4.2 *If S is a Lebesgue measurable subset of an open set $V \subseteq \mathbb{R}^n$, on which $\mathbf{h}$ is defined and C^1, then $\mathbf{h}(S)$ is $\mathcal{H}^n$ measurable.*

Proof: Let $S_k = S \cap B(\mathbf{0}, k), k \in \mathbb{N}$. By inner regularity of Lebesgue measure, there exists a set, F, which is the countable union of compact sets and a set T with $m_n(T) = 0$ such that

$$F \cup T = S_k.$$

Then $\mathbf{h}(F) \subseteq \mathbf{h}(S_k) \subseteq \mathbf{h}(F) \cup \mathbf{h}(T)$. By continuity of $\mathbf{h}$, $\mathbf{h}(F)$ is a countable union of compact sets and so it is Borel. By Lemma 12.4.1, $\mathcal{H}^n(\mathbf{h}(T)) = 0$ and so $\mathbf{h}(S_k)$ is $\mathcal{H}^n$ measurable because of completeness of Hausdorff measure, which comes from $\mathcal{H}^n$ being obtained from an outer measure. Now $\mathbf{h}(S) = \cup_{k=1}^{\infty}\mathbf{h}(S_k)$ and so it is also true that $\mathbf{h}(S)$ is $\mathcal{H}^n$ measurable. This proves the lemma.

The following lemma, depending on the Brouwer fixed point theorem and found in Rudin [34], will be important for the following arguments. The idea is that if a continuous function mapping a ball in $\mathbb{R}^k$ to $\mathbb{R}^k$ doesn't move any point very much, then the image of the ball must contain a slightly smaller ball.

Lemma 12.4.3 *Let $B = B(\mathbf{0}, r)$, a ball in $\mathbb{R}^k$ and let $\mathbf{F} : \overline{B} \to \mathbb{R}^k$ be continuous and suppose for some $\varepsilon < 1$,*

$$|\mathbf{F}(\mathbf{v}) - \mathbf{v}| < \varepsilon r \tag{12.16}$$

for all $\mathbf{v} \in \overline{B}$. Then

$$\mathbf{F}(B) \supseteq B(\mathbf{0}, r(1-\varepsilon)).$$

Proof: Suppose $\mathbf{a} \in B(\mathbf{0}, r(1-\varepsilon)) \setminus \mathbf{F}(B)$.

I claim that $\mathbf{a} \neq \mathbf{F}(\mathbf{v})$ for all $\mathbf{v} \in \overline{B}$, not just for $\mathbf{v} \in B$. Here is why. By the assumption $\mathbf{a} \notin \mathbf{F}(B)$, if $\mathbf{F}(\mathbf{v}) = \mathbf{a}$, then $|\mathbf{v}| = r$ and so

$$|\mathbf{F}(\mathbf{v}) - \mathbf{v}| = |\mathbf{a} - \mathbf{v}| \geq |\mathbf{v}| - |\mathbf{a}| > r - r(1-\varepsilon) = r\varepsilon,$$

a contradiction to 12.16.

Now letting $\mathbf{G} : \overline{B} \to \overline{B}$, be defined by

$$\mathbf{G}(\mathbf{v}) \equiv \frac{r(\mathbf{a} - \mathbf{F}(\mathbf{v}))}{|\mathbf{a} - \mathbf{F}(\mathbf{v})|},$$

it follows from what was just shown $\mathbf{G}$ is continuous. Then by the Brouwer fixed point theorem, $\mathbf{G}(\mathbf{v}) = \mathbf{v}$ for some $\mathbf{v} \in \overline{B}$. Using the formula for $\mathbf{G}$, it follows

$$|\mathbf{v}| = |\mathbf{G}(\mathbf{v})| = \left| \frac{r(\mathbf{a} - \mathbf{F}(\mathbf{v}))}{|\mathbf{a} - \mathbf{F}(\mathbf{v})|} \right| = r$$

Taking the inner product with $\mathbf{v}$,

$$\begin{aligned}
(\mathbf{G}(\mathbf{v}), \mathbf{v}) &= |\mathbf{v}|^2 = r^2 = \frac{r}{|\mathbf{a} - \mathbf{F}(\mathbf{v})|} (\mathbf{a} - \mathbf{F}(\mathbf{v}), \mathbf{v}) \\
&= \frac{r}{|\mathbf{a} - \mathbf{F}(\mathbf{v})|} (\mathbf{a} - \mathbf{v} + \mathbf{v} - \mathbf{F}(\mathbf{v}), \mathbf{v}) \\
&= \frac{r}{|\mathbf{a} - \mathbf{F}(\mathbf{v})|} [(\mathbf{a} - \mathbf{v}, \mathbf{v}) + (\mathbf{v} - \mathbf{F}(\mathbf{v}), \mathbf{v})] \\
&= \frac{r}{|\mathbf{a} - \mathbf{F}(\mathbf{v})|} \left[(\mathbf{a}, \mathbf{v}) - |\mathbf{v}|^2 + (\mathbf{v} - \mathbf{F}(\mathbf{v}), \mathbf{v}) \right] \\
&\leq \frac{r}{|\mathbf{a} - \mathbf{F}(\mathbf{v})|} \left[r^2(1-\varepsilon) - r^2 + r^2\varepsilon \right] = 0,
\end{aligned}$$

a contradiction to $|\mathbf{v}| = r$. Therefore, $B(\mathbf{0}, r(1-\varepsilon)) \setminus \mathbf{F}(B) = \emptyset$ and this proves the lemma.

Lemma 12.4.4 *If $|P\mathbf{x} - P\mathbf{y}| \leq L|\mathbf{x} - \mathbf{y}|$, then for E a set,*

$$\mathcal{H}^n(PE) \leq L^n \mathcal{H}^n(E).$$

Proof: Without loss of generality, assume $\mathcal{H}^n(E) < \infty$. Let $\delta > 0$ and let $\{C_i\}_{i=1}^\infty$ be a covering of E such that $r(C_i) \leq \delta$ for each i and

$$\sum_{i=1}^\infty \alpha(n) r(C_i)^n \leq \mathcal{H}^n_\delta(E) + \varepsilon.$$

Then $\{PC_i\}_{i=1}^\infty$ is a covering of PE such that $r(PC_i) \leq L\delta$. Therefore,

$$\begin{aligned}
\mathcal{H}^n_{L\delta}(PE) &\leq \sum_{i=1}^\infty \alpha(n) r(PC_i)^n \\
&\leq L^n \sum_{i=1}^\infty \alpha(n) r(C_i)^n \leq L^n \mathcal{H}^n_\delta(E) + L^n\varepsilon \\
&\leq L^n \mathcal{H}^n(E) + L^n\varepsilon.
\end{aligned}$$

Letting $\delta \to 0$,

$$\mathcal{H}^n(PE) \leq L^n \mathcal{H}^n(E) + L^n \varepsilon$$

and since $\varepsilon > 0$ is arbitrary, this proves the Lemma.

Then the following corollary follows from 12.14.

Corollary 12.4.5 *Let $R : \mathbb{R}^n \to \mathbb{R}^m$ where $m \geq n$ and R is linear and preserves distances. Let $T \subseteq \mathbb{R}^m$. Then*

$$\mathcal{H}^n(T) \geq \mathcal{H}^n(RR^*T) = \mathcal{H}^n(R^*T).$$

Now let V be an open set in $\mathbb{R}^n$ and let $\mathbf{h} : V \to \mathbb{R}^m$ be a $C^1(V)$ function. For $D\mathbf{h}(\mathbf{x}) = R(\mathbf{x})U(\mathbf{x})$ the right polar decomposition, suppose $U(\mathbf{x})^{-1}$ exists for all $\mathbf{x} \in U$. By definition of the derivative,

$$\begin{aligned} \mathbf{h}(\mathbf{x}+\mathbf{v}) - \mathbf{h}(\mathbf{x}) &= R(\mathbf{x})U(\mathbf{x})\mathbf{v} + \mathbf{o}(\mathbf{v}) \\ &= D\mathbf{h}(\mathbf{x})\mathbf{v} + \mathbf{o}(\mathbf{v}) \end{aligned} \tag{12.17}$$

and therefore letting $\varepsilon > 0$ be given,

$$\left|U^{-1}R^*(\mathbf{h}(\mathbf{x}+\mathbf{v}) - \mathbf{h}(\mathbf{x})) - \mathbf{v}\right| < \varepsilon |\mathbf{v}|$$

whenever $\mathbf{v}$ is small enough. Thus by Lemma 12.4.3, if $r_{\mathbf{x}}$ is small enough,

$$U^{-1}R^*(\mathbf{h}(\mathbf{x}+B(\mathbf{0},r_{\mathbf{x}})) - \mathbf{h}(\mathbf{x})) \supseteq B(\mathbf{0}, r_{\mathbf{x}}(1-\varepsilon))$$

which implies

$$\mathbf{h}(\mathbf{x}+B(\mathbf{0},r_{\mathbf{x}})) \supseteq RUB(\mathbf{0}, r_{\mathbf{x}}(1-\varepsilon)) + \mathbf{h}(\mathbf{x})$$

or in other words,

$$\mathbf{h}(B(\mathbf{x},r_{\mathbf{x}})) \supseteq D\mathbf{h}(\mathbf{x})B(\mathbf{0}, r_{\mathbf{x}}(1-\varepsilon)) + \mathbf{h}(\mathbf{x}). \tag{12.18}$$

Referring to 12.17 again,

$$R^*(\mathbf{x})(\mathbf{h}(\mathbf{x}+\mathbf{v}) - \mathbf{h}(\mathbf{x})) = U(\mathbf{x})\mathbf{v} + \mathbf{o}(\mathbf{v})$$

$$= U(\mathbf{x})\left(\mathbf{v} + U^{-1}(\mathbf{x})\mathbf{o}(\mathbf{v})\right) = U(\mathbf{x})(\mathbf{v} + \mathbf{o}(\mathbf{v}))$$

It follows that if $r_{\mathbf{x}}$ is sufficiently small,

$$\begin{aligned} R^*(\mathbf{x})(\mathbf{h}(\mathbf{x}+B(\mathbf{0},r_{\mathbf{x}})) - \mathbf{h}(\mathbf{x})) &\subseteq U(B(\mathbf{0},r_{\mathbf{x}}) + B(\mathbf{0},\varepsilon r_{\mathbf{x}})) \\ &\subseteq UB(\mathbf{0},r_{\mathbf{x}}(1+\varepsilon)) \end{aligned}$$

and so

$$\mathbf{h}(\mathbf{x}+B(\mathbf{0},r_{\mathbf{x}})) =$$

$$\mathbf{h}(B(\mathbf{x},r_{\mathbf{x}})) \subseteq D\mathbf{h}(\mathbf{x})B(\mathbf{0},r_{\mathbf{x}}(1+\varepsilon)) + \mathbf{h}(\mathbf{x}) \tag{12.19}$$

12.5 The Area Formula

The following lemma is the first approximation to the area formula.

Lemma 12.5.1 *Let V be a bounded open set in $\mathbb{R}^n$ and let $\mathbf{h} \in C^1(V), \mathbf{h} : V \to \mathbb{R}^m$ for $m \geq n$ be one to one such that for*

$$D\mathbf{h}(\mathbf{x}) = R(\mathbf{x}) U(\mathbf{x}),$$

the right polar decomposition, $U(\mathbf{x})$ is one to one. Assume also

$$|\det U(\mathbf{x})| \leq M, \ \mathbf{x} \in V$$

Also assume $\mathcal{H}^n(\mathbf{h}(V)) < \infty$ and $\mathbf{h}(V)$ is bounded. Let W be a Borel set in $\mathbb{R}^m$ and let A be a Lebesgue measurable subset of V. Then

$$\int_{\mathbf{h}(A)} \mathcal{X}_W(\mathbf{y}) \, d\mathcal{H}^n = \int_A \mathcal{X}_W(\mathbf{h}(\mathbf{x})) \, |\det(U(\mathbf{x}))| \, dm_n$$

Proof: Recall

$$\det(U(\mathbf{x})) = \det\left(D\mathbf{h}(\mathbf{x})^* D\mathbf{h}(\mathbf{x})\right)^{1/2}$$

and so the function, $\mathbf{x} \to \det(U(\mathbf{x}))$ is continuous.

Also let O be an open set containing $\mathbf{h}^{-1}(W) \cap A$ such that $m_n\left(O \setminus \left(\mathbf{h}^{-1}(W) \cap A\right)\right) < \varepsilon$ and $\mathcal{H}^n\left(\mathbf{h}\left(O \setminus \left(\mathbf{h}^{-1}(W) \cap A\right)\right)\right) < \varepsilon$. To do this, let $\{O_m\}$ be a decreasing sequence of open sets containing $\mathbf{h}^{-1}(W) \cap A$ such that $m_n(O_m) \to m_n\left(\mathbf{h}^{-1}(W) \cap A\right)$. Thus

$$m_n\left((\cap_m O_m) \setminus \mathbf{h}^{-1}(W) \cap A\right) = 0$$

and so

$$\mathcal{H}^n\left(\mathbf{h}\left((\cap_m O_m) \setminus \mathbf{h}^{-1}(W) \cap A\right)\right) = 0$$

by Lemma 12.4.1. Therefore, if m is large enough, letting $O = O_m$ gives the desired open set.

Let $\mathbf{x} \in \mathbf{h}^{-1}(W) \cap A$. First note $\mathbf{h}^{-1}(W)$ is a Borel set because

$$\mathcal{S} \equiv \left\{E \in \mathcal{B}(\mathbb{R}^m) : \mathbf{h}^{-1}(E) \in \mathcal{B}(\mathbb{R}^n)\right\}$$

is a σ algebra which contains the open sets due to the fact $\mathbf{h}$ is continuous. Therefore, $\mathcal{S} = \mathcal{B}(\mathbb{R}^m)$. Thus $\mathbf{h}^{-1}(W) \cap A$ is measurable.

There exists $1 > r_{\mathbf{x}} > 0$ small enough that 12.19 and 12.18 both hold. There exists a possibly smaller $r_{\mathbf{x}}$ such that

$$B(\mathbf{x}, r_{\mathbf{x}}) \subseteq O \tag{12.20}$$

and

$$||\det(U(\mathbf{x}_1))| - |\det(U(\mathbf{x}))|| < \varepsilon \tag{12.21}$$

whenever $\mathbf{x}_1 \in B(\mathbf{x}, r_{\mathbf{x}})$.

The collection of such balls is a Vitali cover of $\mathbf{h}^{-1}(W) \cap A$. By Corollary 9.7.5 there is a sequence of disjoint balls $\{B_i\}$ such that for

$$N \equiv \mathbf{h}^{-1}(W) \cap A \setminus \cup_{i=1}^{\infty} B_i,$$

$m_n(N) = 0$. Therefore, renaming N to equal

$$\mathbf{h}^{-1}(W) \cap A \setminus \cup_{i=1}^{\infty} B_i \cap \mathbf{h}^{-1}(W) \cap A,$$

$m_n(N) = 0$ and by Lemma 12.4.1,

$$\mathcal{H}^n(\mathbf{h}(N)) = 0.$$

$$\mathbf{h}^{-1}(W) \cap A = \left(\cup_{i=1}^{\infty} B_i \cap A \cap \mathbf{h}^{-1}(W)\right) \cup N$$

$$W \cap \mathbf{h}(A) = (\cup_{i=1}^{\infty} \mathbf{h}(B_i \cap A) \cap W) \cup \mathbf{h}(N) \tag{12.22}$$

where $m_n(N) = \mathcal{H}^n(\mathbf{h}(N)) = 0$.

Denote by $\mathbf{x}_i$ the center of B_i and r_i the radius. Using 12.18, Lemma 12.4.1 which says $\mathbf{h}$ takes sets of Lebesgue measure zero to sets of $\mathcal{H}^n$ measure zero, the translation invariance of $\mathcal{H}^n$, Lemma 12.3.2 which gives the rule for taking a linear transformation outside the Hausdorff measure of something, 12.21, and the assumption that $\mathbf{h}$ is one to one,

$$\begin{aligned}
\int_{\mathbf{h}(A)} \mathcal{X}_W(\mathbf{y})\, d\mathcal{H}^n &= \int_{\mathbf{h}(A)\cap W} d\mathcal{H}^n = \int_{\cup_{i=1}^{\infty}\mathbf{h}(B_i\cap A)\cap W} d\mathcal{H}^n \\
&= \int_{\mathbf{h}\left(\cup_{i=1}^{\infty} B_i\cap A\cap \mathbf{h}^{-1}(W)\right)} d\mathcal{H}^n \geq \int_{\mathbf{h}\left(\cup_{i=1}^{\infty} B_i\right)} d\mathcal{H}^n - \varepsilon \geq \\
&\sum_{i=1}^{\infty} \mathcal{H}^n(\mathbf{h}(B_i)) - \varepsilon \geq \sum_{i=1}^{\infty} \mathcal{H}^n(D\mathbf{h}(\mathbf{x}_i)(B(\mathbf{0},(1-\varepsilon)r_i))) - \varepsilon \\
&= \sum_{i=1}^{\infty} |\det(U(\mathbf{x}_i))|\, m_n(B(\mathbf{0},(1-\varepsilon)r_i)) - \varepsilon \\
&= (1-\varepsilon)^n \sum_{i=1}^{\infty} |\det(U(\mathbf{x}_i))|\, m_n(B(\mathbf{x}_i,r_i)) - \varepsilon \\
&\geq (1-\varepsilon)^n \sum_{i=1}^{\infty} \left(\int_{B_i} |\det(U(\mathbf{x}))|\, dm_n - \varepsilon m_n(B_i)\right) - \varepsilon
\end{aligned}$$

$$\begin{aligned}
&\geq (1-\varepsilon)^n \sum_{i=1}^{\infty} \int_{B_i\cap A\cap \mathbf{h}^{-1}(W)} |\det(U(\mathbf{x}))|\, dm_n - (1-\varepsilon)^n \varepsilon m_n(V) - \varepsilon \\
&= (1-\varepsilon)^n \int_V \mathcal{X}_{\cup_{i=1}^{\infty}\mathbf{h}(B_i\cap A\cap \mathbf{h}^{-1}(W))}(\mathbf{h}(\mathbf{x}))\, |\det(U(\mathbf{x}))|\, dm_n - (1-\varepsilon)^n \varepsilon m_n(V) - \varepsilon \\
&= (1-\varepsilon)^n \int_V \mathcal{X}_{W\cap \mathbf{h}(A)}(\mathbf{h}(\mathbf{x}))\, |\det(U(\mathbf{x}))|\, dm_n - (1-\varepsilon)^n \varepsilon m_n(V) \\
&\quad - (1-\varepsilon)^n \int_V \mathcal{X}_N(\mathbf{x})\, |\det(U(\mathbf{x}))|\, dm_n - \varepsilon \\
&= (1-\varepsilon)^n \int_A \mathcal{X}_W(\mathbf{h}(\mathbf{x}))\, |\det(U(\mathbf{x}))|\, dm_n - (1-\varepsilon)^n \varepsilon m_n(V) - \varepsilon
\end{aligned}$$

The last three lines follows from 12.22. Recall $m_n(N) = 0$. Since $\varepsilon > 0$ is arbitrary, this shows

$$\int_{\mathbf{h}(A)} \mathcal{X}_W(\mathbf{y})\, d\mathcal{H}^n \geq \int_A \mathcal{X}_W(\mathbf{h}(\mathbf{x}))\, |\det(D\mathbf{h}(\mathbf{x}))|\, dm_n$$

The opposite inequality can be established in exactly the same way using 12.19 instead of 12.18 and turning all the inequalities around featuring $(1+\varepsilon)$ instead of $(1-\varepsilon)$, much as was done in the proof of Lemma 9.9.1. Thus

$$\begin{aligned}
\int_{\mathbf{h}(A)} \mathcal{X}_W(\mathbf{y})\, d\mathcal{H}^n &= \int_{\mathbf{h}(A)\cap W} d\mathcal{H}^n = \sum_{i=1}^{\infty} \int_{\mathbf{h}(B_i\cap A)\cap W} d\mathcal{H}^n \\
&= \sum_{i=1}^{\infty} \mathcal{H}^n(\mathbf{h}(B_i\cap A)\cap W) \leq \sum_{i=1}^{\infty} \mathcal{H}^n(\mathbf{h}(B_i)) \\
&\leq \sum_{i=1}^{\infty} \mathcal{H}^n(D\mathbf{h}(\mathbf{x}_i)(B(\mathbf{0},(1+\varepsilon)r_i))) \\
&= \sum_{i=1}^{\infty} |\det(U(\mathbf{x}_i))|\, m_n(B(\mathbf{0},(1+\varepsilon)r_i)) \\
&= (1+\varepsilon)^n \sum_{i=1}^{\infty} |\det(U(\mathbf{x}_i))|\, m_n(B(\mathbf{x}_i,r_i)) \\
&\leq (1+\varepsilon)^n \sum_{i=1}^{\infty} \left(\int_{B_i} |\det(U(\mathbf{x}))|\, dm_n + \varepsilon m_n(B_i)\right)
\end{aligned}$$

$$\begin{aligned}
&\leq (1+\varepsilon)^n \textstyle\sum_{i=1}^{\infty} \int_{B_i\cap A\cap \mathbf{h}^{-1}(W)} |\det(U(\mathbf{x}))|\, dm_n + (1+\varepsilon)^n \varepsilon m_n(V) \\
&\qquad + (1+\varepsilon)^n \textstyle\int_{O\setminus(A\cap \mathbf{h}^{-1}(W))} |\det(U(\mathbf{x}))|\, dm_n \\
&\leq (1+\varepsilon)^n \textstyle\int_V \mathcal{X}_{\cup_{i=1}^{\infty}\mathbf{h}(B_i\cap A)\cap W}(\mathbf{h}(\mathbf{x}))\, |\det(U(\mathbf{x}))|\, dm_n \\
&\qquad + (1+\varepsilon)^n \varepsilon m_n(V) + \varepsilon(1+\varepsilon)^n M \\
&= (1+\varepsilon)^n \textstyle\int_V \mathcal{X}_{W\cap \mathbf{h}(A)}(\mathbf{h}(\mathbf{x}))\, |\det(U(\mathbf{x}))|\, dm_n + (1+\varepsilon)^n \varepsilon m_n(V) \\
&\qquad + \varepsilon(1+\varepsilon)^n M - (1+\varepsilon)^n \textstyle\int_V \mathcal{X}_N(\mathbf{x})\, |\det(U(\mathbf{x}))|\, dm_n \\
&= (1+\varepsilon)^n \textstyle\int_A \mathcal{X}_W(\mathbf{h}(\mathbf{x}))\, |\det(U(\mathbf{x}))|\, dm_n + (1+\varepsilon)^n \varepsilon m_n(V) + \varepsilon(1+\varepsilon)^n M
\end{aligned}$$

Since ε is arbitrary, this proves the lemma.

Next the Borel sets will be enlarged to $\mathcal{H}^n$ measurable sets.

Lemma 12.5.2 *Let V be a bounded open set in $\mathbb{R}^n$ and let $\mathbf{h} \in C^1(V), \mathbf{h}: V \to \mathbb{R}^m$ for $m \geq n$ be one to one such that for*

$$D\mathbf{h}(\mathbf{x}) = R(\mathbf{x})\, U(\mathbf{x}),$$

the right polar decomposition, $U(\mathbf{x})$ is one to one. Assume also

$$|\det U(\mathbf{x})| \leq M,\ \mathbf{x} \in V$$

Also assume $\mathcal{H}^n(\mathbf{h}(V)) < \infty$ and $\mathbf{h}(V)$ is bounded. Let W be a $\mathcal{H}^n$ measurable set in $\mathbb{R}^m$ and let A be a Lebesgue measurable subset of V. Then

$$\int_{\mathbf{h}(A)} \mathcal{X}_W(\mathbf{y})\, d\mathcal{H}^n = \int_A \mathcal{X}_W(\mathbf{h}(\mathbf{x}))\, |\det(U(\mathbf{x}))|\, dm_n \tag{12.23}$$

Proof: By Theorem 12.1.5 there exists a Borel set, F containing W such that $\mathcal{H}^n(F\setminus W) = 0$. By Lemma 12.4.1 $m_n\left(\mathbf{h}^{-1}((F\setminus W)\cap \mathbf{h}(A))\right) = 0$. Therefore, from Lemma 12.5.1

$$\begin{aligned}
\int_{\mathbf{h}(A)} \mathcal{X}_W(\mathbf{y})\, d\mathcal{H}^n &= \int_{\mathbf{h}(A)} \mathcal{X}_F(\mathbf{y})\, d\mathcal{H}^n \\
&= \int_A \mathcal{X}_F(\mathbf{h}(\mathbf{x}))\, |\det(U(\mathbf{x}))|\, dm_n \\
&= \int_A \mathcal{X}_{F\setminus W}(\mathbf{h}(\mathbf{x}))\, |\det(U(\mathbf{x}))|\, dm_n \\
&\quad + \int_A \mathcal{X}_W(\mathbf{h}(\mathbf{x}))\, |\det(U(\mathbf{x}))|\, dm_n \\
&= \int_A \mathcal{X}_W(\mathbf{h}(\mathbf{x}))\, |\det(U(\mathbf{x}))|\, dm_n.
\end{aligned}$$

Note that

$$\mathcal{X}_W(\mathbf{h}(\mathbf{x}))\, |\det(U(\mathbf{x}))| + \mathcal{X}_{F\setminus W}(\mathbf{h}(\mathbf{x}))\, |\det(U(\mathbf{x}))| = \mathcal{X}_F(\mathbf{h}(\mathbf{x}))\, |\det(U(\mathbf{x}))|$$

and the second term on the left equals $\mathcal{X}_{\mathbf{h}^{-1}(F\setminus W)}(\mathbf{x})$ and is measurable because $\mathbf{h}^{-1}(F\setminus W)$ has Lebesgue measure zero by Lemma 12.4.1 while the term on the right is Lebesgue measurable because F is Borel so $\mathcal{X}_F \circ \mathbf{h}$ is measurable because it is a Borel measurable function composed with a continuous function (why?). Therefore, $\mathbf{x} \to \mathcal{X}_W(\mathbf{h}(\mathbf{x}))\, |\det(U(\mathbf{x}))|$ is also measurable. This proves the theorem.

You don't need to assume the open sets are bounded and you don't need to assume a bound on $|\det U(\mathbf{x})|$.

Corollary 12.5.3 *Let V be an open set in $\mathbb{R}^n$ and let $\mathbf{h} \in C^1(V)$ be one to one and also for $D\mathbf{h}(\mathbf{x}) = R(\mathbf{x})U(\mathbf{x})$ the right polar decomposition, $U(\mathbf{x})$ is one to one. Let E be $\mathcal{H}^n$ measurable and let $A \subseteq V$ be Lebesgue measurable. Then*

$$\int_{\mathbf{h}(A)} \mathcal{X}_E(\mathbf{y})\, d\mathcal{H}^n = \int_A \mathcal{X}_E(\mathbf{h}(\mathbf{x}))\, |\det(U(\mathbf{x}))|\, dm_n.$$

Proof: For each $\mathbf{x} \in A$, there exists $r_{\mathbf{x}}$ such that $\overline{B(\mathbf{x}, r_{\mathbf{x}})} \subseteq V$ and $r_{\mathbf{x}} < 1$. Then by the mean value inequality Theorem 6.4.2 and the observation that $||D\mathbf{h}(\mathbf{x})||$ is bounded on the compact set $\overline{B(\mathbf{x}, r_{\mathbf{x}})}$, it follows $\mathbf{h}(B(\mathbf{x}, r_{\mathbf{x}}))$ is also bounded. Also $|\det U(\mathbf{x})|$ is bounded on the compact set $\overline{B(\mathbf{x}, r_{\mathbf{x}})}$. These balls are a Vitali cover of A. By Corollary 9.7.5 there is a sequence of these disjoint balls $\{B_i\}$ such that $m_n(A \setminus \cup_{i=1}^{\infty} B_i) = 0$ and so

$$A = \cup_{i=1}^{\infty} (B_i \cap A) \cup N$$

where $N = A \setminus \cup_{i=1}^{\infty} B_i$ has measure zero.

It follows from Lemma 12.4.1 that $\mathbf{h}(N)$ also has measure zero. Then from Lemma 12.5.2 applied to B_i,

$$\begin{aligned}
\int_{\mathbf{h}(A)} \mathcal{X}_E(\mathbf{y})\, d\mathcal{H}^n &= \sum_i \int_{\mathbf{h}(B_i \cap A)} \mathcal{X}_{E \cap \mathbf{h}(B_i \cap A)}(\mathbf{y})\, d\mathcal{H}^n \\
&= \sum_i \int_{B_i \cap A} \mathcal{X}_E(\mathbf{h}(\mathbf{x}))\, |\det(U(\mathbf{x}))|\, dm_n \\
&= \int_A \mathcal{X}_E(\mathbf{h}(\mathbf{x}))\, |\det(U(\mathbf{x}))|\, dm_n.
\end{aligned}$$

This proves the corollary.

With this corollary, the main theorem follows.

Theorem 12.5.4 *Let V be an open set in $\mathbb{R}^n$ and A is a Lebesgue measurable subset of V. Let $\mathbf{h} \in C^1(V)$ be one to one and also for $D\mathbf{h}(\mathbf{x}) = R(\mathbf{x})U(\mathbf{x})$ the right polar decomposition, $U(\mathbf{x})$ is one to one. Then if g is any nonnegative $\mathcal{H}^n$ measurable function,*

$$\int_{\mathbf{h}(A)} g(\mathbf{y})\, d\mathcal{H}^n = \int_A g(\mathbf{h}(\mathbf{x}))\, |\det(U(\mathbf{x}))|\, dm_n. \tag{12.24}$$

Proof: From Corollary 12.5.3, 12.24 holds for any nonnegative simple function in place of g. In general, let $\{s_k\}$ be an increasing sequence of simple functions which converges to g pointwise. Then from the monotone convergence theorem

$$\begin{aligned}
\int_{\mathbf{h}(A)} g(\mathbf{y})\, d\mathcal{H}^n &= \lim_{k \to \infty} \int_{\mathbf{h}(A)} s_k d\mathcal{H}^n = \lim_{k \to \infty} \int_A s_k(\mathbf{h}(\mathbf{x}))\, |\det(U(\mathbf{x}))|\, dm_n \\
&= \int_A g(\mathbf{h}(\mathbf{x}))\, |\det(U(\mathbf{x}))|\, dm_n.
\end{aligned}$$

This proves the theorem.

Of course this theorem implies the following corollary obtained by splitting the function into the positive and negative parts of the real and imaginary parts.

Corollary 12.5.5 *Let V be an open set in $\mathbb{R}^n$ and A is a Lebesgue measurable subset of V. Let $\mathbf{h} \in C^1(V)$ be one to one and also for $D\mathbf{h}(\mathbf{x}) = R(\mathbf{x})U(\mathbf{x})$ the right polar decomposition, $U(\mathbf{x})$ is one to one. Then if g is any function in $L^1(\mathbf{h}(V))$,*

$$\int_{\mathbf{h}(A)} g(\mathbf{y})\, d\mathcal{H}^n = \int_A g(\mathbf{h}(\mathbf{x}))\, |\det(U(\mathbf{x}))|\, dm_n.$$

You don't need to assume $U(\mathbf{x})$ is one to one. The following lemma is like Sard's lemma presented earlier. However, it might seem a little easier and if so, it is because the area formula above is available and Hausdorff measures are in some ways easier to work with since they only depend on distance.

Lemma 12.5.6 *Let V be an open set in $\mathbb{R}^n$ and let $\mathbf{h}: V \to \mathbb{R}^m$ be a C^1 function such that the right polar decomosition of the derivative is $D\mathbf{h}(\mathbf{x}) = R(\mathbf{x})U(\mathbf{x})$. Let E denote the set*

$$E \equiv \left\{\mathbf{x} \in V : U(\mathbf{x})^{-1} \text{ does not exist}\right\}$$

Then $\mathcal{H}^n(\mathbf{h}(E)) = 0$.

Proof: Recall the notation $J(\mathbf{x}) \equiv |\det(U(\mathbf{x}))|$ discussed earlier. Modify it slightly as

$$J\mathbf{h}(\mathbf{x}) \equiv |\det(U(\mathbf{x}))|$$

where $D\mathbf{h}(\mathbf{x}) = R(\mathbf{x})U(\mathbf{x})$. First suppose V is bounded and so is $||D\mathbf{h}(\mathbf{x})||$. Define $\mathbf{k}_\varepsilon : \mathbb{R}^n \to \mathbb{R}^m \times \mathbb{R}^n$

$$\mathbf{k}_\varepsilon(\mathbf{x}) \equiv \begin{pmatrix} \mathbf{h}(\mathbf{x}) \\ \varepsilon\mathbf{x} \end{pmatrix}$$

Thus

$$\mathbf{k}_\varepsilon(\mathbf{x}+\mathbf{v}) - \mathbf{k}_\varepsilon(\mathbf{x}) = \begin{pmatrix} \mathbf{h}(\mathbf{x}+\mathbf{v}) - \mathbf{h}(\mathbf{x}) \\ \varepsilon\mathbf{v} \end{pmatrix}$$

$$= \begin{pmatrix} D\mathbf{h}(\mathbf{x})\mathbf{v} \\ \varepsilon\mathbf{v} \end{pmatrix} + \mathbf{o}(\mathbf{v})$$

and so

$$D\mathbf{k}_\varepsilon(\mathbf{x}) = \begin{pmatrix} D\mathbf{h}(\mathbf{x}) \\ \varepsilon\,\mathrm{id} \end{pmatrix} \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^m \times \mathbb{R}^n)$$

It is left as an exercise to explain how

$$D\mathbf{k}_\varepsilon(\mathbf{x})^* = \begin{pmatrix} D\mathbf{h}(\mathbf{x})^* & \varepsilon\,\mathrm{id} \end{pmatrix} \in \mathcal{L}(\mathbb{R}^m \times \mathbb{R}^n, \mathbb{R}^n)$$

and

$$J\mathbf{k}_\varepsilon(\mathbf{x})^2 \equiv \det\left(D\mathbf{k}_\varepsilon(\mathbf{x})^* D\mathbf{k}_\varepsilon(\mathbf{x})\right) = \det\left(D\mathbf{h}^*(\mathbf{x})D\mathbf{h}(\mathbf{x}) + \varepsilon^2\,\mathrm{id}\right)$$

Now there is an orthonormal basis, $\{\mathbf{v}_k\}$ for $\mathbb{R}^n$ such that

$$D\mathbf{h}^*(\mathbf{x})D\mathbf{h}(\mathbf{x}) = \sum_{k=1}^n a_k \mathbf{v}_k\mathbf{v}_k,\ \varepsilon^2\,\mathrm{id} = \varepsilon^2\sum_{k=1}^n \mathbf{v}_k\mathbf{v}_k.$$

where each $a_k \geq 0$ and on E, at least one equals 0. Thus the determinant above is the determinant of a diagonal matrix which has all positive entries on the diagonal but at least one of them is ε^2. Since $||D\mathbf{h}(\mathbf{x})||$ is bounded, this shows there exists a constant C independent of $\mathbf{x} \in V$ such that on E, $J\mathbf{k}_\varepsilon(\mathbf{x}) \leq \varepsilon C$. Now by the earlier area formula, Theorem 12.5.4,

$$\begin{aligned}\int_{\mathbf{k}_\varepsilon(V)} \mathcal{X}_{\mathbf{k}_\varepsilon(E)}(\mathbf{z})\,d\mathcal{H}^n &= \int_V \mathcal{X}_{\mathbf{k}_\varepsilon(E)}(\mathbf{k}_\varepsilon(\mathbf{x}))\,J\mathbf{k}_\varepsilon(\mathbf{x})\,dm_n \\ &= \int_V \mathcal{X}_E(\mathbf{x})\,J\mathbf{k}_\varepsilon(\mathbf{x})\,dm_n\end{aligned}$$

Thus

$$\mathcal{H}^n(\mathbf{k}_\varepsilon(E)) \leq \varepsilon C m_n(E)$$

However, $|\mathbf{h}(\mathbf{x}) - \mathbf{h}(\mathbf{x}_1)| \leq |\mathbf{k}_\varepsilon(\mathbf{x}) - \mathbf{k}_\varepsilon(\mathbf{x}_1)|$ and so by Lemma 12.4.4 $\mathcal{H}^n(\mathbf{k}_\varepsilon(E)) \geq \mathcal{H}^n(\mathbf{h}(E))$. Let $P\begin{pmatrix} \mathbf{y} & \mathbf{x} \end{pmatrix} \equiv \mathbf{y}$. Thus

$$\mathcal{H}^n(\mathbf{h}(E)) \leq \varepsilon C m_n(E)$$

and since ε is arbitrary, this shows $\mathcal{H}^n(\mathbf{h}(E)) = 0$.

In the general case when V might not be bounded define for each $k \in \mathbb{N}$ sufficiently large that the sets are nonempty

$$V_k \equiv B(\mathbf{0},k) \cap \left\{\mathbf{x} \in V : \operatorname{dist}\left(\mathbf{x},V^C\right) > 1/k\right\}$$

Then $\overline{V_k}$ is compact and if $E_k \equiv E \cap V_k$ and V_k plays the role of V above, it follows $||D\mathbf{h}(\mathbf{x})||$ is bounded on V_k and $\mathcal{H}^n(\mathbf{h}(E_k)) = 0$. Now let $k \to \infty$ to conclude $\mathcal{H}^n(\mathbf{h}(E)) = 0$. This proves the lemma.

Now with this lemma it is easy to give a fairly general version of the area formula.

Theorem 12.5.7 *Let V be an open set in $\mathbb{R}^n$ and A is a Lebesgue measurable subset of V. Let $\mathbf{h} \in C^1(V)$ be one to one. Then if g is any nonnegative $\mathcal{H}^n$ measurable function,*

$$\int_{\mathbf{h}(V)} g(\mathbf{y})\, d\mathcal{H}^n = \int_V g(\mathbf{h}(\mathbf{x}))\, |\det(U(\mathbf{x}))|\, dm_n. \tag{12.25}$$

Proof: Let $E = \left\{\mathbf{x} \in V : U(\mathbf{x})^{-1} \text{ does not exist}\right\}$ which is the same as the set where $\det U(\mathbf{x}) = 0$. Then by Lemma 12.5.6 $\mathcal{H}^n(\mathbf{h}(E)) = 0$ and so

$$\begin{aligned}
\int_{\mathbf{h}(V)} g(\mathbf{y})\, d\mathcal{H}^n &= \int_{\mathbf{h}(V)} \mathcal{X}_{\mathbf{h}(V\setminus E)}(\mathbf{y})\, g(\mathbf{y})\, d\mathcal{H}^n \\
&= \int_V \mathcal{X}_{\mathbf{h}(V\setminus E)}(\mathbf{h}(\mathbf{x}))\, g(\mathbf{h}(\mathbf{x}))\, |\det(U(\mathbf{x}))|\, dm_n \\
&= \int_V \mathcal{X}_{V\setminus E}(\mathbf{x})\, g(\mathbf{h}(\mathbf{x}))\, |\det(U(\mathbf{x}))|\, dm_n \\
&= \int_V g(\mathbf{h}(\mathbf{x}))\, |\det(U(\mathbf{x}))|\, dm_n
\end{aligned}$$

This proves the theorem.

Note that if $m = n$, this reduces to the usual change of variables formula because $\mathcal{H}^n = m_n$ on $\mathbb{R}^n$ as was shown above.

As before, there is an obvious corollary obtained by splitting up the function in L^1 into positive and negative parts of real and imaginary parts, noting the desired result holds for each of these pieces and then adding them together.

Corollary 12.5.8 *Let V be an open set in $\mathbb{R}^n$ and A is a Lebesgue measurable subset of V and let $\mathbf{h} \in C^1(V)$ be one to one. Then if g is any function in $L^1(\mathbf{h}(A))$,*

$$\int_{\mathbf{h}(A)} g(\mathbf{y})\, d\mathcal{H}^n = \int_A g(\mathbf{h}(\mathbf{x}))\, |\det(U(\mathbf{x}))|\, dm_n.$$

12.6 Area Formula For Mappings Which Are Not One To One

Now suppose $\mathbf{h}$ is only C^1, not necessarily one to one. For

$$V_+ \equiv \{\mathbf{x} \in V : |\det U(x)| > 0\}$$

and Z the set where $|\det U(x)| = 0$, Lemma 12.5.6 implies $\mathcal{H}^n(\mathbf{h}(Z)) = 0$.

Now the following lemma is very interesting for its own sake.

Lemma 12.6.1 *For $\mathbf{x} \in V_+$, there exists an open ball $B_{\mathbf{x}} \subseteq V_+$ such that $\mathbf{h}$ is one to one on $B_{\mathbf{x}}$.*

Proof: Let $D\mathbf{h}(\mathbf{x}) = R(\mathbf{x})U(\mathbf{x})$ be the right polar decomposition. Recall that $U(\mathbf{x})$ is self adjoint and satisfies $U(\mathbf{x})\mathbf{v}\cdot\mathbf{v} \geq \delta|\mathbf{v}|^2$ for some $\delta > 0$ where δ is the smallest eigenvalue of $U(\mathbf{x})$, the square root of the smallest eigenvalue of $U(\mathbf{x})^2 = D\mathbf{h}(\mathbf{x})^* D\mathbf{h}(\mathbf{x})$. Let $r > 0$ such that $\overline{B(\mathbf{x},r)} \subseteq V_+$. Then for $\mathbf{y} \in \overline{B(\mathbf{x},r)}$, the continuity of $\mathbf{y} \to U(\mathbf{y})^2$, resulting for the assumption that $\mathbf{h}$ is C^1, implies

$$\begin{aligned} U(\mathbf{y})^2\mathbf{v}\cdot\mathbf{v} &= U(\mathbf{x})^2\mathbf{v}\cdot\mathbf{v} + \left(U(\mathbf{y})^2 - U(\mathbf{x})^2\right)\mathbf{v}\cdot\mathbf{v} \\ &\geq \delta^2|\mathbf{v}|^2 - \frac{\delta^2}{2}|\mathbf{v}|^2 = \frac{\delta^2}{2}|\mathbf{v}|^2 \end{aligned}$$

provided r is sufficiently small. Thus for small enough r, the eigenvalues of $U(\mathbf{y})^2$ for $\mathbf{y} \in \overline{B(\mathbf{x},r)}$ are at least as large as $\delta^2/2$ and so the eigenvalues of $U(\mathbf{y})$ for these values of $\mathbf{y}$ are lat least as large as $\delta/\sqrt{2}$. Thus for $\mathbf{y} \in \overline{B(\mathbf{x},r)}$,

$$U(\mathbf{y})\mathbf{v}\cdot\mathbf{v} \geq \frac{\delta}{\sqrt{2}}|\mathbf{v}|^2.$$

Now make r still smaller if necessary such that for $\mathbf{y},\mathbf{z} \in \overline{B(\mathbf{x},r)}$,

$$||D\mathbf{h}(\mathbf{y}) - D\mathbf{h}(\mathbf{z})|| < \frac{\delta}{2}.$$

Then for any $\mathbf{y},\mathbf{z}$ of this sort,

$$|\mathbf{h}(\mathbf{z}) - \mathbf{h}(\mathbf{y}) - D\mathbf{h}(\mathbf{y})(\mathbf{z}-\mathbf{y})| < \frac{\delta}{2}|\mathbf{z}-\mathbf{y}|. \tag{12.26}$$

This follows from the mean value inequality, Theorem 6.4.2 because if you define for such a fixed $\mathbf{y} \in \overline{B(\mathbf{x},r)}$

$$\mathbf{F}(\mathbf{z}) \equiv \mathbf{h}(\mathbf{z}) - \mathbf{h}(\mathbf{y}) - D\mathbf{h}(\mathbf{y})(\mathbf{z}-\mathbf{y}),$$

it follows $\mathbf{F}(\mathbf{y}) = \mathbf{0}$ and $D\mathbf{F}(\mathbf{z}) = D\mathbf{h}(\mathbf{z}) - D\mathbf{h}(\mathbf{y})$.

Then for $\mathbf{y},\mathbf{z} \in \overline{B(\mathbf{x},r)}$,

$$R^*(\mathbf{y})(\mathbf{h}(\mathbf{z}) - \mathbf{h}(\mathbf{y})) = U(\mathbf{y})(\mathbf{z}-\mathbf{y}) + R^*\mathbf{o}(\mathbf{z}-\mathbf{y}) \tag{12.27}$$

where $\mathbf{o}(\mathbf{z}-\mathbf{y})$ is of the form

$$\mathbf{h}(\mathbf{z}) - \mathbf{h}(\mathbf{y}) - D\mathbf{h}(\mathbf{y})(\mathbf{z}-\mathbf{y})$$

Then from 12.26,

$$|R^*\mathbf{o}(\mathbf{z}-\mathbf{y})| \leq |\mathbf{o}(\mathbf{z}-\mathbf{y})| \leq \frac{\delta}{2}|\mathbf{z}-\mathbf{y}|.$$

If $\mathbf{h}(\mathbf{z}) = \mathbf{h}(\mathbf{y})$, then taking the inner product of both sides of 12.27 with $\mathbf{z}-\mathbf{y}$,

$$\mathbf{0} \geq (U(\mathbf{y})(\mathbf{z}-\mathbf{y}),(\mathbf{z}-\mathbf{y})) - \frac{\delta}{2}|\mathbf{z}-\mathbf{y}|^2 \geq \left(\frac{\delta}{\sqrt{2}} - \frac{\delta}{2}\right)|\mathbf{z}-\mathbf{y}|^2$$

showing $\mathbf{z} = \mathbf{y}$. This proves the lemma.

Let $\{B_i\}$ be a countable subset of $\{B_{\mathbf{x}}\}_{\mathbf{x}\in V_+}$ such that $V_+ = \cup_{i=1}^{\infty} B_i$. Let $E_1 = B_1$. If $E_1, \cdots, E_k$ have been chosen, $E_{k+1} = B_{k+1} \setminus \cup_{i=1}^{k} E_i$. Thus

$$\cup_{i=1}^{\infty} E_i = V_+, \ \ \mathbf{h} \text{ is one to one on } E_i, \ \ E_i \cap E_j = \emptyset,$$

and each E_i is a Borel set contained in the open set B_i. Now define

$$n(\mathbf{y}) \equiv \sum_{i=1}^{\infty} \mathcal{X}_{\mathbf{h}(E_i \cap A)}(\mathbf{y}) + \mathcal{X}_{\mathbf{h}(Z)}(\mathbf{y}).$$

Thus

$$\sum_{i=1}^{\infty} \mathcal{X}_{\mathbf{h}(E_i \cap A)}(\mathbf{y}) = \mathcal{X}_{A_+}$$

where $A_+ \equiv V_+ \cap A$. The set, $\mathbf{h}(E_i), \mathbf{h}(Z)$ are $\mathcal{H}^n$ measurable by Lemma 12.4.2. Thus $n(\cdot)$ is $\mathcal{H}^n$ measurable.

Lemma 12.6.2 *Let V be an open set, $F \subseteq \mathbf{h}(V)$ be $\mathcal{H}^n$ measurable and let A be a Lebesgue measurable subset of V. Then*

$$\int_{\mathbf{h}(A)} n(\mathbf{y}) \mathcal{X}_F(\mathbf{y}) dm_n = \int_A \mathcal{X}_F(\mathbf{h}(\mathbf{x})) |\det D\mathbf{h}(\mathbf{x})| dm_n.$$

Proof: Using Lemma 12.5.6 and the Monotone Convergence Theorem

$$\begin{aligned}
\int_{\mathbf{h}(A)} n(\mathbf{y}) \mathcal{X}_F(\mathbf{y}) d\mathcal{H}^n &= \int_{\mathbf{h}(A)} \left(\sum_{i=1}^{\infty} \mathcal{X}_{\mathbf{h}(E_i)}(\mathbf{y}) + \overbrace{\mathcal{X}_{\mathbf{h}(Z)}(\mathbf{y})}^{\mathcal{H}^n(\mathbf{h}(Z))=0} \right) \mathcal{X}_F(\mathbf{y}) d\mathcal{H}^n \\
&= \sum_{i=1}^{\infty} \int_{\mathbf{h}(A)} \mathcal{X}_{\mathbf{h}(E_i)}(\mathbf{y}) \mathcal{X}_F(\mathbf{y}) d\mathcal{H}^n \\
&= \sum_{i=1}^{\infty} \int_{\mathbf{h}(B_i \cap A)} \mathcal{X}_{\mathbf{h}(E_i \cap A)}(\mathbf{y}) \mathcal{X}_F(\mathbf{y}) d\mathcal{H}^n \\
&= \sum_{i=1}^{\infty} \int_{B_i \cap A} \mathcal{X}_{E_i \cap A}(\mathbf{x}) \mathcal{X}_F(\mathbf{h}(\mathbf{x})) |\det U(\mathbf{x}))| dm_n \\
&= \sum_{i=1}^{\infty} \int_A \mathcal{X}_{E_i \cap A}(\mathbf{x}) \mathcal{X}_F(\mathbf{h}(\mathbf{x})) |\det U(\mathbf{x})| dm_n \\
&= \int_A \sum_{i=1}^{\infty} \mathcal{X}_{E_i \cap A}(\mathbf{x}) \mathcal{X}_F(\mathbf{h}(\mathbf{x})) |\det U(\mathbf{x})| dm_n
\end{aligned}$$

$$= \int_{A_+} \mathcal{X}_F(\mathbf{h}(\mathbf{x})) |\det U(\mathbf{x})| dm_n = \int_A \mathcal{X}_F(\mathbf{h}(\mathbf{x})) |\det U(\mathbf{x})| dm_n.$$

The integrand in the integrand on the left was shown to be Lebesgue measurable from the above argument. Therefore, the integrand of the integral on the right is also Lebesgue measurable because it equals

$$\mathcal{X}_F(\mathbf{h}(\mathbf{x})) |\det U(\mathbf{x})| \mathcal{X}_{A_+} + 0 \mathcal{X}_Z(\mathbf{x})$$

and both functions in the sum are measurable. This proves the lemma.

Definition 12.6.3 *For $\mathbf{y} \in \mathbf{h}(A)$, define a function, $\#$, according to the formula*

$$\#(\mathbf{y}) \equiv \textit{number of elements in } \mathbf{h}^{-1}(\mathbf{y}).$$

Observe that

$$\#(\mathbf{y}) = n(\mathbf{y})\ \mathcal{H}^n \text{ a.e.} \tag{12.28}$$

because $n(\mathbf{y}) = \#(\mathbf{y})$ if $\mathbf{y} \notin \mathbf{h}(Z)$, a set of $\mathcal{H}^n$ measure 0. Therefore, $\#$ is a measurable function because of completeness of $\mathcal{H}^n$.

Theorem 12.6.4 *Let $g \geq 0$, g is $\mathcal{H}^n$ measurable, and let $\mathbf{h}$ be $C^1(V)$ and A is a Lebesgue measurable subset of V. Then*

$$\int_{\mathbf{h}(A)} \#(\mathbf{y}) g(\mathbf{y}) dm_n = \int_A g(\mathbf{h}(\mathbf{x}))|\det D\mathbf{h}(\mathbf{x})| dm_n. \tag{12.29}$$

The integrand on the right is Lebesgue measurable.

Proof: From 12.28 and Lemma 12.6.2, 12.29 holds for all g, a nonnegative simple function. Approximating an arbitrary measurable nonnegative function, g, with an increasing pointwise convergent sequence of simple functions and using the monotone convergence theorem, yields 12.29 for an arbitrary nonnegative measurable function, g. This proves the theorem.

12.7 The Coarea Formula

In the coarea formula, $\mathbf{h}$ maps $V \subseteq \mathbb{R}^n$ to $\mathbb{R}^m$ where $m \leq n$. It is possible to obtain this formula from the area formula and some interesting linear algebra.

Lemma 12.7.1 *Let $A \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^m)$. Then the nonzero eigenvalues of AA^* and A^*A are the same and occur with the same algebraic multiplicities.*

Proof: This follows from Theorem 3.6.5 on Page 49 applied to the matrices of A and A^*.

Corollary 12.7.2 *Let $A \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^m)$. Then*

$$\det(\mathrm{id} + A^*A) = \det(\mathrm{id} + AA^*).$$

Proof: This follows from Lemma because

$$\mathrm{id} + A^*A = \sum_{k=1}^{r} (1 + a_k)\, \mathbf{w}_k \mathbf{w}_k + \sum_{k=r+1}^{n} 1 \mathbf{w}_k \mathbf{w}_k$$

while $\mathrm{id} + AA^*$ is of the form

$$\sum_{k=1}^{r} (1 + a_k)\, \mathbf{v}_k \mathbf{v}_k + \sum_{k=r+1}^{m} 1 \mathbf{v}_k \mathbf{v}_k.$$

Therefore, both of these determinants equal

$$\prod_{k=1}^{r} (1 + a_k).$$

This proves the corollary.

Next is a lemma which leads to some conclusions about measurability.

Lemma 12.7.3 *Let* $\mathbf{h} : V \subseteq \mathbb{R}^p \to \mathbb{R}^m$ *be continuous and* $\delta > 0$. *Then if* $A \subseteq \mathbb{R}^p$ *is either open or compact,*

$$\mathbf{y} \to \mathcal{H}^s_\delta\left(A \cap \mathbf{h}^{-1}(\mathbf{y})\right)$$

is Borel measurable.

Proof: Suppose first that A is compact and suppose for $\delta > 0$,

$$\mathcal{H}^s_\delta\left(A \cap \mathbf{h}^{-1}(\mathbf{y})\right) < t$$

Then there exist sets S_i, satisfying

$$r(S_i) < \delta,\ A \cap \mathbf{h}^{-1}(\mathbf{y}) \subseteq \cup_{i=1}^\infty S_i,$$

and

$$\sum_{i=1}^\infty \alpha(s)\left(r(S_i)\right)^s < t.$$

I claim these sets can be taken to be open sets. Choose $\lambda > 1$ but close enough to 1 that

$$\sum_{i=1}^\infty \alpha(s)\left(\lambda r(S_i)\right)^s < t$$

Replace S_i with $S_i + B(0,\eta_i)$ where η_i is small enough that

$$\operatorname{diam}(S_i) + 2\eta_i < \lambda \operatorname{diam}(S_i).$$

Then

$$\operatorname{diam}(S_i + B(0,\eta_i)) \le \lambda \operatorname{diam}(S_i)$$

and so $r(S_i + B(0,\eta_i)) \le \lambda r(S_i)$. Thus

$$\sum_{i=1}^\infty \alpha(s)\, r(S_i + B(0,\eta_i))^s < t.$$

Hence you could replace S_i with $S_i + B(0,\eta_i)$ and so one can assume the sets S_i are open.

Claim: If $\mathbf{z}$ is close enough to $\mathbf{y}$, then $A \cap \mathbf{h}^{-1}(\mathbf{z}) \subseteq \cup_{i=1}^\infty S_i$.

Proof: If not, then there exists a sequence $\{\mathbf{z}_k\}$ such that

$$\mathbf{z}_k \to \mathbf{y},$$

and

$$\mathbf{x}_k \in \left(A \cap \mathbf{h}^{-1}(\mathbf{z}_k)\right) \setminus \cup_{i=1}^\infty S_i.$$

By compactness of A, there exists a subsequence still denoted by k such that

$$\mathbf{z}_k \to \mathbf{y},\ \mathbf{x}_k \to \mathbf{x} \in A \setminus \cup_{i=1}^\infty S_i.$$

Hence

$$\mathbf{h}(\mathbf{x}) = \lim_{k\to\infty} \mathbf{h}(\mathbf{x}_k) = \lim_{k\to\infty} \mathbf{z}_k = \mathbf{y}.$$

But $\mathbf{x} \notin \cup_{i=1}^\infty S_i$ contrary to the assumption that $A \cap \mathbf{h}^{-1}(\mathbf{y}) \subseteq \cup_{i=1}^\infty S_i$.

It follows from this claim that whenever $\mathbf{z}$ is close enough to $\mathbf{y}$,

$$\mathcal{H}^s_\delta\left(A \cap \mathbf{h}^{-1}(\mathbf{z})\right) < t.$$

This shows

$$\left\{\mathbf{z}\in\mathbb{R}^p : \mathcal{H}^s_\delta\left(A\cap\mathbf{h}^{-1}(\mathbf{z})\right)<t\right\}$$

is an open set and so $\mathbf{y}\to\mathcal{H}^s_\delta\left(A\cap\mathbf{h}^{-1}(\mathbf{y})\right)$ is Borel measurable whenever A is compact. Now let V be an open set and let

$$A_k\uparrow V,\ A_k \text{ compact.}$$

Then

$$\mathcal{H}^s_\delta\left(V\cap\mathbf{h}^{-1}(\mathbf{y})\right)=\lim_{k\to\infty}\mathcal{H}^s_\delta\left(A_k\cap\mathbf{h}^{-1}(\mathbf{y})\right)$$

so $\mathbf{y}\to\mathcal{H}^s_\delta\left(V\cap\mathbf{h}^{-1}(\mathbf{y})\right)$ is Borel measurable for all V open. This proves the lemma.

Lemma 12.7.4 *Let $\mathbf{h}: V\subseteq\mathbb{R}^p\to\mathbb{R}^m$ be Lipschitz continuous, satisfying an inequality of the form*

$$|\mathbf{h}(\mathbf{x})-\mathbf{h}(\mathbf{y})|\le \mathrm{Lip}(\mathbf{h})\,|\mathbf{x}-\mathbf{y}|$$

where $\mathrm{Lip}(\mathbf{h})$ is a positive constant. Suppose A is either open or compact in $\mathbb{R}^p$. Then $\mathbf{y}\to\mathcal{H}^s\left(A\cap\mathbf{h}^{-1}(\mathbf{y})\right)$ is also Borel measurable and

$$\int_{\mathbb{R}^m}\mathcal{H}^s\left(A\cap\mathbf{h}^{-1}(\mathbf{y})\right)dm_m\le 2^m\left(\mathrm{Lip}(\mathbf{h})\right)^m\frac{\alpha(s)\,\alpha(m)}{\alpha(s+m)}\mathcal{H}^{s+m}(A)$$

In particular, if $s=n-m$ and $p=n$

$$\int_{\mathbb{R}^m}\mathcal{H}^{n-m}\left(A\cap\mathbf{h}^{-1}(\mathbf{y})\right)dm_m\le 2^m\left(\mathrm{Lip}(\mathbf{h})\right)^m\frac{\alpha(n-m)\,\alpha(m)}{\alpha(n)}m_n(A)\tag{12.30}$$

Proof: From Lemma 12.7.3 $\mathbf{y}\to\mathcal{H}^s_\delta\left(A\cap\mathbf{h}^{-1}(\mathbf{y})\right)$ is Borel measurable for each $\delta>0$. Without loss of generality, $\mathcal{H}^{s+m}(A)<\infty$. Now let C_i be closed sets with $r(C_i)<\delta, A\subseteq\cup_{i=1}^\infty C_i$, and

$$\mathcal{H}^{s+m}_\delta(A)+\varepsilon>\sum_{i=1}^\infty\alpha(s+m)\,r(C_i)^{s+m}.$$

Note each C_i is compact so $\mathbf{y}\to\mathcal{H}^s_\delta\left(C_i\cap\mathbf{h}^{-1}(\mathbf{y})\right)$ is Borel measurable. Thus

$$\begin{aligned}
&\int_{\mathbb{R}^m}\mathcal{H}^s_\delta\left(A\cap\mathbf{h}^{-1}(\mathbf{y})\right)dm_m\\
\le\ &\int_{\mathbb{R}^m}\sum_i\mathcal{H}^s_\delta\left(C_i\cap\mathbf{h}^{-1}(\mathbf{y})\right)dm_m\\
=\ &\sum_i\int_{\mathbb{R}^m}\mathcal{H}^s_\delta\left(C_i\cap\mathbf{h}^{-1}(\mathbf{y})\right)dm_m\\
\le\ &\sum_i\int_{\mathbf{h}(C_i)}\mathcal{H}^s_\delta(C_i)\,dm_m\\
=\ &\sum_i m_m\left(\mathbf{h}(C_i)\right)\mathcal{H}^s_\delta(C_i)
\end{aligned}$$

Now $\mathbf{h}(C_i)$ is contained in a ball of radius $2r(C_i)$ and so

$$\begin{aligned}
\le\ &\sum_i\left(\mathrm{Lip}(\mathbf{h})\right)^m 2^m\alpha(m)\,r(C_i)^m\,\alpha(s)\,r(C_i)^s\\
=\ &\left(\mathrm{Lip}(\mathbf{h})\right)^m\frac{\alpha(m)\,\alpha(s)}{\alpha(m+s)}2^m\sum_i\alpha(s+m)\,r(C_i)^{m+s}\\
\le\ &\left(\mathrm{Lip}(\mathbf{h})\right)^m\frac{\alpha(m)\,\alpha(s)}{\alpha(m+s)}2^m\left(\mathcal{H}^{s+m}_\delta(A)+\varepsilon\right)
\end{aligned}$$

Since $\varepsilon > 0$ is arbitrary,

$$\int_{\mathbb{R}^m} \mathcal{H}_\delta^s\left(A \cap \mathbf{h}^{-1}(\mathbf{y})\right) dm_m \leq (\mathrm{Lip}(\mathbf{h}))^m \frac{\alpha(m)\,\alpha(s)}{\alpha(m+s)} 2^m \mathcal{H}_\delta^{s+m}(A)$$

Taking a limit as $\delta \to 0$ this proves the lemma.

Next I will show that whenever A is Lebesgue measurable,

$$\mathbf{y} \to \mathcal{H}^{n-m}\left(A \cap \mathbf{h}^{-1}(\mathbf{y})\right)$$

is m_m measurable and the above estimate holds.

Lemma 12.7.5 *Let A be a Lebesgue measurable subset of V an open set and let $\mathbf{h} : V \subseteq \mathbb{R}^n \to \mathbb{R}^m$ be Lipschitz continuous,*

$$|\mathbf{h}(\mathbf{x}) - \mathbf{h}(\mathbf{y})| \leq \mathrm{Lip}(\mathbf{h})\,|\mathbf{x} - \mathbf{y}|.$$

Then

$$\mathbf{y} \to \mathcal{H}^{n-m}\left(A \cap \mathbf{h}^{-1}(\mathbf{y})\right)$$

is Lebesgue measurable. Furthermore

$$\int_{\mathbb{R}^m} \mathcal{H}^{n-m}\left(A \cap \mathbf{h}^{-1}(\mathbf{y})\right) dm_m \leq 2^m (\mathrm{Lip}(\mathbf{h}))^m \frac{\alpha(n-m)\,\alpha(m)}{\alpha(n)} m_n(A)$$

Proof: Let A be a bounded Lebesgue measurable set in $\mathbb{R}^n$. Then by inner and outer regularity of Lebesgue measure there exists an increasing sequence of compact sets, $\{K_k\}$ contained in A and a decreasing sequence of open sets, $\{V_k\}$ containing A such that $m_n(V_k \setminus K_k) < 2^{-k}$. Thus $m_n(V_1) \leq m_n(A) + 1$. By Lemma 12.7.4

$$\int_{\mathbb{R}^m} \mathcal{H}_\delta^{n-m}\left(V_1 \cap \mathbf{h}^{-1}(\mathbf{y})\right) dm_m < 2^m (\mathrm{Lip}(\mathbf{h}))^m \frac{\alpha(n-m)\,\alpha(m)}{\alpha(n)} (m_n(A) + 1).$$

Also

$$\begin{gathered}\mathcal{H}_\delta^{n-m}\left(K_k \cap \mathbf{h}^{-1}(\mathbf{y})\right) \leq \\ \mathcal{H}_\delta^{n-m}\left(A \cap \mathbf{h}^{-1}(\mathbf{y})\right) \leq \mathcal{H}_\delta^{n-m}\left(V_k \cap \mathbf{h}^{-1}(\mathbf{y})\right)\end{gathered} \tag{12.31}$$

By Lemma 12.7.4

$$\begin{aligned}
&= \int_{\mathbb{R}^m} \left(\mathcal{H}_\delta^{n-m}\left(V_k \cap \mathbf{h}^{-1}(\mathbf{y})\right) - \mathcal{H}_\delta^{n-m}\left(K_k \cap \mathbf{h}^{-1}(\mathbf{y})\right)\right) dm_m \\
&= \int_{\mathbb{R}^m} \mathcal{H}_\delta^{n-m}\left((V_k - K_k) \cap \mathbf{h}^{-1}(\mathbf{y})\right) dm_m \\
&\leq 2^m (\mathrm{Lip}(\mathbf{h}))^m \frac{\alpha(n-m)\,\alpha(m)}{\alpha(n)} m_n(V_k \setminus K_k) \\
&< 2^m (\mathrm{Lip}(\mathbf{h}))^m \frac{\alpha(n-m)\,\alpha(m)}{\alpha(n)} 2^{-k}
\end{aligned}$$

Let the Borel measurable functions, g and f be defined by

$$g(\mathbf{y}) \equiv \lim_{k\to\infty} \mathcal{H}_\delta^{n-m}\left(V_k \cap \mathbf{h}^{-1}(\mathbf{y})\right),\ f(\mathbf{y}) \equiv \lim_{k\to\infty} \mathcal{H}_\delta^{n-m}\left(K_k \cap \mathbf{h}^{-1}(\mathbf{y})\right)$$

It follows from the dominated convergence theorem using $\mathcal{H}_\delta^{n-m}\left(V_1 \cap \mathbf{h}^{-1}(\mathbf{y})\right)$ as a dominating function and 12.31 that

$$f(\mathbf{y}) \leq \mathcal{H}_\delta^{n-m}\left(A \cap \mathbf{h}^{-1}(\mathbf{y})\right) \leq g(\mathbf{y})$$

and

$$\int_{\mathbb{R}^m} (g(\mathbf{y}) - f(\mathbf{y}))\, dm_m = 0.$$

By completness of m_m, this establishes $\mathbf{y} \to \mathcal{H}_\delta^{n-m}\left(A \cap \mathbf{h}^{-1}(\mathbf{y})\right)$ is Lebesgue measurable. Then by Lemma 12.7.4 again,

$$\int_{\mathbb{R}^m} \mathcal{H}_\delta^{n-m}\left(A \cap \mathbf{h}^{-1}(\mathbf{y})\right) dm_m \leq 2^m (\mathrm{Lip}(\mathbf{h}))^m \frac{\alpha(n-m)\,\alpha(m)}{\alpha(n)} m_n(A).$$

Letting $\delta \to 0$ and using the monotone convergence theorem yields the desired inequality for $\mathcal{H}^{n-m}\left(A \cap \mathbf{h}^{-1}(\mathbf{y})\right)$.

The case where A is not bounded can be handled by considering $A_r = A \cap B(\mathbf{0}, r)$ and letting $r \to \infty$. This proves the lemma.

By fussing with the isodiametric inequality one can remove the factor of 2^m in the above inequalities obtaining more attractive formulas. This is done in [13]. See also [27] which follows [13] and [16]. This last reference probably has the most complete treatment of these topics.

With these lemmas, it is now possible to give a proof of the coarea formula.

Define $\Lambda(n,m)$ as all possible ordered lists of m numbers taken from $\{1, 2, \cdots, n\}$.

Lemma 12.7.6 *Let A be a Lebesgue measurable set in V and let $\mathbf{h} : V \subseteq \mathbb{R}^n \to \mathbb{R}^m$ where $m \leq n$ is C^1 and also a Lipschitz map and for which*

$$J\mathbf{h}(\mathbf{x}) \equiv \det\left(D\mathbf{h}(\mathbf{x})\, D\mathbf{h}(\mathbf{x})^*\right)^{1/2} \neq 0.$$

Then the following formula holds along with all measurability assertions needed for it to make sense.

$$\int_{\mathbb{R}^m} \mathcal{H}^{n-m}\left(A \cap \mathbf{h}^{-1}(\mathbf{y})\right) dy = \int_A J\mathbf{h}(\mathbf{x})\, dx \tag{12.32}$$

Proof: For $\mathbf{x} \in \mathbb{R}^n$, and $\mathbf{i} \in \Lambda(n,m)$, with $\mathbf{i} = (i_1, \cdots, i_m)$, define $\mathbf{x_i} \equiv (x_{i_1}, \cdots, x_{i_m})$, and $\pi_{\mathbf{i}}\mathbf{x} \equiv \mathbf{x_i}$. Also for $\mathbf{i} \in \Lambda(n,m)$, let $\mathbf{i}_c \in \Lambda(n, n-m)$ consist of the remaining indices taken in order. For $\mathbf{h} : \mathbb{R}^n \to \mathbb{R}^m$ where $m \leq n$, define $J\mathbf{h}(\mathbf{x}) \equiv \det\left(D\mathbf{h}(\mathbf{x})\, D\mathbf{h}(\mathbf{x})^*\right)^{1/2}$. For each $\mathbf{i} \in \Lambda(n,m)$, define $\mathbf{h}^{\mathbf{i}} : \mathbb{R}^n \to \mathbb{R}^m \times \mathbb{R}^{n-m}$ by

$$\mathbf{h}^{\mathbf{i}}(\mathbf{x}) \equiv \begin{pmatrix} \mathbf{h}(\mathbf{x}) \\ \mathbf{x}_{\mathbf{i}_c} \end{pmatrix}.$$

As in Lemma 12.6.1, since $\mathbf{h}^{\mathbf{i}}$ is in $C^1(V)$ there exist Borel sets $\left\{F_k^{\mathbf{i}}\right\}$ which are disjoint, $F_k^{\mathbf{i}} \subseteq B_k^{\mathbf{i}}$, $\mathbf{h}^{\mathbf{i}}$ is one to one on the open ball $B_k^{\mathbf{i}}$ with a C^1 inverse defined on $\mathbf{h}^{\mathbf{i}}\left(B_k^{\mathbf{i}}\right)$ and

$$\cup_k F_k^{\mathbf{i}} = \left\{\mathbf{x} \in \mathbb{R}^n : \det D\mathbf{h}^{\mathbf{i}}(\mathbf{x}) \neq 0\right\}.$$

By assumption, for each $\mathbf{x} \in V$ there exists $\mathbf{i}$ such that $\det\left(D_{\mathbf{x_i}}\mathbf{h}(\mathbf{x})\right) \neq 0$ which implies $\det D\mathbf{h}^{\mathbf{i}}(\mathbf{x}) \neq 0$. This follows from Proposition 3.8.14 which says that $D\mathbf{h}(\mathbf{x})$ has m independent columns. Hence

$$\cup_{\mathbf{i},j} F_j^{\mathbf{i}} = V$$

Thus

$$\cup_{\mathbf{i},j} F_j^{\mathbf{i}} \cap A = A$$

The problem is the $F_j^{\mathbf{i}}$ might not be disjoint. Let $\left\{E_j^{\mathbf{i}}\right\}$ be measurable sets such that $E_j^{\mathbf{i}} \subseteq F_k^{\mathbf{i}} \cap A$ for some k, the sets, $E_j^{\mathbf{i}}$ are disjoint, and their union equals A. Then

$$\int_A J\mathbf{h}(\mathbf{x})\, dx = \sum_{\mathbf{i} \in \Lambda(n,m)} \sum_{j=1}^{\infty} \int_{E_j^{\mathbf{i}}} \det\left(D\mathbf{h}(\mathbf{x})\, D\mathbf{h}(\mathbf{x})^*\right)^{1/2} dx. \tag{12.33}$$

Now each $E_j^{\mathbf{i}}$ is contained in some $B_k^{\mathbf{i}}$ and so $\mathbf{h}^{\mathbf{i}}$ has an inverse on $\mathbf{h}^{\mathbf{i}}\left(B_k^{\mathbf{i}}\right)$ which I will denote by $\mathbf{g}$. Thus letting $\pi_{\mathbf{i}_c}\mathbf{x} \equiv \mathbf{x}_{\mathbf{i}_c}$ and using the definition of $\mathbf{g}$

$$\mathbf{g}\left(\mathbf{h}\left(\mathbf{x}\right), \mathbf{x}_{\mathbf{i}_c}\right) = \mathbf{x}.$$

By the chain rule,

$$D\mathbf{h}^{\mathbf{i}}\left(\mathbf{g}\left(\mathbf{y}\right)\right) D\mathbf{g}\left(\mathbf{y}\right) = I$$

on $\mathbf{h}^{\mathbf{i}}\left(E_j^{\mathbf{i}}\right)$. Changing the variables using the area formula, the expression in 12.33 equals

$$\int_A J\mathbf{h}\left(\mathbf{x}\right) dx =$$

$$\sum_{\mathbf{i}\in\Lambda(n,m)} \sum_{j=1}^{\infty} \int_{\mathbf{h}^{\mathbf{i}}\left(E_j^{\mathbf{i}}\right)} \det\left(D\mathbf{h}\left(\mathbf{g}\left(\mathbf{y}\right)\right) D\mathbf{h}\left(\mathbf{g}\left(\mathbf{y}\right)\right)^*\right)^{1/2} \left|\det D\mathbf{g}\left(\mathbf{y}\right)\right| dy =$$

$$\sum_{\mathbf{i}\in\Lambda(n,m)} \sum_{j=1}^{\infty} \int_{\mathbf{h}^{\mathbf{i}}\left(E_j^{\mathbf{i}}\right)} \det\left(D\mathbf{h}\left(\mathbf{g}\left(\mathbf{y}\right)\right) D\mathbf{h}\left(\mathbf{g}\left(\mathbf{y}\right)\right)^*\right)^{1/2} \left|\det D\mathbf{h}^{\mathbf{i}}\left(\mathbf{g}\left(\mathbf{y}\right)\right)\right|^{-1} dy. \tag{12.34}$$

Note the integrands are all Borel measurable functions because they are continuous functions of the entries of matrices which entries come from taking limits of difference quotients of continuous functions. Thus from 12.33,

$$\int_{E_j^{\mathbf{i}}} \det\left(D\mathbf{h}\left(\mathbf{x}\right) D\mathbf{h}\left(\mathbf{x}\right)^*\right)^{1/2} dx =$$

$$\int_{\mathbb{R}^n} \mathcal{X}_{\mathbf{h}^{\mathbf{i}}\left(E_j^{\mathbf{i}}\right)}\left(\mathbf{y}\right) \det\left(D\mathbf{h}\left(\mathbf{g}\left(\mathbf{y}\right)\right) D\mathbf{h}\left(\mathbf{g}\left(\mathbf{y}\right)\right)^*\right)^{1/2} \left|\det D\mathbf{h}^{\mathbf{i}}\left(\mathbf{g}\left(\mathbf{y}\right)\right)\right|^{-1} dy \tag{12.35}$$

Next this integral is split using Fubini's theorem. Let $\mathbf{y}_1 \in \mathbb{R}^m$ be fixed and now it is necessary to decide where $\mathbf{y}_2$ is. I need

$$\left(\mathbf{y}_1, \mathbf{y}_2\right) \in \mathbf{h}^{\mathbf{i}}\left(E_j^{\mathbf{i}}\right) = \left(\mathbf{h}\left(E_j^{\mathbf{i}}\right), \pi_{\mathbf{i}_c}\left(E_j^{\mathbf{i}}\right)\right)$$

This requires $\mathbf{y}_2 \in \pi_{\mathbf{i}_c}\left(E_j^{\mathbf{i}}\right)$. However, it is also the case that $\mathbf{y}_1$ is given. Now $\mathbf{y}_1 = \mathbf{h}\left(\mathbf{x}\right)$ and so

$$\mathbf{x} = \left(\mathbf{x}_{\mathbf{i}}, \mathbf{x}_{\mathbf{i}_c}\right) = \left(\mathbf{x}_{\mathbf{i}}, \mathbf{y}_2\right) \in \mathbf{h}^{-1}\left(\mathbf{y}_1\right)$$

which implies $\pi_{\mathbf{i}_c}\mathbf{x} = \mathbf{y}_2 \in \pi_{\mathbf{i}_c}\mathbf{h}^{-1}\left(\mathbf{y}_1\right)$. Thus

$$\mathbf{y}_2 \in \pi_{\mathbf{i}_c}\left(\mathbf{h}^{-1}\left(\mathbf{y}_1\right) \cap E_j^{\mathbf{i}}\right)$$

and by Fubini's theorem, the integral in 12.35 is

$$\int_{\mathbb{R}^m} \int_{\pi_{\mathbf{i}_c}\left(\mathbf{h}^{-1}(\mathbf{y}_1)\cap E_j^{\mathbf{i}}\right)} \det\left(D\mathbf{h}\left(\mathbf{g}\left(\mathbf{y}\right)\right) D\mathbf{h}\left(\mathbf{g}\left(\mathbf{y}\right)\right)^*\right)^{1/2} \left|\det D_{\mathbf{x}_{\mathbf{i}}}\mathbf{h}\left(\mathbf{g}\left(\mathbf{y}\right)\right)\right|^{-1} dy_2 dy_1 \tag{12.36}$$

Now consider the inner integral in 12.36 in which $\mathbf{y}_1$ is fixed. The integrand equals

$$\det\left[\begin{pmatrix} D_{\mathbf{x}_{\mathbf{i}}}\mathbf{h}\left(\mathbf{g}\left(\mathbf{y}\right)\right) & D_{\mathbf{x}_{\mathbf{i}_c}}\mathbf{h}\left(\mathbf{g}\left(\mathbf{y}\right)\right) \end{pmatrix} \begin{pmatrix} D_{\mathbf{x}_{\mathbf{i}}}\mathbf{h}\left(\mathbf{g}\left(\mathbf{y}\right)\right)^* \\ D_{\mathbf{x}_{\mathbf{i}_c}}\mathbf{h}\left(\mathbf{g}\left(\mathbf{y}\right)\right)^* \end{pmatrix}\right]^{1/2} \left|\det D_{\mathbf{x}_{\mathbf{i}}}\mathbf{h}\left(\mathbf{g}\left(\mathbf{y}\right)\right)\right|^{-1}. \tag{12.37}$$

I want to massage the above expression slightly. Since $\mathbf{y}_1$ is fixed, and $\mathbf{y}_1 = \mathbf{h}\left(\pi_{\mathbf{i}}\mathbf{g}\left(\mathbf{y}\right), \pi_{\mathbf{i}_c}\mathbf{g}\left(\mathbf{y}\right)\right) = \mathbf{h}\left(\mathbf{g}\left(\mathbf{y}\right)\right)$, it follows

$$\begin{aligned}\mathbf{0} &= D_{\mathbf{x}_\mathbf{i}}\mathbf{h}\left(\mathbf{g}\left(\mathbf{y}\right)\right) D_{\mathbf{y}_2}\pi_{\mathbf{i}}\mathbf{g}\left(\mathbf{y}\right) + D_{\mathbf{x}_{\mathbf{i}_c}}\mathbf{h}\left(\mathbf{g}\left(\mathbf{y}\right)\right) D_{\mathbf{y}_2}\pi_{\mathbf{i}_c}\mathbf{g}\left(\mathbf{y}\right) \\ &= D_{\mathbf{x}_\mathbf{i}}\mathbf{h}\left(\mathbf{g}\left(\mathbf{y}\right)\right) D_{\mathbf{y}_2}\pi_{\mathbf{i}}\mathbf{g}\left(\mathbf{y}\right) + D_{\mathbf{x}_{\mathbf{i}_c}}\mathbf{h}\left(\mathbf{g}\left(\mathbf{y}\right)\right).\end{aligned}$$

because by definition of $\mathbf{g}$ and $\mathbf{h}^{\mathbf{i}}$, $\pi_{\mathbf{i}_c}\mathbf{g}\left(\mathbf{y}\right) = \mathbf{y}_2$ so $D_{\mathbf{y}_2}\pi_{\mathbf{i}_c}\mathbf{g}\left(\mathbf{y}\right) = \mathrm{id}$. Letting $A \equiv D_{\mathbf{x}_\mathbf{i}}\mathbf{h}\left(\mathbf{g}\left(\mathbf{y}\right)\right)$ and $B \equiv D_{\mathbf{y}_2}\pi_{\mathbf{i}}\mathbf{g}\left(\mathbf{y}\right)$ and using the above formula, 12.37 is of the form

$$\begin{aligned}&\det\left[\begin{pmatrix} A & -AB \end{pmatrix}\begin{pmatrix} A^* \\ -B^*A^* \end{pmatrix}\right]^{1/2} \left|\det A\right|^{-1} \\ &= \det\left[AA^* + ABB^*A^*\right]^{1/2} \left|\det A\right|^{-1} \\ &= \det\left[A\left(I + BB^*\right)A^*\right]^{1/2} \left|\det A\right|^{-1} \\ &= \left(\det\left(A\right)\det\left(A^*\right)\right)^{1/2} \det\left(I + BB^*\right)^{1/2} \left|\det A\right|^{-1} \\ &= \det\left(I + BB^*\right)^{1/2},\end{aligned}$$

which, by Corollary 12.7.2, equals $\det\left(I + B^*B\right)^{1/2}$. (Note the size of the identity changes in these two expressions, the first being an $m \times m$ matrix and the second being a $n-m \times n-m$ matrix.)

Since $\pi_{\mathbf{i}_c}\mathbf{g}\left(\mathbf{y}\right) = \mathbf{y}_2$,

$$\det\left(I + B^*B\right)^{1/2} = \det\left[\begin{pmatrix} B^* & I \end{pmatrix}\begin{pmatrix} B \\ I \end{pmatrix}\right]^{1/2}$$

$$= \det\left[\begin{pmatrix} D_{\mathbf{y}_2}\pi_{\mathbf{i}}\mathbf{g}\left(\mathbf{y}\right)^* & D_{\mathbf{y}_2}\pi_{\mathbf{i}_c}\mathbf{g}\left(\mathbf{y}\right)^* \end{pmatrix}\begin{pmatrix} D_{\mathbf{y}_2}\pi_{\mathbf{i}}\mathbf{g}\left(\mathbf{y}\right) \\ D_{\mathbf{y}_2}\pi_{\mathbf{i}_c}\mathbf{g}\left(\mathbf{y}\right) \end{pmatrix}\right]^{1/2}$$

$$= \det\left(D_{\mathbf{y}_2}\mathbf{g}\left(\mathbf{y}\right)^* D_{\mathbf{y}_2}\mathbf{g}\left(\mathbf{y}\right)\right)^{1/2}.$$

Therefore, 12.36 reduces to

$$\int_{E_j^{\mathbf{i}}} \det\left(D\mathbf{h}\left(\mathbf{x}\right) D\mathbf{h}\left(\mathbf{x}\right)^*\right)^{1/2} dx =$$

$$\int_{\mathbb{R}^m} \int_{\pi_{\mathbf{i}_c}\left(\mathbf{h}^{-1}(\mathbf{y}_1)\cap E_j^{\mathbf{i}}\right)} \det\left(D_{\mathbf{y}_2}\mathbf{g}\left(\mathbf{y}\right)^* D_{\mathbf{y}_2}\mathbf{g}\left(\mathbf{y}\right)\right)^{1/2} dy_2 dy_1. \tag{12.38}$$

Recall $\mathbf{g}\left(\mathbf{y}_1, \mathbf{y}_2\right) = \mathbf{g}\left(\mathbf{h}\left(\mathbf{x}\right), \mathbf{y}_2\right) = \mathbf{x}$. Thus

$$\mathbf{g}\left(\mathbf{y}_1, \pi_{\mathbf{i}_c}\mathbf{x}\right) = \mathbf{g}\left(\mathbf{y}_1, \mathbf{y}_2\right) = \mathbf{x}.$$

Thus

$$\mathbf{g}\left(\mathbf{y}_1, \pi_{\mathbf{i}_c}\left(\mathbf{h}^{-1}\left(\mathbf{y}_1\right) \cap E_j^{\mathbf{i}}\right)\right) = \mathbf{h}^{-1}\left(\mathbf{y}_1\right) \cap E_j^{\mathbf{i}}$$

and so the area formula applied to the inside integral in 12.38 yields

$$\int_{\mathbf{h}^{-1}(\mathbf{y}_1)\cap E_j^{\mathbf{i}}} d\mathcal{H}^n = \int_{\pi_{\mathbf{i}_c}\left(\mathbf{h}^{-1}(\mathbf{y}_1)\cap E_j^{\mathbf{i}}\right)} \det\left(D_{\mathbf{y}_2}\mathbf{g}\left(\mathbf{y}\right)^* D_{\mathbf{y}_2}\mathbf{g}\left(\mathbf{y}\right)\right)^{1/2} dy_2 dy_1$$

and so this integral equals

$$\mathcal{H}^{n-m}\left(\mathbf{h}^{-1}\left(\mathbf{y}_1\right) \cap E_j^{\mathbf{i}}\right).$$

It follows

$$\begin{aligned}&\int_{E_j^{\mathbf{i}}} \det\left(D\mathbf{h}\left(\mathbf{x}\right) D\mathbf{h}\left(\mathbf{x}\right)^*\right)^{1/2} dx\\ =\ &\int_{\mathbb{R}^m} \mathcal{H}^{n-m}\left(\mathbf{h}^{-1}\left(\mathbf{y}_1\right) \cap E_j^{\mathbf{i}}\right) dy_1.\end{aligned}$$

Therefore, summing the terms over all $\mathbf{i}$ and j,

$$\int_A \det\left(D\mathbf{h}\left(\mathbf{x}\right) D\mathbf{h}\left(\mathbf{x}\right)^*\right)^{1/2} dx = \int_{\mathbb{R}^m} \mathcal{H}^{n-m}\left(\mathbf{h}^{-1}\left(\mathbf{y}\right) \cap A\right) dy.$$

This proves the lemma.

You don't need to assume $\mathbf{h}$ is Lipschitz.

Corollary 12.7.7 *Let A be a Lebesgue measurable set in V, an open set and let $\mathbf{h} : V \subseteq \mathbb{R}^n \to \mathbb{R}^m$ where $m \leq n$ is C^1 and for which*

$$J\mathbf{h}\left(\mathbf{x}\right) \equiv \det\left(D\mathbf{h}\left(\mathbf{x}\right) D\mathbf{h}\left(\mathbf{x}\right)^*\right)^{1/2} \neq 0.$$

Then the following formula holds along with all measurability assertions needed for it to make sense.

$$\int_{\mathbb{R}^m} \mathcal{H}^{n-m}\left(A \cap \mathbf{h}^{-1}\left(\mathbf{y}\right)\right) dy = \int_A J\mathbf{h}\left(\mathbf{x}\right) dx \tag{12.39}$$

Proof: Consider for each $\mathbf{x} \in A$, a ball, $B\left(\mathbf{x}, r_{\mathbf{x}}\right) \subseteq \overline{B\left(\mathbf{x}, r_{\mathbf{x}}\right)} \subseteq V$ such that $r_{\mathbf{x}} < 1$. Then by the mean value inequality, Theorem 6.4.2, $\mathbf{h}$ is Lipschitz on B_i. Letting $\{B_i\}$ be an open covering of A, let $C_i \equiv B_i \cap A$. Then let $\{A_i\}$ be such that $A_i \subseteq C_i$ but the A_i are disjoint and $\cup_i A_i = A$. The conclusion of Lemma 12.7.6 applies with B_i playing the role of V in that lemma and one can write

$$\int_{\mathbb{R}^m} \mathcal{H}^{n-m}\left(A_i \cap \mathbf{h}^{-1}\left(\mathbf{y}\right)\right) dy = \int_{A_i} J\mathbf{h}\left(\mathbf{x}\right) dx$$

By Lemma 12.4.1 $\mathcal{H}^{n-m}$Then using the monotone convergence theorem,

$$\int_{\mathbb{R}^m} \mathcal{H}^{n-m}\left(A \cap \mathbf{h}^{-1}\left(\mathbf{y}\right)\right) dy = \int_{\mathbb{R}^m} \mathcal{H}^{n-m}\left(\cup_{i=1}^{\infty} A_i \cap \mathbf{h}^{-1}\left(\mathbf{y}\right)\right) dy =$$

$$\int_{\mathbb{R}^m} \sum_{i=1}^{\infty} \mathcal{H}^{n-m}\left(A_i \cap \mathbf{h}^{-1}\left(\mathbf{y}\right)\right) dy = \int_{\mathbb{R}^m} \lim_{M\to\infty} \sum_{i=1}^{M} \mathcal{H}^{n-m}\left(A_i \cap \mathbf{h}^{-1}\left(\mathbf{y}\right)\right) dy$$

$$= \lim_{M\to\infty} \int_{\mathbb{R}^m} \sum_{i=1}^{M} \mathcal{H}^{n-m}\left(A_i \cap \mathbf{h}^{-1}\left(\mathbf{y}\right)\right) dy = \lim_{M\to\infty} \sum_{i=1}^{M} \int_{\mathbb{R}^m} \mathcal{H}^{n-m}\left(A_i \cap \mathbf{h}^{-1}\left(\mathbf{y}\right)\right) dy$$

$$= \lim_{M\to\infty} \sum_{i=1}^{M} \int \mathcal{X}_{A_i}\left(\mathbf{x}\right) J\mathbf{h}\left(\mathbf{x}\right) dx = \int \sum_{i=1}^{\infty} \mathcal{X}_{A_i}\left(\mathbf{x}\right) J\mathbf{h}\left(\mathbf{x}\right) dx$$

$$= \int \mathcal{X}_{\cup_{i=1}^{\infty} A_i}\left(\mathbf{x}\right) J\mathbf{h}\left(\mathbf{x}\right) dx = \int \mathcal{X}_A\left(\mathbf{x}\right) J\mathbf{h}\left(\mathbf{x}\right) dx$$

This proves the theorem.

You don't need to assume $J\mathbf{h}\left(\mathbf{x}\right) \equiv \det\left(D\mathbf{h}\left(\mathbf{x}\right) D\mathbf{h}\left(\mathbf{x}\right)^*\right)^{1/2} \neq 0$.

Corollary 12.7.8 *Let $A \subseteq V$ an open set be measurable and let $\mathbf{h} : V \to \mathbb{R}^m$ be C^1 where $m \leq n$. Then the following formula holds along with all measurability assertions needed for it to make sense.*

$$\int_{\mathbb{R}^m} \mathcal{H}^{n-m}\left(A \cap \mathbf{h}^{-1}(\mathbf{y})\right) dy = \int_A J\mathbf{h}(\mathbf{x})\, dx \tag{12.40}$$

where $J\mathbf{h}(\mathbf{x}) \equiv \det\left(D\mathbf{h}(\mathbf{x})\, D\mathbf{h}(\mathbf{x})^*\right)^{1/2}$.

Proof: By Corollary 12.7.7, this formula is true for all measurable A contained in the open set $\mathbb{R}^n \setminus S$. It remains to verify the formula for all measurable sets, A, whether or not they intersect S.

Consider the case where

$$A \subseteq S \equiv \{\mathbf{x} : J(D\mathbf{h}(\mathbf{x})) = 0\}.$$

Let A be compact so that by Lemma 12.7.4, $\mathbf{y} \to \mathcal{H}^{n-m}\left(A \cap \mathbf{h}^{-1}(\mathbf{y})\right)$ is Borel measurable. For $\varepsilon > 0$, define $\mathbf{k}$, $\mathbf{p} : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^m$ by

$$\mathbf{k}(\mathbf{x},\mathbf{y}) \equiv \mathbf{h}(\mathbf{x}) + \varepsilon\mathbf{y},\ \mathbf{p}(\mathbf{x},\mathbf{y}) \equiv \mathbf{y}.$$

Then

$$D\mathbf{k}(\mathbf{x},\mathbf{y}) = (D\mathbf{h}(\mathbf{x}), \varepsilon I)$$

and so

$$J\mathbf{k}^2 = \det\left((D\mathbf{h}(\mathbf{x}), \varepsilon I)\begin{pmatrix} D\mathbf{h}^* \\ \varepsilon I \end{pmatrix}\right) = \det\left(D\mathbf{h}(\mathbf{x})\, D\mathbf{h}(\mathbf{x})^* + \varepsilon^2 I\right)$$

Now $\left|\left|D\mathbf{h}(\mathbf{x})\, D\mathbf{h}(\mathbf{x})^*\right|\right|$ is bounded on A because A is compact and $D\mathbf{h}$ is continuous. The linear operator is also of the form

$$\sum_{k=1}^{m} \lambda_k \mathbf{z}_k \mathbf{z}_k$$

for each $\lambda_k \geq 0$ because it is self adjoint and

$$D\mathbf{h}(\mathbf{x})\, D\mathbf{h}(\mathbf{x})^* \mathbf{x} \cdot \mathbf{x} = D\mathbf{h}(\mathbf{x})^* \mathbf{x} \cdot D\mathbf{h}(\mathbf{x})^* \mathbf{x} \geq 0.$$

Since $A \subseteq S$, at least one of these λ_k equals zero. However, they are all bounded by some constant, C for all $\mathbf{x} \in A$ due to the existence of an upper bound for $\left|\left|D\mathbf{h}(\mathbf{x})\, D\mathbf{h}(\mathbf{x})^*\right|\right|$. Thus

$$J\mathbf{k}^2 = \prod_{i=1}^{m} \left(\lambda_i^2 + \varepsilon^2\right) \in [\varepsilon^{2m}, C^2\varepsilon^2] \tag{12.41}$$

since one of the λ_i equals 0. Since $J\mathbf{k} \neq 0$, 12.41 implies

$$\varepsilon C m_{n+m}\left(A \times \overline{B(\mathbf{0},1)}\right) \geq \int_{A \times \overline{B(\mathbf{0},1)}} |J\mathbf{k}|\, dm_{n+m}$$

$$= \int_{\mathbb{R}^m} \mathcal{H}^n\left(\mathbf{k}^{-1}(\mathbf{y}) \cap A \times \overline{B(\mathbf{0},1)}\right) dy$$

Which by Lemma 12.7.4 is at least as large as

$$C_{nm} \int_{\mathbb{R}^m} \int_{\mathbb{R}^m} \mathcal{H}^{n-m}\left(\mathbf{k}^{-1}(\mathbf{y}) \cap \mathbf{p}^{-1}(\mathbf{w}) \cap A \times \overline{B(\mathbf{0},1)}\right) dw dy \tag{12.42}$$

where $C_{nm} = \frac{\alpha(n)}{\alpha(n-m)\alpha(m)}$. It is formula 12.30 applied to the situation where $\mathbf{h} = \mathbf{p}$. It is clear $\mathbf{p}$ is Lipschitz continuous with Lipschitz constant 1 since $\mathbf{p}$ is a projection.

Claim:

$$\mathcal{H}^{n-m}\left(\mathbf{k}^{-1}(\mathbf{y}) \cap \mathbf{p}^{-1}(\mathbf{w}) \cap A \times \overline{B(\mathbf{0},1)}\right)$$
$$\geq \mathcal{X}_{\overline{B(\mathbf{0},1)}}(\mathbf{w})\, \mathcal{H}^{n-m}\left(\mathbf{h}^{-1}(\mathbf{y} - \varepsilon\mathbf{w}) \cap A\right).$$

Proof of the claim: If $\mathbf{w} \notin \overline{B(\mathbf{0},1)}$, there is nothing to prove so assume $\mathbf{w} \in \overline{B(\mathbf{0},1)}$. For such $\mathbf{w}$,

$$(\mathbf{x}, \mathbf{w}_1) \in \mathbf{k}^{-1}(\mathbf{y}) \cap \mathbf{p}^{-1}(\mathbf{w}) \cap A \times \overline{B(\mathbf{0},1)}$$

if and only if

$$\mathbf{k}(\mathbf{x}, \mathbf{w}_1) = \mathbf{h}(\mathbf{x}) + \varepsilon\mathbf{w}_1 = \mathbf{y}, \mathbf{p}(\mathbf{x}, \mathbf{w}_1) = \mathbf{w}_1 = \mathbf{w}, \mathbf{x} \in A,$$

if and only if

$$(\mathbf{x}, \mathbf{w}_1) \in \mathbf{h}^{-1}(\mathbf{y} - \varepsilon\mathbf{w}) \cap A \times \{\mathbf{w}\}.$$

Therefore for $\mathbf{w} \in \overline{B(\mathbf{0},1)}$,

$$\mathcal{H}^{n-m}\left(\mathbf{k}^{-1}(\mathbf{y}) \cap \mathbf{p}^{-1}(\mathbf{w}) \cap A \times \overline{B(\mathbf{0},1)}\right)$$
$$\geq \mathcal{H}^{n-m}\left(\mathbf{h}^{-1}(\mathbf{y} - \varepsilon\mathbf{w}) \cap A \times \{\mathbf{w}\}\right) = \mathcal{H}^{n-m}\left(\mathbf{h}^{-1}(\mathbf{y} - \varepsilon\mathbf{w}) \cap A\right).$$

(Actually equality holds in the claim.) From the claim, 12.42 is at least as large as

$$C_{nm} \int_{\mathbb{R}^m} \int_{\overline{B(\mathbf{0},1)}} \mathcal{H}^{n-m}\left(\mathbf{h}^{-1}(\mathbf{y} - \varepsilon\mathbf{w}) \cap A\right) dw dy \tag{12.43}$$
$$= C_{nm} \int_{\overline{B(\mathbf{0},1)}} \int_{\mathbb{R}^m} \mathcal{H}^{n-m}\left(\mathbf{h}^{-1}(\mathbf{y} - \varepsilon\mathbf{w}) \cap A\right) dy dw$$
$$= C_{nm} m_m\left(\overline{B(\mathbf{0},1)}\right) \int_{\mathbb{R}^m} \mathcal{H}^{n-m}\left(\mathbf{h}^{-1}(\mathbf{y}) \cap A\right) dy. \tag{12.44}$$

The use of Fubini's theorem is justified because the integrand is Borel measurable.

Now by 12.44 this has shown

$$\varepsilon C m_{n+m}\left(A \times \overline{B(\mathbf{0},1)}\right) \geq C_{nm} m_m\left(\overline{B(\mathbf{0},1)}\right) \int_{\mathbb{R}^m} \mathcal{H}^{n-m}\left(\mathbf{h}^{-1}(\mathbf{y}) \cap A\right) dy.$$

it follows since $\varepsilon > 0$ is arbitrary,

$$\int_{\mathbb{R}^m} \mathcal{H}^{n-m}\left(A \cap \mathbf{h}^{-1}(\mathbf{y})\right) dy = 0 = \int_A J\mathbf{h}(\mathbf{x})\, dx.$$

Since this holds for arbitrary compact sets in S, it follows from Lemma 12.7.5 and inner regularity of Lebesgue measure that the equation holds for all measurable subsets of S. Thus if A is any measurable set contained in V

$$\begin{aligned}
\int_{\mathbb{R}^m} \mathcal{H}^{n-m}\left(A \cap \mathbf{h}^{-1}(\mathbf{y})\right) dy &= \int_{\mathbb{R}^m} \mathcal{H}^{n-m}\left(A \cap S \cap \mathbf{h}^{-1}(\mathbf{y})\right) dy \\
&\quad + \int_{\mathbb{R}^m} \mathcal{H}^{n-m}\left((A \setminus S) \cap \mathbf{h}^{-1}(\mathbf{y})\right) dy \\
&= \int_{\mathbb{R}^m} \mathcal{H}^{n-m}\left((A \setminus S) \cap \mathbf{h}^{-1}(\mathbf{y})\right) dy \\
&= \int_{A \setminus S} J\mathbf{h}(\mathbf{x})\, dx = \int_A J\mathbf{h}(\mathbf{x})\, dx
\end{aligned}$$

This completes the proof of the coarea formula.

12.8 A Nonlinear Fubini's Theorem

Coarea formula holds for $\mathbf{h} : V \subseteq \mathbb{R}^n \to \mathbb{R}^m, n \geq m$ if whenever A is a Lebesgue measurable subset of the open set V,the following formula is valid.

$$\int_{\mathbb{R}^m} \mathcal{H}^{n-m}\left(A \cap \mathbf{h}^{-1}(\mathbf{y})\right) dy = \int_A J\mathbf{h}(\mathbf{x})\, dx \tag{12.45}$$

where

$$J\mathbf{h}(\mathbf{x}) = \det\left(D\mathbf{h}(\mathbf{x})\, D\mathbf{h}(\mathbf{x})^*\right)^{1/2}.$$

Note this is the same as

$$\int_A J\mathbf{h}(\mathbf{x}))dx = \int_{\mathbf{h}(A)} \mathcal{H}^{n-m}\left(A \cap \mathbf{h}^{-1}(\mathbf{y})\right) dy$$

because if $\mathbf{y} \notin \mathbf{h}(A)$, then $\mathbf{h}^{-1}(\mathbf{y}) = \emptyset$. Now let

$$s(\mathbf{x}) = \sum_{i=1}^{p} c_i \mathcal{X}_{E_i}(\mathbf{x})$$

where E_i is a Lebesgue measurable subset of V and $c_i \geq 0$. Then

$$\begin{aligned}
\int_V s(\mathbf{x})\, J\mathbf{h}(\mathbf{x})\, dx &= \sum_{i=1}^{p} c_i \int_{E_i} J\mathbf{h}(\mathbf{x})\, dx \\
&= \sum_{i=1}^{p} c_i \int_{\mathbf{h}(E_i)} \mathcal{H}^{n-m}\left(E_i \cap \mathbf{h}^{-1}(\mathbf{y})\right) dy \\
&= \int_{\mathbf{h}(V)} \sum_{i=1}^{p} c_i \mathcal{H}^{n-m}\left(E_i \cap \mathbf{h}^{-1}(\mathbf{y})\right) dy \\
&= \int_{\mathbf{h}(V)} \left[\int_{\mathbf{h}^{-1}(\mathbf{y})} s\, d\mathcal{H}^{n-m}\right] dy \\
&= \int_{\mathbf{h}(V)} \left[\int_{\mathbf{h}^{-1}(\mathbf{y})} s\, d\mathcal{H}^{n-m}\right] dy.
\end{aligned} \tag{12.46}$$

Theorem 12.8.1 *Let $g \geq 0$ be Lebesgue measurable and let V be an open subset of $\mathbb{R}^n$.*

$$\mathbf{h} : V \to \mathbb{R}^m,\ n \geq m$$

where $\mathbf{h}$ is C^1.Then

$$\int_V g(\mathbf{x})\, J((D\mathbf{h}(\mathbf{x})))\, dx = \int_{\mathbf{h}(V)} \left[\int_{\mathbf{h}^{-1}(\mathbf{y})} g\, d\mathcal{H}^{n-m}\right] dy.$$

Proof: Let $s_i \uparrow g$ where s_i is a simple function satisfying 12.46. Then let $i \to \infty$ and use the monotone convergence theorem to replace s_i with g. This proves the nonlinear version of Fubini's theorem.

Note that this formula is a nonlinear version of Fubini's theorem. The "$n-m$ dimensional surface", $\mathbf{h}^{-1}(\mathbf{y})$, plays the role of $\mathbb{R}^{n-m}$ and $\mathcal{H}^{n-m}$ is like $n-m$ dimensional Lebesgue measure. The term, $J((D\mathbf{h}(\mathbf{x})))$, corrects for the error occurring because of the lack of flatness of $\mathbf{h}^{-1}(\mathbf{y})$.

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